Coherent pulse propagation, a dispersive, irreversible phenomenon

Mark J. Ablowitz, David J. Kaup, and Alan C. Newell*

Clarkson College of Technology, Potsdam, New York
(Received 12 April 1974)

The initial value problem for the propagation of a pulse through a resonant two-level optical medium is solved by the inverse scattering method. In general, an incident pulse decomposes not only into a special class of pulses to which the medium is transparent but also yields radiation which is absorbed by the medium. In this respect "this problem" has properties markedly different from other dispersive and reversible wave phenomena some of which are tractable by the inverse scattering method. Indeed, it is remarkable that in the present case the method still applies. In particular, we show that, while there are an infinite number of local conservation laws, the integrated densities, and in particular the energy, are only conserved for a very special class of initial conditions. The theoretical results obtained are in close agreement with all the qualitative features observed in the experiments on coherent pulse propagation. Finally, we also show that causality is preserved. Two new and novel features are introduced and briefly discussed. First, we show that if the homogeneous broadening effect is a function of position in the medium, the pulses may speed up and slow down accordingly, without losing their permanent identities. Second, we have found a new kind of solution mode corresponding to a proper eigenvalue of the scattering problem which is not a bound state.

I. INTRODUCTION

Self-induced transparency (SIT), † the effect of a coherent medium response (acting as an attenuator) to an incident electric field, was first discovered by McCall and Hahn.[1,2] More recently, G. Lamb, *et al.*[3–6] have been able to obtain a whole class of special solutions by the inverse scattering method. By assuming both that the eigenvalues of the appropriate scattering problem remain invariant and that there is no continuous spectrum, permanent localized solutions (analogous to the solitons of the Korteweg–deVries equation; in the context of the sine-Gordon equation they have been termed kinks and breathers; colleagues in nonlinear optics refer to them as 2π and 0π pulses) of the relevant Maxwell–Bloch equations are obtained. Propagation heights and speeds are approximated by using the conservation equations.

In short, Lamb has treated only the case of an incident pulse to which the medium is totally transparent and which undergoes pure lossless propagation. In this situation, the incident pulse decomposes only into a sequence of "solitons" which interact with the medium in a very special way so that no net energy is exchanged. In general, however, only a certain portion of the incident pulse forms these special solitons to which the medium is transparent. The rest of the energy, which is mathematically characterized as the continuous spectrum of the appropriate eigenvalue problem (to be introduced in succeeding paragraphs), is "radiation" and is eventually transferred irreversibly to the medium leaving the portion of the medium in which the decomposition of the incident pulse occurs in a permanently excited state. (The eventual decay of these excited atoms through spontaneous emission occurs over a longer time scale and is not incorporated in this mathematical model.)

In this paper we present the procedure for solving the general initial value problem by the inverse scattering technique. We follow closely the ideas laid out in our recent articles.[7,8] Significantly, it is found that many of the aspects of SIT are remarkably different from all of the nonlinear evolution equations solved previously by this method. Most particularly, while there is a sequence of *local* conservation laws,

$$\frac{\partial T_n}{\partial \tau} + \frac{\partial f_n}{\partial x} = 0, \tag{1}$$

the integrated desities $\int f_n \, d\tau$, including the positive definite norm corresponding to energy, *are not necessarily conserved*. This is a consequence of the irreversible losses to the medium. Indeed, for an *arbitrary* incident pulse, the total energy of the electromagnetic field is a monotonically decreasing function of time, decaying to a constant that depends on the number and amplitudes of the permanent localized pulses which emerge from the decomposition of the incident pulse.

Simply stated, SIT has properties in common with known dispersive and reversible wave phenomena, and still others which are essentially irreversible. By irreversible we mean that for any particular initial condition, energy is transferred to the medium. This results in a population inversion which, due to dephasing effects, is exponentially decaying in the direction of propagation. Thus, integration in the reverse direction would be accompanied by exponential growth. [This is not to say that a sequential pulse in the same direction cannot synchronize (rephase) the system and lead to a coherent photon echo, an effect discussed by Hahn[9] and Abella, Kurnit, and Hartman[10]]. Only if the continuous spectrum is absent, is the problem purely dispersive and reversible. It is indeed remarkable, then, that when the irreversible effects are included, the inverse scattering method can still be applied.

In Sec. II, we give the eigenvalue problem, derive the evolution equations for the scattering data of this eigenvalue problem, explicitly solve them, and also give the equations necessary for solving the inverse problem. In Sec. III, we first give a brief review of the typical results obtained by the inverse scattering method. Then we compare and contrast the solutions from SIT with the typical case, and discuss the agreement of these solutions with what is experimentally known about ultrashort coherent pulse propagation. Finally, in Sec. IV, we discuss the unique feature of SIT wherein the "transmission coefficient" is *not* time invariant, and its implication for the conservation laws. Also, by using the evolution equations for the scattering data, a closed form solution for the "conserved" quantities can be obtained.

