Problems: 12, 13, §3.2.3.

Def: Let f be a real function, S a subset of its domain.

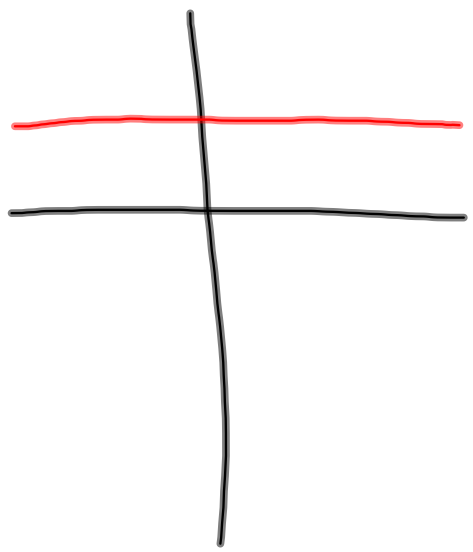
We say f is monotone

$$\left\{\begin{array}{c}\underline{\text{increasing}}\\ \underline{\text{decreasing}}\end{array}\right\} \text{ if } x<y \Rightarrow \left\{\begin{array}{c}f(x)\le f(y)\\ f(x)\ge f(y)\end{array}\right\}$$

Further, f is strictly $\left\{\begin{array}{c}\underline{\text{inc}}\\ \underline{\text{dec}}\end{array}\right\}$ iff

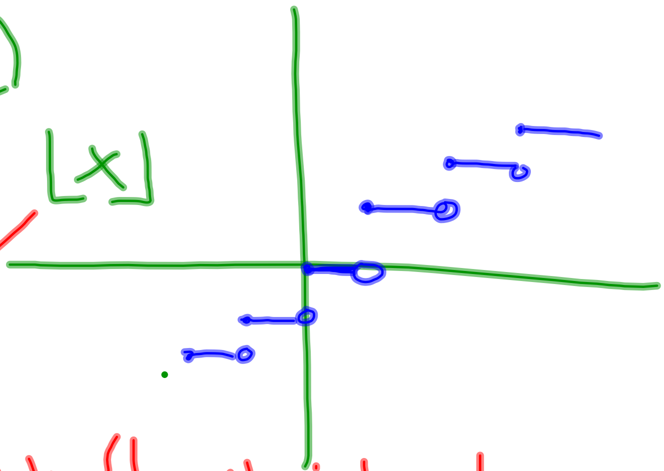
$$x<y \Rightarrow \left\{\begin{array}{c}f(x)<f(y)\\ f(x)>f(y)\end{array}\right\}, \ \forall\ x,y\in S.$$

Ex: 1) $f(x) = c$ (c = constant) is monotone increasing and monotone decreasing. It is neither strictly inc. nor, strictly dec.



Ex 2)

$f(x) = \lfloor x \rfloor$

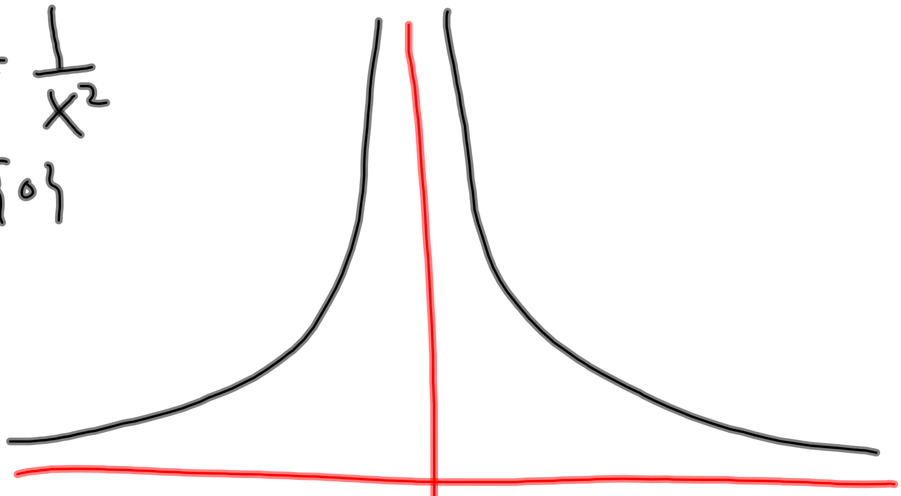


$\lfloor x \rfloor = \{\text{largest integer} \leq x\}$

This is monotone increasing, but <u>not</u> strictly increasing.

