ALGEBRA QUALIFYING EXAMINATION

AUGUST 2014

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say. Rings are taken to be rings with unity.

1A. Let A be an n-by-n matrix in $\mathbb{C}$ for some $n \geq 2$ that is in Jordan canonical form (which by convention is upper-triangular) with minimal polynomial equal to $(x-\lambda)^n$ for some $\lambda \in \mathbb{C}$. Let B be the matrix attained from A by replacing its $(n,1)$-entry by 1. Find the Jordan canonical form of B.

1B. Let k be an algebraically closed field and let M be an $n \times n$ matrix with entries in k. Prove that M can be written as a sum $M = M_s + M_n$ of matrices with k-entries such that
- M_s is semisimple (i.e., its minimal polynomial has distinct roots)
- M_n is nilpotent, and
- M_s and M_n commute.

2A. Let $G = \mathrm{GL}_n(\mathbb{C})$ be the group of n-by-n invertible matrices with complex entries, and let $T < G$ be the subgroup of diagonal matrices. Prove that $N_G(T)/T \cong S_n$, where $N_G(T)$ denotes the normalizer of T in G.

2B. Show that there is a unique isomorphism class of nonabelian groups of order 105, consisting of groups of the form $C \times H$, where C has order 5 and H is nonabelian of order 21.

3A. Let R be the polynomial ring $\mathbb{C}[x,y,z]$.
- (a) Show that every maximal ideal of R has the form $(x-a, y-b, z-c)$ for some $a,b,c \in \mathbb{C}$. You may use the following fact without proof: the only field extension of $\mathbb{C}$ that is finitely generated as a $\mathbb{C}$-algebra is $\mathbb{C}$.
- (b) Let I be the ideal $(x^2-y^2-z^2, xy+1, z^3)$ of R. Find the maximal ideals of the quotient ring R/I, writing each as the image of a maximal ideal of R containing I.

3B. Let R be a commutative ring, and let I be a proper ideal of R.
- (a) Show that if S is a multiplicatively closed subset of R with $S \cap I = \varnothing$, then there exists a prime ideal containing I disjoint from S.
- (b) The *radical* of I is defined to be the set

$$\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n > 0\}.$$

Prove that $\sqrt{I}$ is the intersection of the prime ideals of R that contain I.

4A. Let ζ_{72} be a primitive 72nd root of unity. Find, with proof, all square-free (i.e., not divisible by the square of any prime number) integers $d \neq 1$ such that $\mathbb{Q}(\sqrt{d})$ is contained in $\mathbb{Q}(\zeta_{72})$.

4B. Let $f(x) = x^4 + ax^3 + bx^2 + ax + 1 \in \mathbb{Q}[x]$, or in other words, let f be a monic polynomial of degree 4 with $f(x) = x^4 f(x^{-1})$. Suppose that f is irreducible. Show that the Galois group of the splitting field of f over $\mathbb{Q}$

is isomorphic to the cyclic group of order 4, the Klein-four group, or the dihedral group of order 8.

5A. Give an example of a commutative ring R and a finitely generated R-module M with the following properties.

- M is torsion-free, i.e., if $rm = 0$ with $r \in R$ and $m \in M$ then either $r = 0$ or $m = 0$, and
- M is *not* a free R-module.

5B. Let F be a field, and let K be a finite Galois extension of F. Let $\alpha \in K$, and set $n = [F(\alpha) : F]$. Show that $K \otimes_F F(\alpha)$ and K^n are isomorphic rings.