ANALYSIS QUALIFYING EXAM

JANUARY 2018

Please show all of your work. GOOD LUCK!

(1) A real-valued function f on $[a,b]$ is said to be locally bounded if for every $x \in [a,b]$ there is a $\delta > 0$ such that $f(x)$ is bounded on $(x-\delta, x+\delta) \cap [a,b]$. Prove that if f is locally bounded on $[a,b]$ then f is bounded on $[a,b]$.

(2) i) Prove the following estimate:

$$\int_1^\infty \frac{\sqrt[3]{1+x}}{x^2}\, dx \leq \sqrt[3]{6}\,.$$

ii) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Let f and g be measurable functions on X that are positive μ-a.e.. Suppose that $1 \leq f(x)g(x)$ for μ-a.e. $x \in X$. Prove that

$$1 \leq \int_X f\, d\mu \cdot \int_X g\, d\mu$$

(3) Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable, real-valued functions on $[0,1]$.

i) Suppose there exists positive numbers C and ϵ for which

$$\int_0^1 f_n(x)^2\, dx \leq Cn^{2-\epsilon} \quad \text{for all } n \geq 1.$$

Show that $g_n = \frac{f_n}{n}$ converges to zero in measure.

ii) Suppose $f_n = \chi_{E_n}$ with $E_n \subset [0,1]$ for all $n \geq 1$. Show that if f_n converges to f in L^1, then f is also the characteristic function of a measurable set.

(4) Let $f \in L^1([0,1])$ with $f \notin L^2([0,1])$. Consider the following:

$$\lim_{n\to\infty} \int_0^1 n \ln\left(1 + \frac{|f(x)|^2}{n^2}\right) dx$$

Determine whether or not this limit exists. If it exists, calculate it. If it does not, explain.

(5) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let f be a non-negative, measurable function on X. Define a function on $[0, \infty)$ by

$$F(t) = \mu(\{x : f(x) > t\})$$

Prove that for all $\alpha \geq 0$, $t^\alpha F(t)$ is integrable with respect to Lebesgue measure on $[0, \infty)$ if and only if $f \in L^{1+\alpha}(X, \mu)$.
Hint: express $F(t)$ as an integral.

(6) For non-negative integers k, define

$$S_k = \{f \in L^2([0,1]) : \int_0^1 f(x) \, \exp(2\pi i n x) \, dx = 0, \; if \; |n| \leq k\}$$

Let P_k be the orthogonal projection onto the closed subspace S_k. Prove or disprove each of the following statements:
i) $||P_k|| \to 0$ as $k \to \infty$.

ii) $\forall f \in L^2([0,1])$, we have $||P_k f|| \to 0$ as $k \to \infty$.