Short Pulse Evolution Equation

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Abstract We introduce short pulse evolution equation (SPEE), first derived in a non-optical context in the 80s, the universal equation describing the propagation of short pulses in media which have weak dispersion in the propagation direction. We show how it connects with the first canonical examples of nonlinear wave propagation, the Korteweg–de Vries and nonlinear Schrödinger equations and argue that, in contexts for which SPEE is most useful, modifications of the latter simply do not capture the correct pulse behavior. We discuss some of SPEE's main properties and, in particular, look at its potential singular behaviors in which both the electric field gradient and its amplitude can become large. Finally, we address the practical challenge of whether very high intensity femtosecond pulses can travel significant distances in gases such as the earth's atmosphere.

1 Introduction

The main goal of my Montreal lecture was, and of this short paper is, to show that the principal propagation characteristics of short electromagnetic pulses (10–100 fs in duration) can be captured by a very simple and beautiful evolution equation which we call short pulse evolution equation (SPEE). An additional goal is to ask whether ultra short pulses might be tailored so that they reach high intensities at a specified but distant point along their trajectory paths. The ideas behind the derivation of SPEE are not new. They have their origins in the 60s when there was widespread recognition (see discussions in [1, 2]) that the propagation of waves and wave envelopes occurring in many different contexts had, because of shared symmetries, very similar propagation characteristics and could be well described by what are essentially multinomial Taylor expansions in field amplitudes and their gradients.

There are two granddaddy examples. One is the Korteweg–de Vries (KdV) equation and the other is the nonlinear Schrödinger (NLS) equation. From these starting points sprang many second generation evolution equations which added

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A.D. Bandrauk et al. (eds.), *Laser Filamentation*, CRM Series in Mathematical Physics, DOI 10.1007/978-3-319-23084-9_1

to the KdV equation the effects of diffraction or transverse dispersion (the Kadomtsev–Petviashvili or KP equation), the Benney–Roskes–Davey–Stewartson modifications of the NLS equation, and the three wave interaction (3WI) equations. A variation on the KP equation which embodies the symmetry of field reversal was originally introduced in the 80s by Kuznetsov et al. [3] in the context of sound waves in ferromagnets, elaborated on by Turitsyn and Falkovich [4] and discussed more recently in [5, 6] in the context of electromagnetic fields for which the medium crystalline structure has the centrosymmetric and isotropic properties which introduce the field reversal symmetry. This universal equation, SPEE, also describes the evolution of intense short EM pulses and those are the subject of this short paper.

Why is the study of such an equation in the context of intense ultra short pulses useful? First, it relates the behaviors of light pulses to similar behaviors which occur in different contexts and leads to a cross fertilization of understanding between waves which have similar characteristics, but arise in very different situations. The most important common ingredients are (1) weak dispersion in the direction of ray propagation, (2) almost parallel rays, and (3) weak nonlinearity in a sense we will define. Second, there is much interest in propagating high intensity laser pulses with finite energies (high intensity plus finite energy necessitates short pulse duration) through gases such as the atmosphere. There are many reasons for this, ranging from use as light bullet weapons to ionization and light emissions in the THZ ranges. For details we refer to recent pioneering works and reviews by Kosareva et al. [7], Debayle et al. [8], and Kolesik and Moloney [9]. Third, we want to develop a canonical description which allows us to describe the evolution of femtosecond pulses which range in duration from less than tens to up to a thousand cycles centered about a typical central frequency of 2.35×10^{15} rad/sec or wavelengths of 800 nm. As we shall see, such wave pulses have the potential of developing very large electric field gradients and, in regions of anomalous longitudinal dispersion, very large intensities. We want to be able to understand such interesting and potentially useful behaviors in as simple a way as possible. Fourth, whereas numerical simulations, such as UPPE [10], are extraordinarily valuable, they are more so when complemented by analytical descriptions which allow us to see which balances between ingredients produce which effects. SPEE is also useful because it allows us to suggest possible ways of using, overcoming, or circumventing some of the less desirable by-products such as ionization produced by localized high intensities. Fifth, SPEE allows us to see clearly under what circumstances, a modified NLS description of the pulse envelope, the default and knee jerk position of many in the optical sciences, is satisfactory. As we shall see, under most circumstances when intensities of ionization level obtain, NLS does not provide a uniformly valid description.

2 The Nature of SPEE

We begin by exploring and understanding the anatomies of KdV and NLS. KdV describes the deformation of each of the Riemann invariants F and G of the solution $u(x,t) = F(x-ct) + G(x+ct)$ of the wave equation (or any other linear hyperbolic system with distinct wavespeeds) under the additional influences of weak dispersion and weak nonlinearity, each of order ε, $0 < \varepsilon \ll 1$. The shapes F and G, invariant under the action of the linear wave equation, deform over long times ε^{-1} according to

$$v_\tau + 6vv_\zeta + v_{\zeta\zeta\zeta} = 0, \tag{1}$$

where $v(\zeta,\tau) = F(\zeta = x-ct, \tau = \varepsilon t)$. One can think of v as the right going wave in a narrow channel of shallow water. If weak variations with respect to a transverse direction y are included, (1) becomes (2), the KP equation,

$$\frac{\partial}{\partial\zeta}(v_\tau + 6vv_\zeta + v_{\zeta\zeta\zeta}) \pm v_{yy} = 0. \tag{2}$$

In the shallow water case (think very long oceanic waves triggered by an earthquake), the plus sign obtains. In other cases, such as ours where the weak transverse dependence is due to diffraction, the minus sign obtains, and if the transverse direction is two-dimensional, v_{yy} is replaced by $\nabla_\perp^2 v$, the two-dimensional transverse Laplacian. If the original wave equation has two speeds which are close, e.g., $c \pm \varepsilon$, then the corresponding pairs satisfy the Boussinesq equations. If the original field is constrained by the symmetry of field reversal such as in centrosymmetric crystals, then the leading order nonlinearity is cubic and (2) becomes (3), the modified KP equation (MKP).

$$\frac{\partial}{\partial\zeta}(v_\tau + 6v^2v_\zeta + v_{\zeta\zeta\zeta}) = \nabla_\perp^2 v. \tag{3}$$