II. EVOLUTION OF THE SCATTERING DATA

We consider the following initial value problem. An incident electromagnetic wavetrain within the confines of a spatially modulated envelope impinges on a medium at $x=0$. Measuring time τ in a frame moving with the phase speed of the incident wave pulse, the SIT equations (following Ref. 6) are, in nondimensional form,

$$\epsilon_x = \langle\lambda\rangle, \tag{2}$$

$$\lambda_\tau + 2i\alpha\lambda = \epsilon N, \tag{3a}$$

$$N_\tau = -\tfrac{1}{2}(\epsilon^*\lambda + \epsilon\lambda^*). \tag{3b}$$

Here ϵ is the complex electric field envelope, λ is the out-of-phase and in-phase components of the induced polarization (also complex), N is the normalized population inversion, and $\langle\lambda\rangle \equiv \int_{-\infty}^{\infty} g(\alpha)\lambda(\alpha,x,\tau)\,d\alpha$, where $g(\alpha)$ characterizes the inhomogeneous broadening of the medium and is normalized to unit area. The initial conditions are the values of $\epsilon(x=0,\tau)$ (which is assumed to decay sufficiently rapidly as $\tau\to\pm\infty$), $\lambda(\tau\to-\infty)\to 0$, and $N(\tau\to-\infty)\to -1$. We remark that given $\epsilon(x=0,\tau)$, only one set of boundary conditions $(\tau\to-\infty)$ can be prescribed for the "Bloch equations" (3).

Following Ref. 6, consider the eigenvalue problem

$$v_{1\tau} + i\zeta v_1 = \tfrac{1}{2}\epsilon v_2, \tag{4a}$$

$$v_{2\tau} - i\zeta v_2 = -\tfrac{1}{2}\epsilon^* v_1, \tag{4b}$$

on the interval $-\infty<\tau<\infty$ (subscripts in τ and x denote partial differentiation). Using the ideas in Refs. 7 and 8, we now show how the x dependencies of v_1 and v_2,

$$v_{1x} = A(\zeta,x,\tau)v_1 + B(\zeta,x,\tau)v_2, \tag{5a}$$

$$v_{2x} = C(\zeta,x,\tau)v_1 - A(\zeta,x,\tau)v_2, \tag{5b}$$

can be used to construct $\epsilon(x,\tau)$ with the above initial and boundary conditions.

Equations (4), (5) require the integrability conditions

$$A_\tau = \tfrac{1}{2}\epsilon C + \tfrac{1}{2}\epsilon^* B, \tag{6a}$$

$$B_\tau + 2i\zeta B = \tfrac{1}{2}\epsilon_x - A\epsilon, \tag{6b}$$

$$C_\tau - 2i\zeta C = -\tfrac{1}{2}\epsilon_x^* - A\epsilon^*, \tag{6c}$$

which ensure that the eigenvalue ζ is independent of x. With ζ real, it is straightforward to show that the choices

$$A(\zeta,x,\tau) = \frac{i}{4}\left\langle\frac{N}{\zeta-\alpha}\right\rangle = \frac{i}{4}P\int_{-\infty}^{\infty}\frac{N(\alpha,x,\tau)g(\alpha)}{\zeta-\alpha}\,d\alpha, \tag{7a}$$

$$B(\zeta,x,\tau) = -\frac{i}{4}\left\langle\frac{\lambda}{\zeta-\alpha}\right\rangle = -\frac{i}{4}P\int_{-\infty}^{\infty}\frac{\lambda(\alpha,x,\tau)g(\alpha)}{\zeta-\alpha}\,d\alpha, \tag{7b}$$

$$C(\zeta,x,\tau) = -\frac{i}{4}\left\langle\frac{\lambda^*}{\zeta-\alpha}\right\rangle = -\frac{i}{4}P\int_{-\infty}^{\infty}\frac{\lambda^*(\alpha,x,\tau)g(\alpha)}{\zeta-\alpha}\,d\alpha, \tag{7c}$$

where $P\int_{-\infty}^{\infty}$ denotes the Cauchy principal value integral, satisfy (6) because of (2) and (3). [As might be expected, the consistent choice of principal value or indenting the contour under (over) the singularity $\alpha=\zeta$ leads to the same final results.] The unique features of this problem are manifested in the mathematical fact that, as $\tau\to+\infty$, A,B,C need not be equal to their respective values as $\tau\to-\infty$ (unlike all other nonlinear evolution equations previously solved by the inverse scattering method). These results are simply seen by noting that (3), given ϵ at any x, constitute ordinary linear (in λ, N) differential equations in τ, the solutions of which are uniquely determined by the conditions $N(\tau\to-\infty)\to-1$ and $\lambda(\tau\to-\infty)\to 0$. Naturally, N and λ do not, in general, take on these values as $\tau\to+\infty$.

Indeed, the quantities N, λ, and λ^*, as shown by Lamb,[6] are related to the fundamental solutions of (4). We define

$$\phi = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}, \quad \bar\phi = \begin{bmatrix}\bar\phi_1\\ \bar\phi_2\end{bmatrix}$$

to be independent solutions of (2), which satisfy the boundary conditions

$$\left.\begin{aligned}\bar\phi &\to \begin{bmatrix}1\\0\end{bmatrix}\exp(-i\zeta\tau) &(8a)\\ \bar\phi &\to \begin{bmatrix}0\\-1\end{bmatrix}\exp(i\zeta\tau) &(8b)\end{aligned}\right\}\text{as } \tau\to-\infty \text{ for all } x.$$