$f(x) = \frac{1}{x^2}$

$\mathbb{R} - \{0\}$



* f is strictly increasing on $(-\infty, 0)$
* ______ decreasing on $(0, \infty)$

Thm: Spse f is a differentiable function on $S \subseteq \mathbb{R}$. If $\begin{Bmatrix} f'(x) > 0 \\ f'(x) < 0 \end{Bmatrix}$ for all $x \in S$. then

f is strictly $\begin{Bmatrix} \text{increasing} \\ \text{decreasing} \end{Bmatrix}$ on S

Pf: Use the mean value theorem:
If f is diff. on (a,b) and cts. on $[a,b]$ then $\exists\, c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Suppose $\exists\ a, b \in S$ with $f(a) = f(b)$ but $a \neq b$. Then MVT $\Rightarrow \exists\ c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. Thus, if $f'(x) \neq 0\ \forall x$, then $f(b) \neq f(a)$

..... (finish proof)

$\blacksquare$

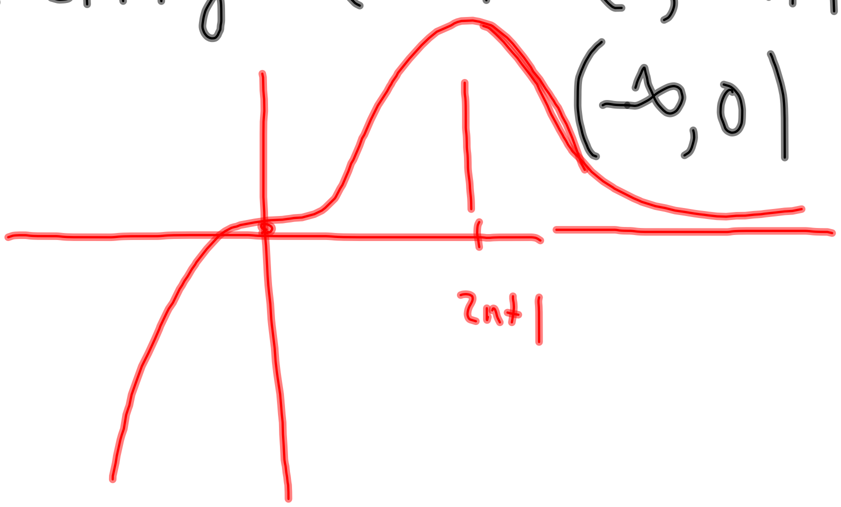
Ex: $f_n(x) = x^{2n+1}e^{-x}$, $n \in \mathbb{Z}_{\geq 0}$

Find all intervals in $\mathbb{R}$ on which f_n is monotone, and determine which kind (i.e. inc/dec). Is f strictly inc/dec on these intervals.

$$f_n'(x) = (2n+1)x^{2n}e^{-x} - x^{2n+1}e^{-x}$$

$$= \frac{x^{2n}}{e^x}(2n+1-x)$$

- f_n is strictly dec on $(2n+1, \infty)$
- f_n is strictly inc. on $(0, 2n+1)$, $(-\infty, 0)$



Thm: If f is strictly inc/dec on S then f is 1-1 on S. In particular, f^{-1} exists on $f(S)$.

