Geometry-Topology Qualifying Exam

Spring 2015

Problem 1
Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{a-\cos\theta},$$

if $a > 1$ is a constant. Hint: Use $z = e^{i\theta}$ substitution.

Problem 2 Find all critical points and all critical values of the function

$$F : \mathbb{R}^4 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \mapsto \begin{pmatrix} x^3+y^3+z^3+t^3 \\ xyzt \end{pmatrix}.$$

For which of the following values $c \in \mathbb{R}^2$ is the level set $F^{-1}(c)$ a regular submanifold?

$$(i)\ c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad (ii)\ c = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \qquad (iii)\ c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Why?

Problem 3
For any smooth one-form ω and any pair of smooth vector fields X and Y

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

Generalize this formula for any smooth two-form, i.e. complete the following equality:

$$d\omega(X,Y,Z) = \dots$$

Prove your general formula.
Note: You can either use the local expression for the exterior derivative or its definition as an antiderivation of degree 1 (acting on the graded algebra $\Omega^*(M)$) satisfying $d^2 = 0$, such that for any smooth function f and any smooth vector field X

$$df(X) = Xf.$$

Problem 4

Let p and q be two distinct points on a two-torus T^2. Compute the fundamental group $\pi_1(X, x_0)$ of the space

$$X = T^2/(p \sim q)$$

obtained from the two-torus T^2 by identifying these two points p and q.

Problem 5

Consider

1. a two-torus $T^2 = S^1 \times S^1$ with a cycle $\beta = S^1 \times p$ and a cycle $\gamma = q \times S^1$, and
2. an annulus $D = \{x \in \mathbb{R}^2 | 1 \leq |x| \leq 2\}$. The boundary of the annulus is $\partial A = a - b$ consists of two circles $a = \{x \in \mathbb{R}^2 || x| = 2\}$ and $b = \{x \in \mathbb{R}^2 || x| = 1\}$.

The space Y is obtained by identifying a with β and also identifying $b \cdot b$ (which is the concatenation of b with itself) with γ :

$$Y = T^2 \sqcup D/\{a \sim \beta, b \cdot b \sim \gamma\}.$$

Compute the singular homology groups of Y.

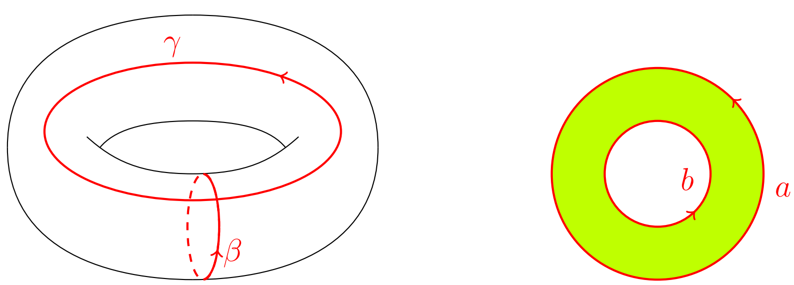

Figure 1: A torus and a disk.

Problem 6

Prove that any pointed map

$$\Phi : (X, x_0) \to (Y, y_0),$$

induces a homomorphism of the fundamental groups

$$\Phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

Compute Φ_* for the inclusion map $\Phi : S^1 \times S^1 \to S^1 \times D^2$, with the second factor $S^1 = \partial D^2$.