ANALYSIS QUALIFYING EXAM

AUGUST 2020

Please show all of your work. GOOD LUCK!

(1) Prove that the series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{x}{n^2}\right)$$

converges to a continuous function on $\mathbb{R}$.

(2) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f : X \to [0, \infty)$ be measurable. If the following limit exists, determine its value:

$$\lim_{n\to\infty} \int_X e^{-nf(x)}\, d\mu(x)\,.$$

If it does not, explain. In either case, justify your claims.

(3) Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable. Show that: if $f(a) = 0$, then

$$\int_a^b |f(x)|^2 dx \le \frac{(b-a)^2}{2} \int_a^b |f'(x)|^2 dx\,.$$

(4) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $g : X \to \mathbb{R}$ be measurable. Suppose that there is $T > 0$ for which:

$$\mu\left(\{x \in X : |g(x)| > t\}\right) = \frac{1}{t^2} \quad \text{for all } t \ge T.$$

Find all values of p, with $1 \le p \le \infty$, for which $g \in L^p(X, \mu)$. **Hint:** Fubini's theorem may be helpful.

(5) Let X be the set of all tuples $(x_1 \ldots x_n)$ with $x_j = \pm 1$. Let $\mathcal{M}$ be the set of all subsets of X, and let a measure μ on the sigma-algebra $\mathcal{M}$ be defined by the formula: $\mu(X) = 2^{-n}|X|$ where $|X|$ is the cardinality of X. Evaluate

$$\int_X (x_1 + \cdots + x_n)^2 d\mu.$$

(6) Let $f :\ \mathbb{R}^2 \to \mathbb{R}$ be a mapping that is given by the formula $f(x_1, x_2) = |x_1| + |x_2|$. Define a Borel measure on $\mathbb{R}$:

$$\mu(X) = m_2(f^{-1}(X))$$

where m_2 is the two-dimensional Lesbegue measure. Prove that μ is absolutely continuous with respect to the Lesbegue measure m on $\mathbb{R}$ and find the Radon–Nikodym derivative $d\mu/dm$.