(Although not the subject of this paper, it is intriguing to ask why it is that many of these universal equation representing multinomial Taylor expansions in amplitude and derivative are exactly integrable!) Equation (3) is, for all intents and purposes, SPEE. Let $\mathbf{E}(\mathbf{x} = (x,y), z, t) = \hat{e}E(\mathbf{x}, z, \tau = t - n_0 z/c)$ be linearly polarized. Then $E(\mathbf{x}, z, \tau)$ evolves along the propagation direction z as

$$\frac{\partial}{\partial\tau}\left(\frac{\partial E}{\partial z} + \frac{3\pi\hat{\chi}^{(3)}}{n_0 c}E^2\frac{\partial E}{\partial\tau} - \frac{1}{2n_0 c}\frac{\partial}{\partial\tau}\left(\hat{\chi}(0) - \hat{\chi} - \hat{\chi}\left(\mathrm{i}\frac{\partial}{\partial\tau}\right)\right)E\right) = \frac{c}{2n_0}\nabla_\perp^2 E. \tag{4}$$

If we approximate the susceptibility $\hat{\chi}(\omega)$ as $\hat{\chi}(0) + \mathrm{i}a(\omega t_0) + b(\omega t_0)^2$ (since it varies very slowly over the frequency range of interest),

$$\frac{\partial}{\partial \tau}\left(\frac{\partial E}{\partial z} + \frac{3\pi\hat{\chi}^{(3)}}{n_0 c}E^2\frac{\partial E}{\partial \tau} - \frac{bt_0^2}{2n_0 c}\frac{\partial^3 E}{\partial \tau^3} - \frac{at_0}{2n_0 c}\frac{\partial^2 E}{\partial \tau^2}\right) = \frac{c}{2n_0}\nabla_\perp^2 E. \tag{5}$$

The form (5) makes contact with earlier work as in (3) but the form (4) is more useful. In (4) and (5), τ is retarded time, $\hat{\chi}(\omega)$ is the linear susceptibility (which for air in the frequency range of interest is 5.5×10^{-4}; for future consideration $\hat{\chi}(3\omega) = 5.94 \times 10^{-4}$, $n_0 = \sqrt{1 + \hat{\chi}(0)}$, t_0 is pulse width (10–100 fs), b and a are dimensionless. The Kramers–Kronig relations demand that there is a weak absorption, namely $\hat{\chi}(\omega)$ has both a real and imaginary part, which in (5) is manifested by the Burgers' like second derivative on the left-hand side. The effects of plasma induced diffraction and absorption can and will be added later. Because $\hat{\chi}(\omega)$ is well tabulated in the literature, it is better when using numerical simulations of (4) which exchange $E(\mathbf{x}, z, \tau) = \big(1/(2\pi)\big)\int \hat{E}(\mathbf{x}, z, \omega)\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\omega$ for its Fourier transform $\hat{E}(\mathbf{x}, z, \omega)$ to use $\hat{\chi}$ directly rather than any power series representation.

The various terms in (4), (5) arise as follows: The first term $\partial^2 E/(\partial\tau\partial z)$ is simply the wave operator $\partial^2/\partial z^2 - (n_0^2/c^2)(\partial^2/\partial t^2)$ written in the longitudinal (z) and retarded time $\tau = t - (n_0/c)z$ coordinates. The second term is the nonlinear term and arises from the Kerr (cubic) dependence of the polarization on the electric field. The factor $\hat{\chi}^{(3)}$ is the Fourier transform of the nonlinear susceptibility and, in (4), is treated as constant. The nonlinearity is weak because the product $\hat{\chi}^{(3)}E^2$ is small. The balance of the first and second terms means that $E^2(\mathbf{x}, z, \tau)$ behaves as an arbitrary function f of $\tau - \big(3\pi\hat{\chi}^{(3)}/(n_0 c)\big)E^2 z$ where f is the initial shape of E^2 from which one may deduce that $(E^2)_\tau$ will become singular in finite time. The third term in (4) represents both dispersion in the direction of propagation and attenuation. Its balance with $\partial E/\partial z$ gives rise to a weak dispersion of waves with different frequencies and a Burgers' like loss but nonlinearity can suppress the dispersion (walk off in optical parlance) because, when b in (5) is negative, the balance of the first three terms gives rise to a coherent soliton in which structure all frequencies travel at the same speed. The fourth term is diffraction. When combined with focussing nonlinearity, localized objects (collapses) with both τ and x, y dependence can result.

Whereas KdV and its cousins KP, MKP, Boussinesq, and SPEE are multinomials about the origin $(0, 0, 0, 0)$ in $\partial/\partial\tau$, $\partial/\partial\zeta$, $\nabla_\perp^2$ and v (in which in (3), for example, the relative sizes are ε^3, ε, ε^4 and ε (so that all terms balance)) which describe the slow deformation of the Riemann invariant v under the joint influences of weak dispersion, nonlinearity and diffraction, the NLS equation describes the deformation of the envelope $A(x, y, z, t)$ of a strongly dispersive, weakly nonlinear (amplitude ε) and almost monochromatic (bandwidth ε) carrier wave $u = \mathrm{Re}\{A\exp(\mathrm{i}\mathbf{k}_0 \cdot \mathbf{x} - \mathrm{i}\omega(\mathbf{k}_0)t)\}$. It takes the form

$$\frac{\partial A}{\partial t} + \nabla_{\mathbf{k}}\omega \cdot \nabla A - \frac{\mathrm{i}\varepsilon}{2}\nabla^\top \Omega \nabla A - \mathrm{i}\gamma\varepsilon A^2 A^* = 0, \tag{6}$$

where $\nabla_{\mathbf{k}}\omega$ is the group velocity of the wavevector $\mathbf{k}_0$, $\nabla^{\top}, \nabla$ is the row (column) vector $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$, and Ω is the dispersion tensor $\partial^2\omega/(\partial k_i \partial k_j)$, running over x, y, z. The system is strongly dispersive only when the eigenvalues of Ω are nonzero.

For example, if $\omega = \omega(k = |\mathbf{k}|)$, $\nabla_{\mathbf{k}}\omega = \omega' \hat{k}$ and $\mathbf{k}_0$ is the carrier wavevector, then $\Omega = \operatorname{diag}(\omega'/k, \omega'/k, \omega'')$, $' = \mathrm{d}/\mathrm{d}k$. If ω'' is small so that the dispersion in the direction of propagation is weak, then when we are expanding the original field u as $\operatorname{Re}\{A \exp(\mathrm{i}k_0 z - \mathrm{i}\omega_0 t)\}$ + corrections, the corrections, assuming the field reversal symmetry obtains, will contain the third harmonic $\exp(3\mathrm{i}k_0 z - 3\mathrm{i}\omega_0 t)$ with the factor $A^3/(\omega(3k_0) - 3\omega(k_0))$ and in general the $(2n-1)$th harmonic with the factor $A^{2n-1}/(\omega((2n-1)k_0) - (2n-1)\omega(k_0))$. If dispersion in the propagation direction is weak, then the denominators $\omega\big((2n-1)k_0\big) - (2n-1)\omega(k_0)$, $n = 1, 2, \ldots$ can be very small. Thus the asymptotic expansion for u will only be uniformly valid if the amplitude of A, ε, is very small indeed. If, on the other hand, $\varepsilon^2/\big(\omega(3k_0) - 3\omega(k_0)\big)$ is of order unity, as we shall see shortly it is for short pulses with intensities of most interest, even those with many oscillations under their envelopes and having relatively narrow spectra around k_0 and ω_0, the NLS description is not valid because the corrections are as large as the leading order approximation.

