10/12
407

Problems. #41, p. 100.

Inverses of real functions

Let $f : A \to B$ be any function that is 1-1.
Recall the graph of f is

$$\Gamma(f) = \{ (x, y) \mid x \in A, y \in B, y = f(x) \} \subseteq A \times B$$

We define $f^{-1} : f(A) \to A$ to be the ~~function~~ unique function with graph

$$\Gamma(f^{-1}) = \{ (y, x) \mid y \in f(A), x \in A, \text{ and } (x, y) \in \Gamma(f) \} \subseteq f(A) \times A \subseteq B \times A.$$

In other words: $y = f(x) \iff x = f^{-1}(y)$

For all functions $f : \mathbb{R} \to \mathbb{R}$, we observe:

$$\Gamma(f^{-1}) = \{ (y, x) \mid (x, y) \in \Gamma(f) \}$$

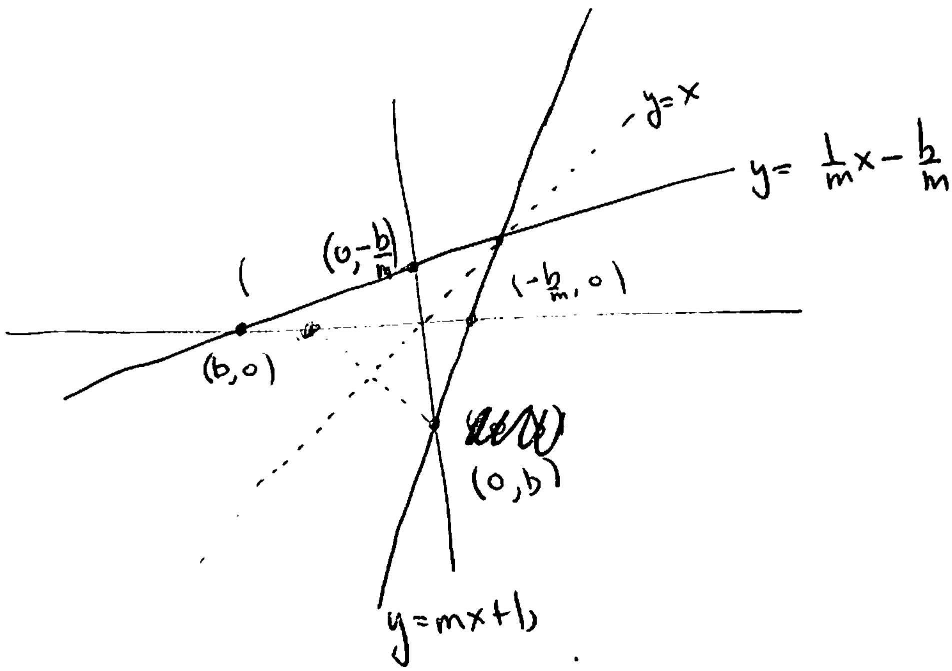
= the reflection of $\Gamma(f)$ about the line $y = x$ (interchange y and x).

Ex: Find $f^{-1}(x)$ for $y = f(x) = mx + b$.

Sol: We interchange x and y in the formula for $f(x)$: $y = mx + b \rightsquigarrow x = my + b$

Then solve for y:

$$y = \frac{1}{m}x - \frac{b}{m} = f^{-1}(x).$$



<u>Consequence</u>: The "horizontal line test":

Let $S \subseteq \mathbb{R} \times \mathbb{R}$ be a subset of the plane. Then we know that S is the graph of a real function iff S passes the "vertical line test": no vertical line intersects S in more than <u>one</u> point.