Then we can identify N and λ with ϕ and $\bar\phi$ as follows:

$$N = [\phi_1(\zeta,x,\tau)\bar\phi_2(\zeta,x,\tau) + \bar\phi_1(\zeta,x,\tau)\phi_2(\zeta,x,\tau)]\,|_{\zeta=\alpha}, \tag{9a}$$

$$\lambda = 2\phi_1(\zeta,x,\tau)\bar\phi_1(\zeta,x,\tau)\,|_{\zeta=\alpha}. \tag{9b}$$

Note that (9) gives N and λ in terms of ϕ and $\bar\phi$ at $\zeta=\alpha$. When ζ is real.

$$\bar\phi = \begin{bmatrix}\phi_2^*\\ -\phi_1^*\end{bmatrix}, \tag{10}$$

and it is the second independent solution of (4) with the above boundary condition (8b). In accordance with the usual scattering procedure[11] let, as $\tau\to+\infty$,

$$\phi \to \begin{bmatrix}a(\zeta,x)\exp(-i\zeta\tau)\\ b(\zeta,x)\exp(i\zeta\tau)\end{bmatrix}, \tag{11a}$$

$$\bar\phi \to \begin{bmatrix}\bar b(\zeta,x)\exp(-i\zeta\tau)\\ -\bar a(\zeta,x)\exp(i\zeta\tau)\end{bmatrix}, \tag{11b}$$

where for ζ real, $a\bar a + b\bar b = 1$, $\bar a = a^*$, $\bar b = b^*$. By using these results, as $\tau\to+\infty$, N can be concisely written as,

$$N(\alpha,x,\tau\to+\infty)\to -1 + 2bb^*(\alpha,x), \tag{12}$$

and

$$\lambda(\alpha,x,\tau\to+\infty)\to 2ab^*(\alpha,x)\exp(-2i\alpha\tau). \tag{13}$$

Notice if $b\equiv 0$ (no continuous spectrum), then $N(\tau\to+\infty)\to -1$, and $\lambda(\tau\to+\infty)\to 0$. (12) and (13) indicate that in general the medium is left in an excited state. The τ dependency of the polarization (λ) is a reflection of the fact that the oscillators return to their natural frequency; 2α is a measure of the difference between the carrier wave frequency of the incident pulse and the natural frequency corresponding to the difference in energy levels of the broadened two level medium.

Since it is the quantities $\phi\exp(A_- x)$ and $\bar\phi\exp(-A_- x)$ which satisfy (5), then

$$\phi_x=\begin{bmatrix} A-A_- & B \\ C & -(A+A_-)\end{bmatrix}\phi, \tag{14a}$$

$$\overline{\phi}_x=\begin{bmatrix} A+A_- & B \\ C & -A+A_-\end{bmatrix}\overline{\phi}, \tag{14b}$$

where

$$A_-(\zeta,x)\equiv\lim_{\tau\to-\infty} A(\zeta,x,\tau). \tag{15}$$

Then, in the limit of $\tau\to+\infty$, by using (7), (11), (12), (13), (14) and (15), the evolution equations for $a(\zeta,x)$ and $b(\zeta,x)$ are

$$a_x=\frac{i}{4}\left[\left\langle\frac{-1+2b^*b(\alpha,x)}{\zeta-\alpha}\right\rangle - \left\langle\frac{-1}{\zeta-\alpha}\right\rangle\right] -\frac{i}{4}\left[\lim_{\tau\to+\infty}\left\langle\exp[2i(\zeta-\alpha)]\tau\ \frac{2b^*a(\alpha,x)}{\zeta-\alpha}\right\rangle\right]b, \tag{16a}$$

$$b_x=-\frac{i}{4}\left[\lim_{\tau\to+\infty}\left\langle\exp[2i(\alpha-\zeta)\tau\ \frac{2a^*b(\alpha,x)}{\zeta-\alpha}\right\rangle\right]a -\frac{i}{4}\left[\left\langle\frac{-1+2b^*b(\alpha,x)}{\zeta-\alpha}\right\rangle + \left\langle\frac{-1}{\zeta-\alpha}\right\rangle\right]b, \tag{16b}$$

where, for ζ real,

$$\left\langle\frac{(\circ\circ\circ)}{\zeta-\alpha}\right\rangle \equiv P\int_{-\infty}^{\infty}\frac{(\circ\circ\circ)g(\alpha)}{\zeta-\alpha}\,d\alpha.$$

Note that the singular point $\alpha=\zeta$ is removable and therefore any choice for $\langle(\circ\circ\circ)/(\zeta-\alpha)\rangle$, applied consistently, yields the same analytic function.

Using well-known results when ζ is real, we find

$$\lim_{\tau\to\infty}\left\langle\frac{ab^*\exp[2i(\zeta-\alpha)\tau]}{\zeta-\alpha}\right\rangle = i\pi a(\zeta,x)b^*(\zeta,x)g(\zeta), \tag{17a}$$

$$\lim_{\tau\to\infty}\left\langle\frac{a^*b\exp[2i(\alpha-\zeta)\tau]}{\zeta-\alpha}\right\rangle = -i\pi a^*(\zeta,x)b(\zeta,x)g(\zeta). \tag{17b}$$

Thus, (16) reduces to

$$a_x=\frac{i}{2}a\left(\left\langle\frac{bb^*}{\zeta-\alpha}\right\rangle - i\pi bb^*g\right) =\frac{i}{2}a\int_{C_u}\frac{bb^*}{\zeta-\alpha}g(\alpha)\,d\alpha, \tag{18a}$$

$$b_x=\frac{i}{2}b\left(\left\langle\frac{aa^*}{\zeta-\alpha}\right\rangle + i\pi aa^*g\right) =\frac{i}{2}b\int_{C_a}\frac{aa^*}{\zeta-\alpha}g(\alpha)\,d\alpha. \tag{18b}$$

In (15), $\int_{C_u}(\int_{C_a})$ refer to the contours along the real axis indenting under (over) the pole at $\alpha=\zeta$.