From the previous discussion, we draw some conclusions and introduce some important new results. It is convenient to rescale the variables

$$E \to e_0 E, \quad \tau \to t_0 \tau, \quad z \to c t_0 z, \quad x, y \to L(x, y)$$

and introduce the following length scale ratios (all of which are small)

$$\begin{aligned} \varepsilon_{\mathrm{NL}} &= \frac{ct_0}{z_{\mathrm{NL}}} = \frac{3\pi \hat{\chi}^{(3)} e_0^2}{n_0}, & \varepsilon_{\mathrm{disp}} &= \frac{ct_0}{z_{\mathrm{disp}}} = \frac{|b|}{2n_0}, \\ \varepsilon_{\mathrm{diff}} &= \frac{ct_0}{z_{\mathrm{diff}}} = \frac{(ct_0)^2}{2n_0 L^2}, & \varepsilon_{\mathrm{att}} &= \frac{ct_0}{z_{\mathrm{att}}} = \frac{a}{2n_0} \end{aligned} \tag{7}$$

so that (5) becomes ($s = \operatorname{sgn} b$)

$$\frac{\partial}{\partial \tau}\left(\frac{\partial E}{\partial z} + \varepsilon_{\mathrm{NL}} E^2 \frac{\partial E}{\partial \tau} - s\varepsilon_{\mathrm{disp}} \frac{\partial^3 E}{\partial \tau^3} - \varepsilon_{\mathrm{att}} \frac{\partial^2 E}{\partial \tau^2}\right) = \varepsilon_{\mathrm{diff}} \nabla_{\perp}^2 E. \tag{8}$$

Remark 1. For $1 \gg \varepsilon_{\mathrm{NL}} \gg \varepsilon_{\mathrm{disp}}, \varepsilon_{\mathrm{att}}, \varepsilon_{\mathrm{diff}}$, the principal response is to steepen all the forward facing (in τ) faces for $E > 0$ and all the backward facing ones for $E < 0$. Without regularization from dispersion and attenuation, the slopes would become infinite and thereafter the electric field multivalued. With regularization they will form shocks dominated either by diffusion (shocks are narrow discontinuities) or dispersion (shocks are wide with lots of wiggles). An initial spectrum will become broad and continuous. Credit must go to Alterman and Rauch [11] and Schaffer and Wayne [12] who, unaware of the original work of Kuznetsov et al., clearly recognized that the cubic dependence of polarization on the electric field combined with weak or essentially zero dispersion in the direction of propagation could lead to potential shocks.

I want to emphasize that, to this point, the shape $E(\mathbf{x}, z = 0, \tau = t)$ can be very short with only a few oscillations and therefore a correspondingly broad spectrum. The dynamics will steepen certain slopes and broaden others with the net effect of significantly broadening the spectrum even further. But the shape can also be envelope like with a narrow spectrum supported near some carrier frequency ω_0. It is this shape that we address with the second remark.

Remark 2. If the initial pulse has the shape of an envelope under which there are many oscillations of a basic carrier wave $\exp(\mathrm{i}\omega_0\tau)$, then a natural response is to ask if its evolution can be captured by developing an equation for the slowly varying envelope of the leading harmonic. Let us apply this idea to SPEE written in the form (4). Write E as $a\mathrm{e}^{\mathrm{i}\omega_0\tau} + a^*\mathrm{e}^{-\mathrm{i}\omega_0\tau}$ + corrections.

To leading order we find

$$a_z - \frac{\mathrm{i}\omega_0}{2n_0c}\big(\hat{\chi}(0) - \hat{\chi}(\omega_0)\big)a = 0$$

whose solution is

$$a = b\exp\frac{\mathrm{i}\omega_0}{2n_0c}\big(\hat{\chi}(0) - \hat{\chi}(\omega_0)\big)z.$$

Note that when combined with $\exp \mathrm{i}\omega_0\tau$ this gives

$$b\exp\left(\mathrm{i}\omega_0 t - \frac{\mathrm{i}n_0\omega_0}{c}z + \frac{\mathrm{i}\omega_0}{2n_0c}\big(\hat{\chi}(0) - \hat{\chi}(\omega_0)\big)z\right)$$

and the exponent is simply

$$\left(\mathrm{i}\omega_0 t - \sqrt{1 + \chi(\omega_0)}\frac{\omega_0}{c}z\right)$$

when we write $\hat{\chi}(\omega_0)$ as $\hat{\chi}(0) - \big(\hat{\chi}(0) - \hat{\chi}(\omega_0)\big)$ and expand. Now let us calculate the correction to E which reflects the third harmonic $a_3\mathrm{e}^{3\mathrm{i}\omega_0\tau}$ which will be generated by the cubic term in (4). We find

$$a_{3z} - (\hat{\chi}(0) - \hat{\chi}(3\omega_0))a_3 = \frac{3\pi\chi^{(3)}}{n_0c}\mathrm{i}\omega_0 b^3\exp\left(\frac{3\mathrm{i}\omega_0}{2n_0c}\big(\hat{\chi}(0) - \hat{\chi}(\omega_0)\big)z\right).$$

Solving, we find that the amplitude of a_3 divided by the amplitude of b (or a) which we designate as e_0 is

$$\frac{2\pi\chi^{(3)}e_0^2}{\hat{\chi}(3\omega_0) - \hat{\chi}(\omega_0)}.$$

For air, $n_2 = \chi^{(3)}/(2n_0)$ is roughly 3×10^{-23} m^2/W and the intensity at which air begins to ionize is 5×10^{17} W/m^2 (and we of course want to be able to describe the envelope at these high intensities), and $\hat{\chi}(\omega_0) \simeq 5 \times 10^{-4}$ and $\hat{\chi}(3\omega_0) - \hat{\chi}(\omega_0) = 0.4 \times 10^{-4}$ (for lower frequencies, the difference is even less) which makes the above ratio approximately 2. At the very least, we can say it is order one.

What happens therefore to such a pulse?