~~Since the reflection of a vertical line about $y=x$ is a horizontal line, we conclude and a real function f is 1-1 iff $\exists f^{-1}$~~ exists

Since a real function f is 1-1 iff

$$\{(y,x) \mid y = f(x)\} \subseteq \mathbb{R}^2$$

is the graph of a function and the reflection of vertical lines about $y=x$ are horizontal, we conclude

* A real function f is 1-1 iff no horizontal line intersects $\Gamma(f)$ at more than one point

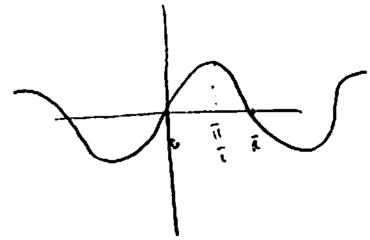
It is often possible to restrict the domain of a real function f to make it 1-1:

Ex: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not 1-1. However, $f|_{\mathbb{R}_{\geq 0}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $x \mapsto x^2$ is 1-1, with inverse

$$f|_{\mathbb{R}_{\geq 0}}^{-1}(x) = \sqrt{1-x^2}.$$

Also, $f|_{\mathbb{R}_{\leq 0}} : \mathbb{R}_{\leq 0} \to \mathbb{R}$, $x \mapsto x^2$ is 1-1, with inverse $(f|_{\mathbb{R}_{\leq 0}})^{-1}(x) = -\sqrt{1-x^2}$

<u>Ex</u>: $f: \mathbb{R} \to \mathbb{R}$ is <u>not</u> 1-1
$x \mapsto \sin(x)$



<u>But</u> for any $k \in \mathbb{Z}$,

$$f_k : \left[k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right] \longrightarrow \mathbb{R}$$
$$x \longmapsto \sin(x)$$

<u>is</u> 1-1. <u>Notice</u> $f_k^{-1}(x) = f_0^{-1}(x) + k\pi$

<u>Def</u>: $\arcsin(x) = \sin^{-1}(x) \overset{\text{def}}{:=} f_0^{-1}$

It is a function $[-1, 1] \longrightarrow [-\pi/2, \pi/2]$

<u>Q</u>. Are there <u>other</u> restrictions of f which have inverses?

<u>A</u>: Yes, but none if we require

- range of restriction = range of f
- restriction is continuous on its domain

<u>Derivatives</u>: Let f be a 1-1 real function s.t. f and f^{-1} are differentiable. Then

$$f(f^{-1}(x)) = x \quad \text{for all } x \in \text{range of } f$$

By chain rule, $f'(f^{-1}(x)) \cdot (f^{-1}(x))' = 1$

So

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

<u>Ex</u>: $$\frac{d}{dx} \arcsin(x) = \frac{1}{\cos(\arcsin(x))} \quad \text{for } x \in [-1, 1]$$

$$= \frac{1}{\sqrt{1 - x^2}} \quad \text{for } x \in [-1, 1]$$

since $\arcsin(x) \in [-\pi/2, \pi/2)$ and $\cos(x) \geq 0$ for such x.

<u>Exponents and Logs</u>

$b \neq 1$, $y = b^x : \mathbb{R} \to \mathbb{R}$, range is $\mathbb{R}_{\geq 0}$

$y = \log_b(x) : \mathbb{R}_{\geq 0} \to \mathbb{R}$, range is $\mathbb{R}$

are <u>inverses</u>: $x = b^{\log_b(x)}$, $\log_b(b^x) = x$.

The usual rules for exponents:

$$b^x \cdot b^y = b^{x+y} \iff \log_b(xy) = \log_b(x) + \log_b(y)$$

$$\Rightarrow \frac{b^x}{b^y} = b^{x-y} \iff \log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

$$(b^x)^y = b^{xy} \iff \log_b(x^y) = y\log_b(x)$$

$$\uparrow$$

$$y = b^x \iff x = \log_b y$$

~~Exmpl~~

Spse $b, c > 0$, $b, c \neq 1$. Then

$$\log_c(b^y) = y \log_c(b) = \log_b(b^y) \cdot \log_c(b)$$

Setting $x = b^y$ gives

$$\boxed{\log_c(x) = \log_b(x) \log_c(b) \quad \forall x > 0}$$

i.e. $\log_c(x)$ is a <u>constant</u> <u>multiple</u> of $\log_b(x)$.

~~Defn: $e :=$ the positive number > 1 s.t. $\int_1^e \frac{1}{x} = 1$~~