To complete the solution of (18), we need the x dependency of aa^* for real ζ ($bb^*=1-aa^*$). This follows directly from (18a). Defining

$$\mathscr{A}\equiv aa^*, \tag{19}$$

then (18a) gives

$$\mathscr{A}_x=\mathscr{A}(1-\quad)\pi g, \tag{20}$$

or

$$\mathscr{A}(\alpha,x)=\frac{\mathscr{A}_0}{\mathscr{A}_0+(1-\mathscr{A}_0)\exp(-\pi gx)}, \tag{21}$$

where $\mathscr{A}_0=\mathscr{A}(\alpha,0)$. Consequently, the solution of (18) is

$$a(\zeta,x)=a(\zeta,0)\exp[-i\Omega(\zeta,x)], \tag{22a}$$

$$b(\zeta,x)=\frac{b(\zeta,0)\exp(i\Omega)\exp(-\pi gx)}{\mathscr{A}_0+(1-\mathscr{A}_0)\exp(-\pi gx)}\exp\left(i\frac{x}{2}\int_{C_u}\frac{g\,d\alpha}{\zeta-\alpha}\right), \tag{22b}$$

where

$$\Omega(\zeta,x)\equiv\frac{1}{2\pi}\int_{C_u}\frac{d\alpha}{\zeta-\alpha}\ln[\mathscr{A}_0+(1-\mathscr{A}_0)\exp(-\pi gx)]. \tag{23}$$

In order to determine ϵ, N, and λ for $x>0$ via the inverse scattering method, we do not need the general result given by (22) and (23), but only the x dependence of (i) b^*/a for real ζ, (ii) the bound state eigenvalues (ζ_k) in the upper half ζ-plane [which are found from the eigenvalue problem (4) and are the zeros of a], and (iii) $\overline{C}_k$. (When b^* is analytically extendable into the upper half ζ-plane, $\overline{C}_k$ is simply the residue of b^*/a at the eigenvalue $\zeta=\zeta_k$.) First, the x independence of the eigenvalues [assumed by Lamb[6] and required by (6)] can immediately be seen from (22a) and (23). Since, in the upper half ζ-plane, Ω is analytic, the zeros of a do not move (furthermore, new zeros do not appear), and the eigenvalues will therefore remain independent of x. From (22) and (23) [or also from (18)] we have

$$\frac{b^*}{a}(\xi,x)=\frac{b^*}{a}(\xi,0)\exp\left[-\frac{i}{2}x\int_{C_u}\frac{g(\alpha)}{\xi-\alpha}\,d\alpha\right] \tag{24}$$

and

$$\overline{C}_k(x)=\overline{C}_k(0)\exp\left[-\frac{i}{2}x\int_{C_u}\frac{g(\alpha)}{\zeta_k-\alpha}\,d\alpha\right]. \tag{25}$$

To complete the solution, one continues as given in Ref. 11. First, solve the eigenvalue problem (4) for the bound state eigenvalues (ζ_k), and $\overline{C}_k$, and also for b^*/a ($\zeta=$real), all at $x=0$. Then, using (24) and (25), construct

$$F(y)=-i\sum_k\overline{C}_k(x)\exp(-i\zeta_k y)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b^*}{a}(\zeta,x)\times\exp(-i\zeta y)\,d\zeta. \tag{26}$$

Solve the inhomogeneous linear integral equation

$$K(\tau,\theta)=F(\tau+\theta)-\int_{-\infty}^{\tau}\int_{-\infty}^{\tau}F(\theta+\beta)F^*(\beta+\gamma)K(\tau,\gamma)\,d\beta\,d\gamma; \tag{27}$$

then ϵ is given by

$$\epsilon(x,\tau)=-4K(\tau,\tau;x). \tag{28}$$

Once K is found, N and λ can also be determined.[11]

In concluding this section, we note an alternative form for (24) is

$$\frac{b^*}{a}(\xi,x)=\frac{b^*}{a}(\xi,0)\exp\left[-\frac{\pi}{2}g(\xi)x-\frac{i}{2}xP\int_{-\infty}^{\infty}\frac{g(\alpha)}{\xi-\alpha}\,d\alpha\right], \tag{29}$$

which explicitly shows that b^*/a decays exponentially as $x\to\infty$ at a rate proportional to the inhomogeneous broadening.

