Problems: #4 p 100.

Inverses of real functions

Let $f: A \to B$ be any 1-1 function.

Recall $\Gamma(f) = \{(a,b) \mid f(a) = b\} \subseteq A \times B$

Define $f^{-1}: f(A) \to A$ to be the function whose graph is $\Gamma(f^{-1}) = \{(b,a) \mid (a,b) \in \Gamma(f)\} \subseteq f(A) \times A$

i.e. $f(x) = y \Leftrightarrow x = f^{-1}(y)$

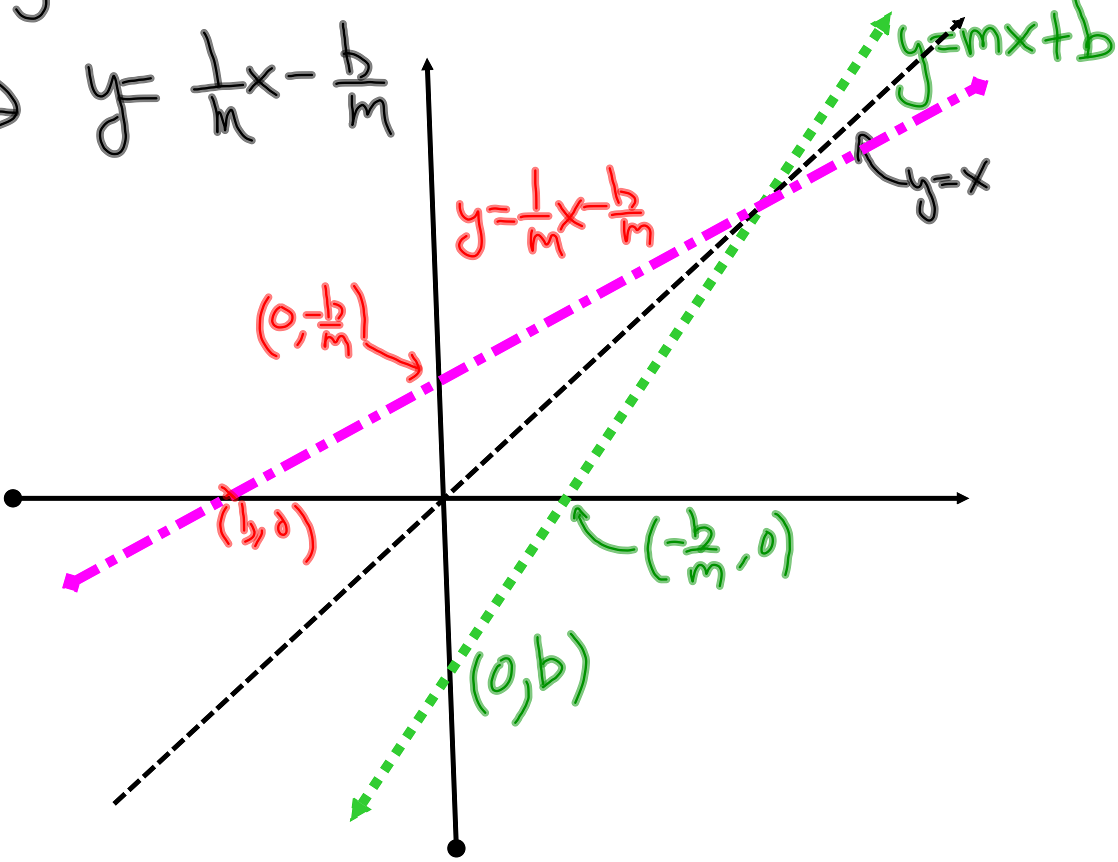
Ex: Find $f^{-1}(x)$ where $y = f(x) = mx + b$

$m \neq 0$.

