# ALGEBRA QUALIFYING EXAMINATION

AUGUST 2020

Do either one of $nA$ or $nB$ for $1 \leq n \leq 5$. Justify all your answers.

1A. Let $V$ be a vector space over $\mathbb{R}$ and $h_V : V \times V \to \mathbb{R}$ be a positive semidefinite bilinear form, i.e. $h_V$ is a symmetric bilinear form ad $h_V(v,v) \geq 0$ for all $v \in V$. Let

$$W = \{v \in V \mid h_V(v, v') = 0 \text{ for all } v' \in V\}.$$

Define a bilinear form on $V/W$ by

$$h_{V/W}(\overline{v}, \overline{v'}) = h_V(v, v'),$$

where $\overline{v}, \overline{v'} \in V/W$ and $v \in V$ (resp. $v' \in V'$) maps to $\overline{v}$ (resp. $\overline{v'}$) under the canonical map $V \to V/W$.

a.) Show that $h_{V/W}$ is well-defined, i.e. it is independent of the choice of $v$ and $v'$.
b.) Show that $h_{V/W}$ is an inner product on $V/W$.

1B. Fix an integer $n \geq 1$ and let $P_n$ be the vector space of polynomials in one variable with complex coefficients and degree at most $n$. Let $T : P_n \to P_n$ be the linear map given by $T(f(x)) = f(x+1)$.

a.) Find all eigenvalues and eigenvectors of $T$.
b.) Determine the Jordan canonical form of $T$.

2A. Let $\mathbb{F}_3$ be the field with three elements. Consider the subgroup of $\mathrm{GL}_2(\mathbb{F}_3)$ given by

$$G = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{F}_3^\times,\ b \in \mathbb{F}_3 \right\}.$$

Give an explicit isomorphism $G \simeq S_3$.

2B. Prove that no simple group has order $280 = 2^3 \cdot 5 \cdot 7$.

3A. Let $R$ be an integral domain and assume that $R[x]$ is a PID. Show that $R$ is a field.

3B. Let $R = \mathbb{C}[x,y]/(x^3 + x^2 - y^4)$.

a.) Prove that $R$ is an integral domain. **Hint:** Consider $f := x^3 + x^2 - y^4$ as a polynomial in $y$ with coefficients in the PID $\mathbb{C}[x]$, and use Eisenstein's Criterion.
b.) Prove that $R$ is not a Unique Factorization Domain.

4A. Let $K = \mathbb{Q}\left(\sqrt[4]{2}(1+i)\right)$. Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

4B. Let $f(x) = x^5 + 20x + 32$ and let $K$ be the splitting field of $f$ over $\mathbb{Q}$. Prove that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to either $A_5$ or to $D_5$. You may use without proof the facts that $f$ is irreducible modulo 3, and the discriminant of $f$ is $2^{18} \cdot 5^6$.

5A. Let $R = \mathbb{F}_2[t]$ and $M$ be the module over $R$ generated by $a, b, c, d$ subjected to the relations

$$(t+1)a + tb + tc + td = t^2 a + (t^2+1)b + tc + (t^2+1)d = 0.$$

Write $M$ and $M \otimes_R M$ as a direct sum of cyclic modules.

5B. Let $a, b, c$ be unknown integers, and let $M$ be the cokernel of the map $\Psi : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by $\Psi(x, y, z) = (6x + 10y + 6z, 4x + 10y + 10z, ax + by + cz)$.

a.) Do there exist integers $a, b, c$ with $M \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(5)$ as abelian groups? Prove your answer.

b.) Do there exist integers $a, b, c$ with $M \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(6)$ as abelian groups? Prove your answer.