III. GENERAL FEATURES OF THE SOLUTION

Of all the steps required for the application of the inverse scattering method, the most important and crucial step is to be able to solve for the time dependence (in this case, the x-dependence) of the scattering data for arbitrary initial scattering data. Once this is done, everything else follows, allowing one to construct the solution at any later time (in this case, x) from the initial data and to determine the form and structure of the general solution. For those familiar with Hamilton—Jacobi theory, the power of the inverse scattering method can best be appreciated as follows: The inverse scattering method is simply a canonical transformation which yields the Hamilton—Jacobi functional differential equation completely separable. Naturally once separation has been achieved, the solution for the resulting "action-angle" variables is trivial. Although complete separation is not achieved in the case of SIT, still the separation is sufficient to allow a solution to be found, as we have just seen. For the rest of this section, we want to give a review of the typical forms and features of solutions obtained via the inverse scattering method, discuss the analogies and distinct differences of the solutions for SIT compared to other inverse scattering solutions, and show the remarkable qualitative agreement between these solutions with what is known experimentally about ultrashort coherent pulse propagation.

Throughout all applications of the inverse scattering method,[7,8,11–13] there are two distinct features of the general solution which have remained invariant. The first is the concept of the "soliton,"[14] which is a stable, localized, permanent waveform which evolves in time by a simple translation. The second is the concept of "radiation" which is not in general localized, does not have a permanent shape, and in general does decay algebraically in time. Any general solution of the evolution equations will always contain a mixture of these two fundamental solutions, and in general, it is impossible to separate (by inspection) a general solution into these two fundamental modes since the mixing is nonlinear. However, when a general solution is "mapped" by the direct scattering problem (which is a nonlinear mapping) into the scattering data, these fundamental modes are then separated. [This is simply a generalization of the well-known technique for solving linear evolution equations by Fourier transformations, whereby one "maps" a function into its Fourier transform. In this case, the evolution equation for the Fourier transform is also separable. One should also note that Eq. (26) is in effect a Fourier transform!] In terms of the scattering data, each "soliton" corresponds to exactly one bound state of the eigenvalue problem and vise versa, while the "radiation" corresponds to the continuous spectra of the eigenvalue problem. These modes are easily seen from the form of F [Eq. (26)]. In (26), each soliton is specified by giving ζ_k and $\overline{C}_k$, where ζ_k gives the velocities of the soliton, while $\overline{C}_k$ essentially specifies the initial position and phase of the soliton. Consequently, the number of solitons is exactly equal to the number of bound states. For the radiation mode, this is represented in (26) by the integral along the real axis over the continuous spectrum, and is specified by giving (b^*/a).

The simplest solution to find via the inverse scattering method is the solution for a single soliton with no radiation present. In this case, when we set $b(\zeta)\equiv 0$, the kernel of (26) becomes completely separable, allowing an explicit solution. Inserting the x-dependence given by (25), then, from (26), (27), and (28), we find

$$\epsilon(x,\tau)=4\eta\exp(-i\phi)\operatorname{sech}\theta, \tag{30}$$

where

$$\zeta_1=\xi+i\eta, \tag{31a}$$

$$\overline{C}_1=-2i\eta\exp(-\theta_0)\exp(+i\phi_0), \tag{31b}$$

$$\theta=\theta_0+\omega_2 x-2\eta\tau, \tag{32a}$$

$$\phi=\phi_0+\omega_1 x-2\xi\tau, \tag{32b}$$

$$\omega_1+i\omega_2=-\frac{1}{2}\int_{-\infty}^{\infty}\frac{g(\alpha)\,d\alpha}{\zeta_1-\alpha}. \tag{33}$$

To relate these variables to physical quantities, we first note that x and τ are not the usual space—time coordinates. Letting the usual space—time coordinates be X and T, then when $c=1$ ($c=$ speed of light)

$$x=X, \tag{34a}$$

$$\tau=T-X. \tag{34b}$$

Thus by (30), (32a), and (34) this soliton has a velocity of

$$v=(1+\omega_2/2\eta)^{-1}, \tag{35}$$

which is less than unity. Before proceeding further, it becomes necessary to choose a model for the inhomogeneous broadening term in (33). A physical and simple model is the Lorentzian line shape

$$g(\alpha)=\frac{1}{\pi}\,\frac{\Gamma}{\alpha^2+\Gamma^2} \tag{36}$$

where 2Γ is the width at half-height. From (33) and (36) we have

$$\omega_1=-\frac{1}{2}\,\frac{\xi}{\xi^2+(\eta+\Gamma)^2}, \tag{37a}$$

$$\omega_2=+\frac{1}{2}\,\frac{\eta+\Gamma}{\xi^2+(\eta+\Gamma)^2}, \tag{37b}$$

and consequently, when $\Gamma\ll\eta$, the velocity is essentially dependent only on the magnitude of ζ_1. On the other hand, the width of the soliton (in time, T) is inversely proportional to the imaginary part of the eigenvalue η, while the amplitude is proportional to η. This is a well-known result for nonlinear waveforms, in that the height, width, and velocity are interrelated.