Sol: We interchange x, y & solve for y:

$$y = mx + b \overset{\text{swap } x,y}{\rightsquigarrow} x = my + b$$

$$\Rightarrow y = \frac{1}{m}x - \frac{b}{m}$$



Consequence: Horizontal line test:

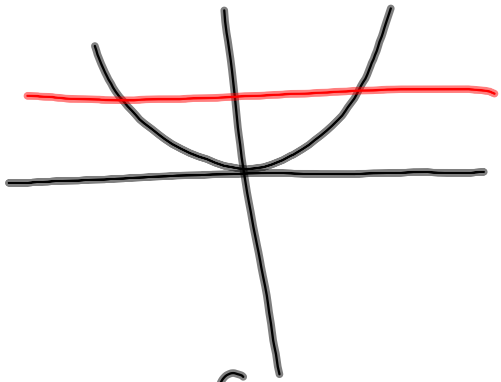
A real function f is 1-1 iff no horizontal line intersects $\Gamma(f)$ in more than one point.

* Let $S \subseteq \mathbb{R} \times \mathbb{R}$ be a subset. Then $S = \Gamma(f)$ for some real function f iff: S passes the vertical line test.

Since a real function f is 1-1 iff $\{(y,x) \mid y = f(x)\} \subseteq \mathbb{R}^2$ is the graph of a function and the reflection of a vertical line about $y = x$ is a horizontal line

Conclude: A real function f is 1-1 iff no horiz. line intersects $\Gamma(f)$ in more than one point.

* It's often possible to restrict the domain of a function to make it 1-1:



Ex: $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$

Not 1-1.

However, the restriction of f

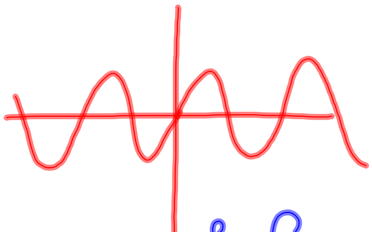
* $f|_{\mathbb{R}_{\geq 0}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$
$x \mapsto x^2$

$(f|_{\mathbb{R}_{\geq 0}})^{-1}(x) = \sqrt{x}$

$f|_{\mathbb{R}_{<0}} : \mathbb{R}_{<0} \to \mathbb{R}$
$x \mapsto x^2$

$(f|_{\mathbb{R}_{<0}})^{-1}(x) = -\sqrt{x}$

Ex: $f:\mathbb{R}\to\mathbb{R}$
$x\mapsto \sin(x)$



Not 1-1.

There are *many* restrictions of f which 1-1:

$$f_k(x) := f\big|_{[k\pi-\frac{\pi}{2},\,k\pi+\frac{\pi}{2}]} : [k\pi-\tfrac{\pi}{2}, k\pi+\tfrac{\pi}{2}] \to \mathbb{R}$$

is 1-1 for any $k\in\mathbb{Z}$.

Define: $\arcsin(x) = \sin^{-1}(x) \overset{\text{def}}{:=} f_0^{-1}(x)$

Then $\sin^{-1}: [-1,1]\to[-\frac{\pi}{2},\frac{\pi}{2}]$.

Q: What is $f_k^{-1}(x)$ in terms of $\sin^{-1}(x)$ for $k\in\mathbb{Z}$?

A:

$$\begin{tikzcd}
{[k\pi-\frac{\pi}{2}, k\pi+\frac{\pi}{2}]} \arrow[r, "f_k"] & {[-1,1]} \arrow[d, "\sin^{-1}=f_0^{-1}"] \\
 & {[-\frac{\pi}{2},\frac{\pi}{2}]} \arrow[ul, "x+k\pi"]
\end{tikzcd}$$

So $f_k^{-1}(x) = \sin^{-1}(x)+k\pi$

$\ast$ $\sin^{-1}(x)$ is the unique inverse of a restriction of $\sin(x)$ that:

1) is continuous
2) has range = domain of $\sin$
3) domain includes all acute angles $(0,\frac{\pi}{2})$

Exponents / logs:

$b \neq 1$, $\quad y = b^x =: \exp_b(x)$

$\exp_b : \mathbb{R} \to \mathbb{R}_{>0}$

$\log_b(x) := \exp_b^{-1}$ (makes sense b/c $\exp_b$ is 1-1)

$\mathbb{R}_{>0} \to \mathbb{R}$

$$\boxed{y = b^x \text{ iff } x = \log_b(y)}$$

$$b^x \cdot b^y = b^{x+y} \Longleftrightarrow \log_b(xy) = \log_b(x) + \log_b($$