First we recognize that all its odd harmonics will be excited significantly. Each lobe (oscillation) in the envelope will steepen on some faces and broaden on others as discussed in Remark 1. Whereas before it reaches these high intensities, the nonlinearity can give rise to third and higher harmonics whose amplitudes are small enough so that the NLS equation will obtain (namely, in the parlance of optics, walk off will separate the first from the third harmonic), that will not be the case once the intensities increase to the levels of interest. At these levels, the harmonics will phase lock and travel together. Some investigators may try to remedy the breakdown of NLS by introducing coupled envelope equations for a, a_3, and so on but this is a clumsy representation. The simplest description is SPEE. It has the capacity to describe the collective behavior of all odd harmonics simultaneously.

Remark 3. Collapses and the initiation of local large intensities. Although the NLS equation is not the correct analytical description for pulses in media with very weak dispersion in the direction of propagation, it nevertheless serves as a useful learning tool to investigate the synergy which can occur between nonlinearity and diffraction/dispersion from which interplay the pulse intensity can become locally infinite. We return to (6), move in the group velocity frame, assume the dispersion matrix to be diagonal and positive (so that $\Omega = \mathrm{diag}(\omega'/k, \omega'/k, \omega'') > 0$), rescale x, y, z, assume A to depend only on the r coordinate in n dimensions, and allow for a nonlinearity of order $2\sigma + 1$ in amplitude to obtain

$$\frac{\partial A}{\partial t} - \mathrm{i}\nabla^2 A - \mathrm{i}\alpha (AA^*)^\sigma A = 0, \tag{9}$$

where

$$\nabla^2 A = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial A}{\partial r}.$$

The power $N = \int_0^\infty AA^* r^{n-1}\, \mathrm{d}r$ is constant. The Hamiltonian, also a motion constant, is given by

$$H = \int \left(\frac{\partial A}{\partial r} \cdot \frac{\partial A^*}{\partial r} - \frac{\alpha}{\sigma + 1} (AA^*)^{\sigma+1} \right) r^{n-1}\, \mathrm{d}r$$

and

$$\frac{\partial A}{\partial t} = -\mathrm{i} \frac{\delta H}{\delta A^*}.$$

The well-known virial theorem gives the result that if

$$V = \int_0^\infty r^2 AA^* r^{n-1}\,\mathrm{d}r, \quad C = 2\mathrm{i}\int_0^\infty r\left(A\frac{\partial A^*}{\partial r} - A^*\frac{\partial A}{\partial r}\right) r^{n-1}\,\mathrm{d}r,$$

then

$$\frac{\mathrm{d}V}{\mathrm{d}t} = C, \quad \frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\mathrm{d}^2 V}{\mathrm{d}t^2} = 8H - \frac{4\alpha}{\sigma+1}(\sigma n - 2)\int_0^\infty (AA^*)^{\sigma+1} r^{n-1}\,\mathrm{d}r. \tag{10}$$

Thus, if $\alpha > 0$ (focussing nonlinearity) and the initial state is chosen so that $H < 0$, then for $\sigma n \geq 2$, the second time derivative of a positive quantity is negative which, after a finite time, leads to the contradiction that V eventually becomes negative. Instead what happens is that at some intermediate time t_0, the analysis is invalidated by the occurrence of a singularity in A whose modulus takes the shape $\Lambda^{1/\sigma} f(\Lambda r)$ where $\Lambda(t_0 - t)$ tends to infinity as $t \to t_0$. The power or number of particles is then given by $N = \Lambda^{-(\sigma n - 2)/\sigma} N_\mathrm{c}$, $N_\mathrm{c} = \int_0^\infty f^2(\zeta)\zeta^{n-1}\,\mathrm{d}\zeta$, a pure number. The case $\sigma n = 2$ is called critical collapse because the number of particles or power is conserved. For $\sigma n > 2$, the collapse is supercritical and particles and power are continuously radiated from the singular region as the pulse collapses. Of course, the original derivation of NLS required weak nonlinearity so that the analytical theory breaks down before the state of infinite amplitude is reached. Nevertheless, the theory correctly predicts the local intensification of the pulse envelope.

Turitsyn and Falkovich [4] showed that a similar result obtains for SPEE when attenuation is neglected. We can write (8) as

$$\frac{\partial E}{\partial z} = \frac{\partial}{\partial \tau} \cdot \frac{\delta K}{\delta E}$$

with Hamiltonian

$$\frac{\varepsilon_\mathrm{diff}}{3} K = \hat{K} = \int \left(\frac{D}{2}E_\tau^2 + \frac{\varepsilon_\mathrm{diff}}{2}(F_x^2 + F_y^2) - \frac{\varepsilon_\mathrm{NL}}{12}E^4\right)\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}y,$$

where $F_\tau = E$ and $D = -s\varepsilon^\mathrm{diss}$. For $s = \operatorname{sgn} b = +1$ dispersion is normal; for $s = -1$, it is anomalous.

A straightforward computation yields

$$\frac{\partial^2}{\partial z^2}\int (x^2 + y^2)E^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}y = 16\varepsilon_\mathrm{diff}\left(\hat{K} - \frac{D}{2}\int E_\tau^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}y\right). \tag{11}$$

For the case of anomalous dispersion $D > 0$, i.e. $s = \operatorname{sgn} b = -1$, and the choice of initial conditions such that $\hat{K}$, conserved under the evolution, is negative, the intrinsically positive quantity on the left-hand side of (11) will eventually become negative. This contradiction can only be avoided if E becomes infinite at some intermediate time. Thus, in the absence of dissipation, (8) can give rise to locally collapsing filaments.

Before we discuss the nature of the collapse, let us remind ourselves that for most frequencies in air and in many gases, the dispersion in the direction of propagation, while very small, is not anomalous but normal; namely, $s = +1$. In this case, we cannot make the same conclusion that the pulse will develop arbitrarily large intensities. Indeed, previous work by us [13] in the case of NLS has shown that in the context of three-dimensional signs $(+,+,-)$ ($\omega'' < 0$ with $\alpha > 0$ means the dispersion in the propagation is normal) and $\sigma = 1$, the two-dimensional collapse with critical power $N = \Lambda^{-(\sigma n-2)/\sigma} N_c = N_c$ when $\sigma n = 2$, the collapse is arrested when the normal dispersion $\omega''(\partial^2 A/\partial z^2)$ term is turned on. The mechanism for the gradual erosion of the critical power pulse is one of four wave mixing in which, if k_0 is the carrier wavenumber under the collapsing envelope pulse, wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ with $2k_0\hat{z} = \mathbf{k}_1 + \mathbf{k}_2$, $\omega(2k_0) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)$ are excited and carry the power of the pulse away from the filament. Such behavior is similar to what is experienced by a deep water wave soliton with elevation $\eta = A\exp\left(\mathrm{i}k_0 x - \mathrm{i}\sqrt{gk_0}t\right) + (*)$, where $A = 2a\,\mathrm{sech}\,a\left(x - \frac{1}{2}\sqrt{g/k_0}t\right)$. When rescaled, the envelope A satisfies

$$\frac{\partial A}{\partial t} - \frac{\mathrm{i}}{2}\left(\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial y^2}\right) - \mathrm{i}\alpha A^2 A^* = 0$$

and is unstable to waves $\mathbf{k}_1(k_{1x}, k_{1y}), \mathbf{k}_2(k_{2x}, k_{2y} = -k_{2x})$ which travel away from the original propagation direction $\hat{x}$ and are generated by four wave mixing.