In addition to single soliton solutions, multiple soliton solutions can also be explicitly given.[11,13] A special type of a multiple soliton solution occurs when more than one soliton have the same velocity, and these have been called "multi-soliton bound states." These solutions in general have a very complicated and oscillating waveform. A simple example of a multisoliton bound state for the SIT equations is the analogy of the "breather" (called a 0π pulse by Lamb) solution of the sine-Gordon equation.[7] In the special case where $\mathrm{Re}\lambda$, N, and g are even in α, $\mathrm{Im}\lambda$ is odd in α, and $\epsilon(0,\tau)$ is real, then one can show that $\epsilon(x,\tau)$ remains real for all x and the discrete eigenvalues must occur either on the imaginary axis $\xi=0$, whence we have either a simple soliton (kink)

$$\epsilon(x,\tau)=4\eta\operatorname{sech}\theta \tag{38}$$

or in complex conjugate pairs (ζ and $-\zeta^*$) whence we have a breather

$$\epsilon(x,\tau)=8\,\frac{\eta}{\xi}\,\frac{\xi\cosh\theta\sin\phi+\eta\sinh\theta\cos\phi}{\cosh^2\theta+\eta^2/\xi^2\cos^2\phi}$$
$$=4\,\frac{\partial}{\partial\tau}\tan^{-1}\left(\frac{\eta}{\xi}\,\frac{\cos\phi}{\cosh\theta}\right), \tag{39}$$

with θ and ϕ as given by (32).

Usually, the computation of solutions when the radiation (the continuous spectrum) is present is very difficult.[15] The exact manner in which this part of the spectrum evolves in time depends on the specific problem being solved, but one can still make some general statements concerning it. This fundamental mode of the solution is invariably characterized by a series of oscillations which propagate away from the initial disturbance (whence the name "radiation"). In all other cases (except SIT), these oscillations decay only algebraically in time, usually approaching some special decaying nonlinear oscillating state. Consequently, all of these systems evolve toward a general final state consisting of free solitons, multisoliton bound states, and decaying radiation, with the soliton states eventually ordering themselves according to their velocities.

Much more could be said about the inverse scattering solutions, but it is now perhaps best to refer the reader to the literature in this area,[11–15] and instead go on to discuss some of the specifics of the solutions for SIT.

Many of the features of SIT are very similar to the general case discussed above in that we have these two fundamental modes consisting of solitons and radiation. However, SIT is distinctly different from all other previous systems solved by the inverse scattering method in that the x dependence of the continuous spectrum [Eq. (29)] is not simply oscillatory, but is damped! This has the physical consequence that the medium will act as a "filter," and will only allow the discrete spectrum (the solitons) to be propagated through. Of course, this is exactly what is observed experimentally. To see what has happened to the continuous spectrum, let us consider an arbitrary initial pulse incident on a medium at $x=0$. Knowing the shape of the initial pulse, we can solve the eigenvalue problem (4) for the bound state parameters (ζ_k, $\overline{C_k}$, $k=1,2,\ldots,N$), the "transmission coefficient", a, and the "reflection coefficient", b, for real ζ. Let us now look at N and λ in the limit of $\tau\to+\infty$, which corresponds to the respective values after the initial pulse has passed. Directly from (12), (13), (22), and (23), we find

$$N(\alpha,x)\to-1+\frac{2(1+N_0)\exp(-\pi gx)}{1-N_0+(1+N_0)\exp(-\pi gx)}\,, \tag{40}$$

$$\lambda(\alpha,x,\tau)\to\frac{2\lambda(\alpha,0,\tau)\exp(-\pi gx/2)\exp(-i\chi)}{1-N_0+(1+N_0)\exp(-\pi gx)}\,, \tag{41}$$

where N_0 is N at $x=0$ as $\tau\to+\infty$ and χ is a real phase given by

$$\chi=\Omega+\Omega^*+\frac{x}{2}\,P\int_{-\infty}^{\infty}\frac{g(\alpha)\,d\alpha}{\xi-\alpha}\,, \tag{42}$$

with Ω given by (23). Equations (40) and (41) exhibit two more well-known but related phenomena: the excitation of the medium and its consequent "ringing" after the initial pulse has passed.[1,2] Since (40) shows that, in general, $N+1$ is not zero as $\tau\to+\infty$, a certain fraction of the atoms remain excited after the initial pulse has passed. In order to do this, energy must be extracted from the initial pulse, and it is then shared coherently between the atoms, causing the ringing as given by (41). Since the solitons will eventually be propagated through, they cannot lose energy, so that the energy must come from the continuous spectrum. Further, the absorption of the continuous spectrum continues until it becomes exponentially small as $x\to\infty$, with both $N+1$ and $\lambda\to 0$ in this limit.

Inspection of (41) reveals a very interesting feature of the ringing. For certain initial pulse profiles, the maximum amplitude of the ringing will not occur at $x=0$, but can occur well inside the medium, at $x=x_r(\alpha)$, given by

$$x_r(\alpha)=\frac{1}{\pi g(\alpha)}\ln\left(\frac{1+N_0(\alpha)}{1-N_0(\alpha)}\right). \tag{43}$$

Naturally, to be physical, x_r must be greater than zero, requiring $N_0(\alpha)>0$, and if $N_0(\alpha)<0$, then the maximum in the physical region occurs at $x=0$. Of course, this is not totally unexpected since as a consequence of (3) and the boundary conditions, we have

$$N^2+\lambda^*\lambda=1, \tag{44}$$

showing that $|\lambda|$ is a maximum when $N=0$. Thus, if the initial pulse gives $N>0$ for a range of α, due to the following absorption of the continuous spectrum, N will monotonically decrease in x, giving the maximum in λ at some $x>0$. What is new about (43) is by solving for the complete x dependence of the scattering data we have an explicit expression for x_r.

