Geometry/Topology Qualifying Exam, January 2019

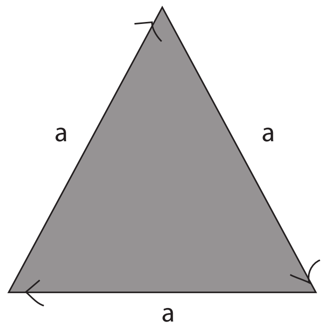
1) Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} dx.$$

2) Consider $X_a = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = a\right\}$ and $Y_b = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+by^2 = z^2+1\right\}$.
For which values of a and b is $X_a \cap Y_b$ a submanifold of $\mathbb{R}^3$? Justify your answer.

3) Consider the torus $S^1 \times S^1$ with coordinate chart (inverse) $\phi : (0,2\pi) \times (0,2\pi) \to S^1 \times S^1$ given by $\phi(x,y) = \left(e^{ix}, e^{iy}\right)$. Show that $dx \wedge dy$ extends to a global 2-form that represents a nontrivial class in $H^2_{dR}\left(S^1 \times S^1\right)$.

4) Consider the topological space Y shown in the following shaded figure with sides identified as indicated.



Calculate the fundamental group of Y. Be sure to specify your choice of basepoint.

5) Let S^n denote the n dimensional sphere (the set of all unit length vectors in $\mathbb{R}^{n+1}$).

a) Show that if the continuous mapping $f : S^n \to S^n$ is not surjective, then $f_* : H_k(S^n) \to H_k(S^n)$ is trivial for all $k > 0$.

b) Give an example of a continuous mapping $f : S^2 \to S^2$ that is surjective, but $f_* : H_k\left(S^2\right) \to H_k\left(S^2\right)$ is trivial for all $k > 0$.

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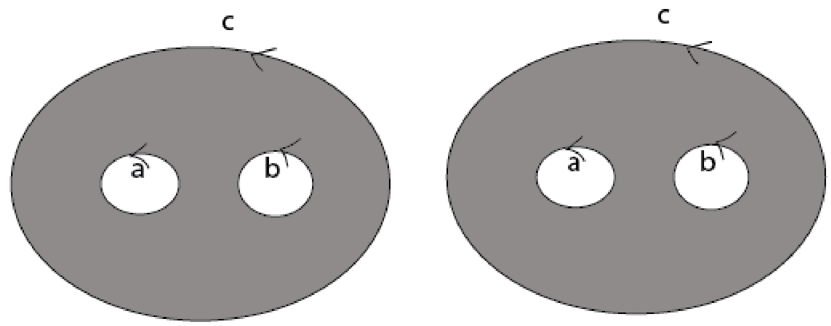
6) Consider the space X obtained by identifying the shaded regions (Region U and Region V) below along the curves a, b, c as shown.



Region U, Region V

Let $\tilde{U}$ and $\tilde{V}$ be open sets in X that deformation retract to U and V, respectively. We consider the Mayer-Vietoris sequence from the decomposition $X = \tilde{U} \cup \tilde{V}$.

a) Calculate $H_2(X)$ and $H_0(X)$.

b) Show that the term in the sequence $\alpha : H_1(\tilde{U}) \oplus H_1(\tilde{V}) \to H_1(X)$ has image isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$,

c) Show that the term in the sequence $\beta : H_0(\tilde{U} \cap \tilde{V}) \to H_0(\tilde{U}) \oplus H_0(\tilde{V})$ has kernel isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

d) Let γ denote the map in the sequence $\gamma : H_1(X) \to H_0(\tilde{U} \cap \tilde{V})$. Show that the following is a short exact sequence that splits:

$$0 \to Ker(\gamma) \to H_1(X) \xrightarrow{\gamma} Im(\gamma) \to 0,$$

and use the sequence to calculate $H_1(X)$. Note: you can use parts b and/or c without proof even if you did not prove them.