But, we should point out, in the case of 2D NLS, the collapse is critical, namely it carries just enough power to sustain collapse. Any perturbation which can remove particles or power will interrupt the collapse. For SPEE, therefore, it is worth looking at the nature of the collapse. In (8), write

$$E(x,y,z,\tau) = (Z - z)^{-\alpha} F\left(X = \frac{x}{(Z-z)^{\beta}}, Y = \frac{y}{(Z-z)^{\beta}}, T = \frac{\tau - \tau^*}{(Z-z)^{\gamma}}\right), \tag{12}$$

where Z is the position of the collapse and τ^* follows the pulse maximum. From (12), the pulse power

$$P = \int E^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau = (Z-z)^{-2\alpha+2\beta+\gamma}\int F^2(X,Y,T)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}T, \tag{13}$$

the latter being a calculable real number. If we balance the four terms in (8) (in the absence of dissipation), we find

$$-\alpha - \gamma - 1 = -3\alpha - 2\gamma = -\alpha - 4\gamma = -\alpha - 2\beta. \tag{14}$$

Equating terms one, two, and four (thereby ignoring dispersion), we find $\gamma = 1-2\alpha$, $\beta = 1-\alpha$, and then $P = (Z-z)^{3-6\alpha}P_c$ so that the collapse is supercritical, critical, or subcritical, depending on whether $\alpha < \frac{1}{2}$, $\alpha = \frac{1}{2}$ or $\alpha > \frac{1}{2}$. This means as the one parameter (α) family of self-similar solutions (12) progresses in z, their power has a z dependence which is not constant but evolves as $(Z-z)^{6-3\alpha}$. If we include dispersion, the parameter α is determined to be $\frac{1}{3}$ and then $\beta = \frac{2}{3}$, $\gamma = \frac{1}{3}$ and $3-6\alpha = 1$ so that, as the filament nears the point at which E becomes infinite, its power decreases linearly as $(Z-z)$.

In the final section of this short paper, we will discuss the ramifications of Remarks 1–3 in the context of ultra short light pulses. In particular, we will be asking if it might be possible to tailor a pulse so that it reaches high intensities at a specified but distant point on its trajectory path. But first, we introduce a derivation of SPEE, originally presented in [6].

3 Derivation of SPEE

The reasons for introducing a brief derivation of SPEE are: (1) To show how it arises from Maxwell's equations and therefore how to include extra effects such as charge and currents arising from plasma formation. (2) To connect with the equations such as UPPE used as a basis for most numerical simulations. (3) To make clear the origins of the various terms in SPEE and to show that the effects of the weak dependency of the electrical susceptibility on frequency, weak nonlinearity (whereas pulse intensities I are large, the nonlinear refractive index n_2 is small and therefore $n_2 I$ is still small), diffraction (for beams wide compared to the dominant wavelength) all lead to long distance modifications of the pulse shape which, under linear wave propagation, would travel unchanged. The result we strictly obtain is for a linearly polarized field $\mathbf{E} = \hat{y}E(x,z,t)$ but, with a little additional work, can be shown to be true for an $E(x,y,z,t)$ which depends on both transverse coordinates (x,y). If $\mathbf{E} = \hat{y}E(x,y,z,t)$, one needs to add a small electrical field component in the propagation direction z so as to satisfy $\nabla\cdot\mathbf{D} = 0$ or $\nabla\cdot\mathbf{D} = \rho$. So, in our derivation, diffraction is strictly $\frac{\partial^2 E}{\partial x^2}$ but in the final answer we write $\nabla_\perp^2 E = \partial^2 E/\partial x^2 + \partial^2 E/\partial y^2$.

We take as constitutive relations $\mathbf{B} = \mu\mathbf{H}$ and

$$\mathbf{D} = \varepsilon_0\hat{y}\left(E + \int_{-\infty}^{t}\chi(t-\tau)E(\tau)\,\mathrm{d}\tau \right.$$
$$\left. + \int_{-\infty}^{t}\chi^{(3)}(t-\tau_1,t-\tau_2,t-\tau_3)E(\tau_1)E(\tau_2)E(\tau_3)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3\right). \quad (15)$$

We write the Fourier transforms of $E(x,z,t)$, the linear and nonlinear susceptibility $\chi(t)$ and $\chi^{(3)}(t_1,t_2,t_3)$ to be $e(x,z,\omega) = \int_{-\infty} E(x,z,t)\mathrm{e}^{\mathrm{i}\omega t}$, $\hat{\chi}(\omega)$ and $\hat{\chi}^{(3)}(\omega_1,\omega_2,\omega_3)$ respectively. From Maxwell's equation $\nabla\times\nabla\times\mathbf{E} = \mu(\partial^2\mathbf{D}/\partial t^2)$ and $\mu\varepsilon_0 = 1/c^2$, we obtain the exact relation

$$\frac{\partial^2 e}{\partial x^2} + \frac{\partial^2 e}{\partial z^2} + \frac{\omega^2}{c^2}(1+\hat{\chi}(\omega))e = -\frac{2\pi\omega^2}{c^2}\int \chi^{(3)}(\omega_1,\omega_2,\omega_3)e(\omega_1)e(\omega_2)e(\omega_3) \times \delta(\omega-\omega_1-\omega_2-\omega_3)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3, \qquad (16)$$

where $\delta(x)$ is the Dirac delta function. Anticipating that we will be taking nonlinearity and diffraction to be small when compared to the linear response $(n^2(\omega)\omega^2/c^2)e$, $n^2(\omega) = 1+\hat{\chi}(\omega)$, we write $e(r,z,\omega)$ as the sum of forward and backward fields. Let

$$e(r,z,\omega) = A(r,z,\omega)\mathrm{e}^{\mathrm{i}k_0(\omega)z} + B(r,z,\omega)\mathrm{e}^{-\mathrm{i}k_0(\omega)z}, \qquad (17)$$

where $k_0^2(\omega) = (\omega^2/c^2)(1+\hat{\chi}_r(\omega))$, $\hat{\chi}_r(\omega)$ the real part of $\hat{\chi}(\omega)$ and make the free choice of a relation between A and B to be

$$\frac{\partial A}{\partial z}\mathrm{e}^{\mathrm{i}k_0(\omega)z} + \frac{\partial B}{\partial z}\mathrm{e}^{\mathrm{i}k_0(\omega)z} = 0$$