Of course, the rate at which the continuous spectrum is absorbed depends only on the inhomogeneous broadening factor $g(\alpha)$. Since g is normalized to have a unit area, the effective absorption rate depends mostly on the width of the level and the width and centering of the incident pulse. If the central frequency of the incident pulse is centered on the resonant frequency and if its width is smaller than the level width, then a maximum filtering effect is achieved. For the model (36), the decay length in this case for the continuous spectrum [see Eq. (29)] is simply $\frac{1}{2}\pi\Gamma$. When the central frequency of the incident pulse is not centered on the resonant frequency by a significant amount, then, in terms of (36), the decay length increases significantly to α^2/Γ, giving inefficient filtering.

In concluding this section, we want to look at the form of the solution as $x\to\infty$, and will direct our attention to the function F in (26). In this limit, the contribution of the radiation term to F becomes exponentially small while the soliton contribution becomes exponentially large, forcing F to approach the form for pure solitons (i.e., no radiation). If one now neglects the radiation contribution, a closed form solution for $\epsilon(x,\tau)$ is possible.[11] As is well known, as $x \to \infty$, this solution approaches a linear sum of the simple soliton solutions, (30), and multisoliton bound states. This illustrates another well-known property of ultrashort coherent pulse propagation called "pulse-reshaping" whereby the incident pulse is "reshaped" into those pulses capable of undergoing lossless propagation (solitons).

Let us now return and consider the radiation contribution to F in this limit. If one uses the method of steepest descent, one finds that the radiation contribution to F does vanish exponentially everywhere, except near the light cone ($\tau=0$). Here, when b^*/a approaches zero only algebraically as $|\zeta| \to \infty$, the radiation field is merely a small "blip." Otherwise, it gives no contribution.

Now, let the initial conditions be such that $\epsilon=0$ if $\tau<0$. Then since (2) is causal, ϵ must remain zero for all x when $\tau<0$. The radiation contribution to F guarantees this, because if $\epsilon=0$ when $\tau<0$ at $x=0$, one can show that b^*/a is analytically extenable into the upper half ζ-plane and that $\overline{C_k}$ is then simply the residue of b^*/a at $\zeta=\zeta_k$. Then, by contour integration, one can show that F, and hence $\epsilon(x,\tau)$, are identically zero for all x when $\tau<0$. In other words, in this case the radiation field is necessary to ensure that the forward tail of the leading soliton does not extend beyond the light cone.

Finally, we point out that the pulse heights and shapes are dependent on the medium parameters, but not on the inhomogeneous broadening $g(\alpha)$. The pulse speeds do depend on this factor. But, returning to the derivations in Sec. II, one sees that, without loss of generality, we could allow $g(\alpha)$ to be also a function of x and still obtain the x dependence of the scattering data. In this case, the solitons would still retain the same heights and shape while changing their velocities as they propagate.

IV. MATHEMATICAL ASPECTS OF SIT

In all other previous examples using the inverse scattering method to solve nonlinear evolution equations, the x dependency of the scattering data was always given by

$$a_x=0, \quad b_x=-2A_0(\zeta)\,b,$$

where $A_0(\zeta)=A_-(\zeta)=A_+(\zeta)$ and was independent of x. The simplicity of these expressions, and in particular the x invariance of a (the "transmission coefficient"), was related to the existence of globally conserved quantities. (For a further and fuller discussion see Ref. 12.) The present problem has this property only when the incident pulse is so special as to decompose into only kinks and breathers with no "radiation", i.e., $b(\zeta)\equiv 0$. The fact that the initial value problem is still tractable when Eq. (16) are fairly complicated leads us to conjecture that the inverse method may be applicable to a wider class of problems than heretofore believed.

In general, $b(\zeta)\neq 0$, and although one still has *local* conservation laws, the *global* quantities are not conserved. As examples, the first two local conservation laws are given by

$$f_1=\tfrac{1}{2}\epsilon^*\epsilon, \tag{45a}$$

$$f_2=\tfrac{1}{4}(\epsilon^*\epsilon)^2-\epsilon^*_\tau\epsilon_\tau, \tag{45b}$$

$$T_1=\langle 1+N\rangle, \tag{46a}$$

$$T_2=\epsilon^*\epsilon\langle N\rangle+2i\epsilon\langle\alpha\lambda^*\rangle-2i\epsilon^*\langle\alpha\lambda\rangle-8\langle\alpha^2(N+1)\rangle, \tag{46b}$$

where (45) and (46) satisfy (1). Defining

$$F_n\equiv\int_{-\infty}^{\infty} f_n\,d\tau, \tag{47}$$

then from (1) we have

$$\frac{dF_n}{dx}=T_n\Big|_{\tau=-\infty}^{+\infty} \tag{48}$$

As shown by Schnack and Lamb,[16] when ϵ vanishes sufficiently rapid as $\tau\to+\infty$, (48) becomes

$$\frac{dF_n}{dx}=K_n\langle\alpha^{2n-2}(N+1)\rangle\Big|_{\tau=-\infty}^{+\infty}, \tag{49}$$

where K_n is a set of numerical coefficients. By using (40), (49) is integrable, and this gives

$$F_n(x)=F_n(0)-\frac{2K_n}{\pi}\int_{-\infty}^{\infty} d\alpha\,\alpha^{2n-2}\times\ln\left(\frac{1-N_0}{2}+\frac{1+N_0}{2}\exp(-\pi g x)\right). \tag{50}$$