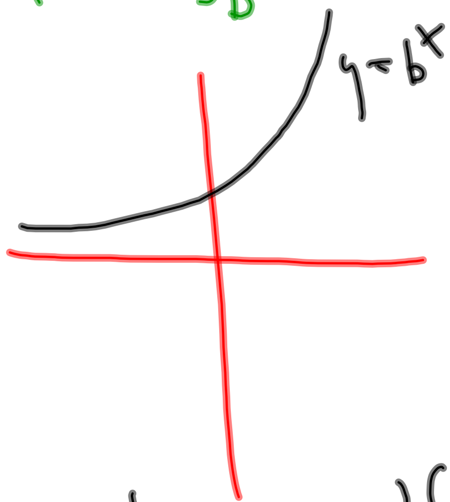
pf: If $x, y \in S$, $x \neq y$, then either $x<y$ or $x>y$, so WOLOG $x<y$. Since f is strictly inc/dec, $f(x)<f(y)$ or $f(x)>f(y)$. In particular, $f(x) \neq f(y)$. Thus, f is 1-1. ∎

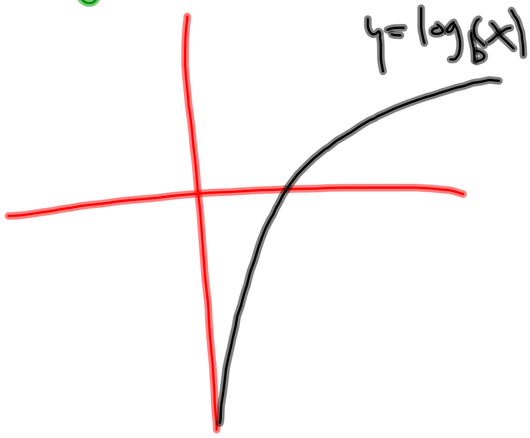
Thm: If f is strictly $\begin{Bmatrix}\text{increasing}\\ \text{decreasing}\end{Bmatrix}$ on S, then f^{-1} is strictly $\begin{Bmatrix}\text{increasing}\\ \text{decreasing}\end{Bmatrix}$ on $f(S)$.

pf: exercise.

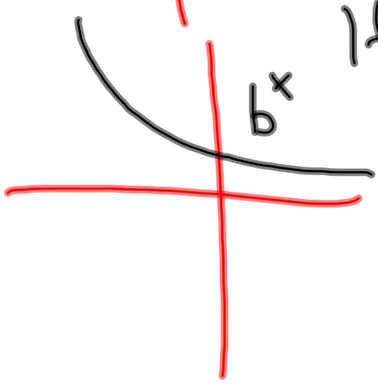
Ex: $b>1$. Then $y=f(x)=b^x$ is strictly inc $\left(f'(x) = \underbrace{b^x}_{>0} \underbrace{\log b}_{>0 \text{ as } b>1} \right)$

$f^{-1}(x) = \log_b(x)$ is strictly inc.





If $0<b<1$



Sequences:

$\{s_n\}_{n\in\mathbb{Z}_{\geq k}}$ a sequence.

$\{s_n\}$ is $\begin{Bmatrix} \text{increasing} \\ \text{decreasing} \end{Bmatrix}$ if $\begin{Bmatrix} s_{n+1} > s_n \\ s_{n+1} < s_n \end{Bmatrix}$

$\forall\, n\in\mathbb{Z}_{\geq k}$.

Ex: Let s_n be defined by $s_0=1$

$s_{n+1}=\frac{1}{8}(3s_n+6)$

n	0	1	2	
s_n	1	$\frac{9}{8}$	$\frac{75}{64}$	$\cdots$

claim s_n is increasing.

pf: $s_{n+1} > s_n \iff \frac{1}{8}(3s_n+6) > s_n$

$\iff 6 > 5s_n$

$\iff \frac{6}{5} > s_n$

Claim: $s_n < \frac{6}{5}$, $\forall n$.

pf: Induction. Base case: $s_0 = 1 < \frac{6}{5}$ ✓

If $s_k < \frac{6}{5}$, then $s_{k+1} = \frac{1}{8}(3s_k+6)$

$< \frac{1}{8}\left(3\frac{6}{5}+6\right)$

$= \frac{6}{5}$ ∎