(cf. method of variation of parameters). Then substituting (17) into (16) and adding and subtracting the free choice, we obtain the exact relations:

$$2\mathrm{i}k_0(\omega)\frac{\partial A}{\partial z} = -\frac{\mathrm{i}\omega^2}{c^2}\hat{\chi}_i(\omega)A - \frac{\mathrm{i}\omega^2}{c^2}\hat{\chi}_i(\omega)B\mathrm{e}^{-2\mathrm{i}k_0(\omega)z} - \frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 B}{\partial x^2}\mathrm{e}^{-2\mathrm{i}k_0(\omega)z} - \frac{2\pi}{c^2}\omega^2 P\mathrm{e}^{-\mathrm{i}k_0(\omega)z}, \qquad (18)$$

where

$$P = \int \hat{\chi}^{(3)}(\omega_1,\omega_2,\omega_3)\delta(\omega-\omega_1-\omega_2-\omega_3)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3 \times(A(\omega_1)\mathrm{e}^{-\mathrm{i}k_0(\omega_1)z}) + B(\omega_1)\mathrm{e}^{-\mathrm{i}k_0(\omega_1)z}(\omega_1\to\omega_2)(\omega_1\to\omega_3), \qquad (19)$$

and an equivalent equation for $\partial B/\partial z$. These equations are exact. There have been no approximations thus far. We will now ask how can we approximate solutions to (18) and the analogous equation for B if the right-hand side is small? Suppose we set $A = A_0 + A_1 + \cdots$, $B = B_0 + B_1 + \cdots$. To leading order, A_0 and B_0 will be independent of z but, in order to remove secular terms (terms growing as z) in the iterates $A_1, B_1, \ldots,$ we will have to choose their slow variations $\partial A_0/\partial z$ and $\partial B_0/\partial z$ accordingly. So the first task is to identify the secular terms in A_1 and B_1. The equation for A_1 is

$$2\mathrm{i}k_0(\omega)\frac{\partial A_1}{\partial z} = -\frac{\mathrm{i}\omega^2}{c^2}\hat{\chi}_i(\omega)A_0 - \frac{\mathrm{i}\omega^2}{c^2}\hat{\chi}_i(\omega)B_0\mathrm{e}^{-2\mathrm{i}k_0(\omega)z} - \frac{\partial^2 A_0}{\partial x^2} - \frac{\partial^2 B_0}{\partial x^2}\mathrm{e}^{-2\mathrm{i}k_0(\omega)z} + P_0\mathrm{e}^{-\mathrm{i}k_0(\omega)z}, \qquad (20)$$

where P_0 is P given in (19) with A replaced by A_0. Since, to leading order, A_0 and B_0 are z independent, it is clear that, by direct integration from $z = 0$ to $z = z$, the first and third terms on the right-hand side of (20) are secular whereas the second and fourth, each of whose fast dependence is $((\mathrm{e}^{-\mathrm{i}k_0(\omega)z} - 1)/\mathrm{i}k_0(\omega))$ are not. The more interesting discussion involves the nonlinear term one member of which, when integrated from 0 to z, is

$$\int \hat{\chi}^{(3)}(\omega_1, \omega_2, \omega_3) A_0(\omega_1) A_0(\omega_2) A_0(\omega_3) \times \left(\frac{\exp\big(\mathrm{i}k_0(\omega_1) + k_0(\omega_2) + k_0(\omega_3) - k_0(\omega)\big)z\big) - 1}{\mathrm{i}\big(k_0(\omega_1) + k_0(\omega_2) + k_0(\omega_3) - k_0(\omega)\big)} \right) \times \delta(\omega - \omega_1 - \omega_2 - \omega_3)\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\omega_3. \tag{21}$$

In order to assess whether this term gives rise to secular behavior, we must ask what is its limiting behavior as z becomes large. To answer this, we require a little mathematics. We state two important results. If $f(x)$ is an ordinary (measurable) function which is absolutely integrable and $h(x)$ is not identically zero over a finite interval in the domain of integration, we know that $\lim_{z\to\infty} \int f(x) \exp(\mathrm{i}h(x)z)\, \mathrm{d}x = 0$ (the Riemann–Lebesgue lemma) and that

$$\lim_{z\to\infty} \int f(x) \Delta\big(h(x)\big)\, \mathrm{d}x = \int \left(\pi \operatorname{sgn} z \delta\big(h(x)\big) + \mathrm{i}P\left(\frac{1}{h(x)}\right) f(x)\, \mathrm{d}x,\right.$$

where $\Delta(h) = (\mathrm{e}^{\mathrm{i}hz} - 1)/\mathrm{i}h$ and P denotes the Cauchy Principal Value. This means that as long as the amplitudes $A_0(\omega), B_0(\omega)$ are ordinary (as opposed to, say, Dirac delta) functions and are absolutely integrable, the nonlinear terms will give a bounded contribution to A_1 and be therefore nonsecular unless $h(\omega_1, \omega_2, \omega_3) = k_0(\omega_1) + k_0(\omega_2) + k_0(\omega_3) - k_0(\omega)$ is identically zero over finite regions of ω_2, ω_3 space (note: $\omega_1 = \omega - \omega_2 - \omega_3$). Namely, the nonlinear term will give no cumulative long distance effect unless h is identically zero. This occurs only when there is no (at least to leading order) longitudinal dispersion. We will comment at the end of this section, what happens if the pulse envelopes are very long and the amplitudes $A_0(\omega)$ have delta function behavior.

For now, we focus on the case of short pulses for which the Fourier transforms $e(x, z, \omega)$ are smooth in ω and for which (21) is only secular when $k_0(\omega_1) + k_0(\omega_2) + k_0(\omega_3) - k_0(\omega)$ is (almost) identically zero; namely, when $k_0(\omega) = n_0(\omega/c)$ for constant $n_0 = \sqrt{1 + \hat{\chi}_r}$. It should now also be clear that any of the other products in P_0 such as those involving $A_0(\omega_1)A_0(\omega_2)B_0(\omega_3)$ with fast z behavior $\Delta\big(k_0(\omega_1) + k_0(\omega_2) - k_0(\omega_3) - k_0(\omega)\big)$ never give rise to secular terms. Thus, the interaction between right and left going pulses is small and does not affect the deformations of either A_0 and B_0.