We note that, as $x\to\infty$, F_n becomes *independent* of the inhomogeneous broadening factor, a result which is contrary to that suggested by Ref. 16. Still, one can use the conservation laws in certain cases to obtain reasonable values for the eigenvalues, although one can easily devise many examples where this technique will fail. For example, let $\epsilon(\tau,0)$ be zero if $\tau<0$ or $\tau>\tau_1$, and a constant value of ϵ_0 between these limits. Then, from (4), it is easy to show that $b(\alpha)\to O(1/\alpha)$ as $|\alpha|\to\infty$, and, by (12), $N_0\to-1+O(1/\alpha)$. Then inspection of (43) shows that $F_n(x\to+\infty)$ is undefined if $n\geq 2$. Thus, in this example, one has only *one* conservation law which can be used, and if the initial profile contains more than one soliton, a unique determination of the eigenvalues is impossible.

In any case, whenever $|\epsilon|\to 0$ faster than $|\tau|^{-1}$ as $\tau\to\pm\infty$, one can *always* determine the eigenvalues by simply solving the eigenvalue problem, Eq. (4). Even in the most complicated cases, numerical determination of the eigenvalues is quite practical with present high speed computers.

Finally, one interesting feature of the eigenvalue problem (4) is the possibility of having $a=0$ ("bound states") on the real axis[12]! For the KdV equation,[13] bound states on the real axis are strictly forbidden, but are allowed by (4) as can be shown easily by specific examples. One can now ask whether or not these modes give anything new for SIT. First, if $a=0$ on the real axis, F as given by (26) has a pole in the integral on the real axis. If one retraces the derivation of (27), one finds that this integral is to be replaced by the Cauchy principle value plus $(-i)\cdot\mathrm{Res}[(b^*/a)\exp(-i\zeta y)]$ at the pole (i.e., when b^* is sufficiently analytic to be extended a certain amount into the upper half ζ-plane, F is always a contour integral above all zeros of a). Taking the limit of large x and using the method of steepest descent, one finds that the contribution to F from a zero on the real axis vanishes exponentially in

x like the radiation does. Meanwhile, the τ dependence of F is in between that of a soliton and radiation, since for small τ it gives zero and for large τ it simply oscillates like $\exp(-i\xi_0\tau)$, where ξ_0 is the zero of a. (Solitons grow exponentially in τ while the radiation decays algebraically.) Due to this x and τ dependence, a zero on the real axis corresponds more to a particular form of radiation than to a soliton. From (35) and (37) we see that if we did consider it to be a soliton, it would have a zero velocity; consequently it will never "detach" itself from the radiation, in agreement with the x and τ dependence of F.

Finally, for a zero of a on the real axis, we note the form of a and b as $x\to\infty$. From (22), in this limit, $|a|\to 1$ and $|b|\to 0$ exponentially everywhere on the real axis except at the zero of a. Here, $|a|\to 0$ and $|b|\to 1$. Consequently, in this limit a and b do not possess a first derivative with respect to ζ, which implies the integral $\int_{-\infty}^{\infty}|\epsilon|(1+|\tau|)\,d\tau$ does not exist as $x\to\infty$.[12] A zero on the real axis also has a consequence for the ringing, since at $\alpha=\zeta_0$, x_r [Eq. (43)] is infinity. However, the width of this ringing about $\alpha=\xi_0$ vanishes exponentially in x, causing the stored energy to also vanish exponentially.

*Supported in part under NSF Grants Nos. GP-32839X2 and GP-43653.

†*Note added in proof*: While the term "Self-induced transparency" literally connotes only lossless propagation, we use the term in the wider context as referring to general coherent pulse propagation.

[1]S. L. McCall and E. L. Hahn, Phys. Rev. Lett. **18**, 908 (1967).
[2]S. L. McCall and E. L. Hahn, Phys. Rev. **103**, 183 (1969).
[3]G. L. Lamb, Jr., Phys. Lett. A **25**, 181 (1967).
[4]G. L. Lamb, Jr., Rev. Mod. Phys. **43**, 99 (1971).
[5]G. L. Lamb, Mo. O. Scully, and F. A. Hopf, Appl. Opt. **11**, 2572 (1972).
[6]G. L. Lamb, Phys. Rev. Lett. **31**, 196 (1973).
[7]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **30**, 1262 (1973).
[8]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **31**, 125 (1973).
[9]E. L. Hahn, Phys. Rev. **80**, 580 (1950).
[10]I. D. Abella, N. A. Kurnit, and S. R. Hartmann, Phys. **141**, 391 (1966).
[11]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].
[12]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, to be published.
[13]C. S. Gardner, J. Greene, M. D. Kruskal, and R. M. Mirua, Phys. Rev. Lett. **19**, 1095 (1967); M. D. Kruskal, R. M. Miura, C. S. Gardner, and N. J. Zabusky, J. Math. Phys. **11**, 952 (1970).
[14]A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE **61**, 1443 (1973).
[15]M. J. Ablowitz and A. C. Newell, J. Math. Phys. **14**, 1277 (1973).
[16]D. D. Schnack and G. L. Lamb, Jr., in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1973), pp. 23–33.