Having established that for short pulses, the nonlinear terms only give a cumulative long distance effect when $\hat{\chi}(\omega)$ is weakly dependent on ω and $k_0(\omega) = n_0\omega/c$ is nondispersive, we now return to (16) and write $\hat{\chi}(\omega)$ as $\hat{\chi}(\omega) - \hat{\chi}(0) + \hat{\chi}(0)$ and assume that $\hat{\chi}(\omega) - \hat{\chi}(0)$ is small compared to $\hat{\chi}(0)$. This small difference will give rise to weak dispersion and attenuation. We recall that since $E(x,z,t)$ and $D(x,z,t)$ are real, $\hat{\chi}(-\omega) = \hat{\chi}(\omega)$ so that $\hat{\chi}_r(\omega)$ is even in ω and $\hat{\chi}_i(\omega)$ is odd in ω. We will then obtain an equation (18) for $A(x,z,\omega)$ in which $k_0(\omega) = n_0(\omega/c) = \sqrt{1+\hat{\chi}_r(0)}$ and the susceptibility deviation from $\hat{\chi}(0)$ will manifest itself as the term $\left(\left(\hat{\chi}(0) - \hat{\chi}(\omega)\right)/c^2\right)\omega^2 A$ on the right-hand side of (18).

When we now repeat the analysis, we find the only secular terms arising in A_1 are those for which all fast dependence on z have been removed. To suppress these secular terms in A_1, we allow A_0 to be slowly varying. The result is the equation which is the Fourier transform of SPEE;

$$2\mathrm{i}n_0\frac{\omega}{c}\frac{\partial A_0}{\partial z} = \frac{\hat{\chi}(0)-\hat{\chi}(\omega)}{c^2}\omega^2 A_0 - \frac{\partial^2 A_0}{\partial x^2} - \frac{2\pi\omega^2}{c^2}\int \hat{\chi}^{(3)}(\omega_1,\omega_2,\omega_3)A_0(\omega_1)A_0(\omega_2)A_0(\omega_3) \times \delta(\omega-\omega_1-\omega_2-\omega_3)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3. \tag{22}$$

It says, in effect, that the evolution of the right going Fourier amplitude of the electric field in frequency space travels without deformation on distances of the order of many wavelengths but is distorted over much longer distances by a combination of dispersion and attenuation, diffraction and nonlinearity.

The facts that the forward and backward going components can be separated and that $\hat{\chi}(\omega)$ is almost constant suggest that we rewrite (22) as an equation for the leading order component of the electric field

$$E_0\left(x, z, \tau = t - \frac{n_0 z}{c}\right) = \frac{1}{2\pi}\int A_0(x,z,\omega)\mathrm{e}^{-\mathrm{i}\omega(t-(n_0 z/c))}\,\mathrm{d}\omega. \tag{23}$$

We note that $-\mathrm{i}\omega A_0$ has its Fourier transform $\partial E_0/\partial\tau$. Then if we take $\hat{\chi}^{(3)}$ constant, (22) becomes

$$\frac{2n_0}{c}\frac{\partial^2 E_0}{\partial\tau\partial z} + \frac{2\pi\hat{\chi}^{(3)}}{c^2}\frac{\partial^2 E_0^3}{\partial\tau^2} + \mathrm{FT}\left\{\frac{\hat{\chi}(0)-\hat{\chi}(\omega)}{c^2}\omega^2 A_0\right\} = \nabla_\perp^2 E_0 \tag{24}$$

which is (4). We have now replaced $\partial^2 E_0/\partial x^2$ by the full horizontal Laplacian.

How might the NLS equation enter this picture? If the pulse is a long envelope, then the Fourier transform $A(x,z,\omega)$ of the right going pulse will be close to a Dirac delta function centered on the first harmonic carrier frequencies ω_0 and $-\omega_0$. In that case, the integral on the right-hand side of (19) becomes an algebraic expression with frequencies $\pm\omega_0, \pm 3\omega_0$ and the exponent we called $h(\omega_1,\omega_2,\omega_3)$ can become zero when $\omega_1 = \omega_0, \omega_3 = -\omega_2 = \pm\omega_0$ leading to the modal interaction term familiar in NLS, $A_0^2(\omega_0)A_0^*(\omega_0)$. But, as we have pointed out in Sect. 2, the terms $A_0^3(\omega_0)$ corresponding to the frequency $3\omega_0$ will give rise to correction A_1 which have the denominator $k_0(3\omega_0) - 3k_0(\omega_0)$ which is small. Indeed as we have already shown, for intensities of interest, the NLS for the envelope of the first harmonic will not correctly describe the field $E(x, z, \tau)$.

We next address, in the final section, what modifications and additions one must make to SPEE in order to include other effects, the most important of which will be the effect of localized plasma generation by a high intensity pulse peak on pulse propagation and absorption. Our goal is to ask whether it might be possible to tailor light pulses to reach high intensities at a specified but distant location on its trajectory path.

4 Challenges, Questions, and Conclusions

Conclusions as to whether SPEE (in forms (4), (5), (8)) in the absence of any losses first leads to infinite gradients or infinite amplitudes are not definitive in all cases. Most evidence suggests that when dispersion in the direction of propagation is normal ($s = 1$ in (8)), electric field gradients dramatically steepen whereas when dispersion is anomalous ($s = -1$ in (8)), the high local intensity singularity obtains. It is therefore important to gain a better understanding on this question.

Without wishing to overmathematize the discussion, let us make several observations. First, if we set $\hat{\chi}(0) - \hat{\chi}(\omega) = 0$ in (4), we obtain dispersionless SPEE (it is just a nonlinear wave equation) and it is easy to see that

$$\frac{\partial}{\partial z}\int (E_\tau)^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\tau = -2\varepsilon_{\mathrm{NL}} \int E^2 E_\tau^2 = -\frac{\varepsilon_{\mathrm{NL}}}{2} \int (E^2)_\tau^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\tau$$

as, when multiplied by E, the diffraction term integrates to zero. Because the right-hand side remains finite and negative, and the left-hand side is the derivative of an intrinsically positive quantity, it is hard not to believe that at some finite z the integrals must fail to exist which means that the electric field gradient becomes singular. Although diffraction can in principle conspire with nonlinearity to produce infinite amplitudes, it cannot stop gradient steepening. So what happens when we reintroduce $\hat{\chi}(0) - \hat{\chi}(\omega)$? Let us discuss this in the context of (8). The steepening gradient will be regularized either by the dispersion term $-s\varepsilon_{\mathrm{disp}}\partial^3 E/\partial\tau^3$ or by the attenuation $-\varepsilon_{\mathrm{att}}\partial^2 E/\partial\tau^2$. Let us imagine the former dominates. Then a second important observation to make is that, if we ignore transverse effects, the resulting modified KdV equation $E_z + \varepsilon_{\mathrm{NL}} E^2 E_\tau - s\varepsilon_{\mathrm{disp}} E_{\tau\tau\tau} = 0$ only has coherent soliton solutions when the dispersion is anomalous ($s = -1$). When dispersion is normal, the steepening gradients do not break into solitons but into fine wiggles with time scales $\tau \approx \sqrt{\varepsilon_{\mathrm{disp}}/\varepsilon_{\mathrm{NL}}}$. This leads to spectral broadening. A third observation then is that the term $-(D/2)\int E_\tau^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\tau$ is positive in the virial theorem (11)

($D = -s\varepsilon_{\text{disp}}$) and is of order unity, independent of the size of $\varepsilon_{\text{disp}}$. The possible negativity of the Hamiltonian $\hat{K}$ is counterbalanced and the argument for the occurrences of high intensities weakened.

If attenuation, the Burgers term $\varepsilon_{\text{att}} E_{\tau\tau}$ in (8), dominates, then gradient regularization is achieved by shock creation. Shocks will be of a width (in τ) proportional to ε_{att}. Again spectral broadening will result. Again, if $s = +1$, the term $(D/\tau) \int E_\tau^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau$ in the virial theorem (11) counterbalances the negativity of $\hat{K}$ and the formation of large intensities.

These conclusions obtain whether or not the original pulse has few or many oscillations under its envelope. As we have already pointed out, the only difference is whether an initially broad spectrum further broadens to a continuous one or whether an initially narrow spectrum develops all its odd harmonics.

We now turn to the case of collapsing filaments in which case the pulse amplitude becomes locally large and eventually singular. Recall that the basic reason for the localization of light intensity is the presence of a self-focussing nonlinearity where the effective refractive index increases with intensity. In that case, the light rays will turn towards those x, y, τ regions of the pulse which have the largest intensity. As a result, the intensity then becomes even larger and attracts even more light rays. In 1D, this self-focussing saturates in a soliton like object. In 2D, it forms a collapsing filament which, if the envelope were to be described by NLS, has the critical amount of power required to sustain collapse. For SPEE, we do not know what happens at $D = 0$. In 3D, with $D = -s\varepsilon_{\text{disp}} > 0$, the collapse is supercritical and its power decays as $Z - z$ as the singular location $z = Z$ is approached.

We would like to address the following questions: What is the nature of the collapse and does it change in any significant way as $D \to 0$; namely, as D approaches zero through positive values? For $D = 0$, and in the absence of any kind of attenuation, which singularly comes first? What does that choice depend on? When the amplitude singularity is the first to appear either for $D > 0$ or $D = 0$, what is the nature of the regularization which occurs? If $D = 0$, and the amplitude singularity occurs, what is the nature of the collapsing filament? We have shown that in that case

$$E = (Z - z)^{-\alpha} F\left(\rho = \frac{r}{(Z - z)^{1-\alpha}}\right), \quad T = \left(\frac{\tau}{(Z - z)^{1-2\alpha}}\right)$$

with α undetermined by a simple balance argument. How is α determined then? By imposing certain restrictions on the solutions of $F(\rho, T)$ such as demanding that $F_\rho(0, T) = 0$ and $F(\rho, T)$ approaches zero without oscillations as $\rho \to \infty$ so that α is determined from a nonlinear eigenvalue problem. Is it possible that, for $D = 0$, $\alpha = \frac{1}{2}$, so that the power P in the collapsing filament does not tend to zero as $z \to Z$ but stays finite?

We now turn to the nature of the regularization when collapse events occur and the pulse locally develops very large amplitudes, large enough to ionize the gas (e.g., air) through which it is traveling. The ionization process creates a plasma in which a current is initiated and this current adds both to defocussing dispersion (acts similar to diffraction) and attenuation and power absorption. If one re-derives the evolution of E of Sect. 3 including these effects, one essentially has to replace $\varepsilon_0(\partial/\partial\tau)E^3$ by $\varepsilon_0(\partial/\partial\tau)E^3 + j$ on the left-hand side of (8) where j is the plasma current in the direction of the polarization of $\mathbf{E}$. How is j to be calculated? There is no rigorous derivation although M. Kolesik in this volume has developed an interesting approach. The efforts to date essentially use the Drude model for which

$$\frac{\partial j}{\partial t} + \gamma j = \frac{q^2}{m}E\rho, \quad \frac{\partial \rho}{\partial t} = W(I)(\rho_a - \rho) + \frac{\sigma}{U}\rho I, \tag{25}$$

where ρ is the density of electrons of charge q and mass m generated by ionization, I is electric field intensity and $W(I) \sim (I/I_c)^\alpha$, $\alpha \simeq 6$–7 describes multiphoton ionization, and as $(\sigma/U)\rho I$ captures avalanche ionization. The collision rate γ is extremely important. The reason is its inverse γ^{-1} measures the time it takes for the plasma to form. That time is generally believed to be in the range of picoseconds (10^{-12} s). Therefore while short pulses of femtosecond duration with local intensities I in excess of the ionization threshold I_c will create plasma, that plasma will be created in the wake of the pulse.

The power containing part of the pulse will not see the plasma directly. Therefore there should be little or no defocussing effects on the lead part of the pulse. Unfortunately, it is not true that the lead part of the pulse will escape the effects of plasma absorption. The energy to ionize the gas has got to come from somewhere. For the sake of achieving high intensity pulses traveling to a distance, it would help if that energy could be provided by part of a supercritical collapsing pulse which is radiated away, or the extra power between the initial power P and the critical power in a critical collapse which is also radiated away computations to date in the case of a supercritical collapsing pulse of NLS suggests that the presence of plasma absorption attenuates the central collapsing filament and arrests the growth of large intensities.

So how might one circumvent the absorption of the plasma? I do not have what I can call a good idea as to how one might achieve this. But I do have a number of suggestions which may be worth exploring.

1. Can one find a way to use the excess power between the initial power and the critical power, or the continuously radiating power in a supercritical collapse, to balance the cost of ionization?
2. If normal dispersion keeps the pulse from collapsing and attaining locally high intensities above the ionization threshold until the pulse reaches the target, can one modify the atmosphere in the neighborhood of a target so that, effectively $\hat{\chi}(0) - \hat{\chi}(\omega)$ is zero there? The size of the region to be so affected would be the collapse time multiplied by c.
3. Depending on the initial transverse shape of the beam, it may be possible to create multiple filaments (along a ring for example). Can one overcome losses from plasma absorption by using the powers in developing filaments with subionization intensities to resupply the power of a neighboring high intensity filament. In principle, because the medium is focussing, it should be possible to do this.

Acknowledgement The author is grateful for support from The Air Force contract FA 9550-10-1-US61 and from NSF DMS 1308862.

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