407
10/19



Problems: 12, 13.

* Change Teams.

Def: Let $f$ be a real function, and $S$ a subset of its domain. We say that $f$ is

monotone
$$\begin{cases} \text{- increasing on } S \text{ iff} & x<y \Rightarrow f(x)\leq f(y), \ \forall x,y\in S \\ \text{- decreasing on } S \text{ iff} & x<y \Rightarrow f(x)\geq f(y), \ \forall x,y\in S. \end{cases}$$

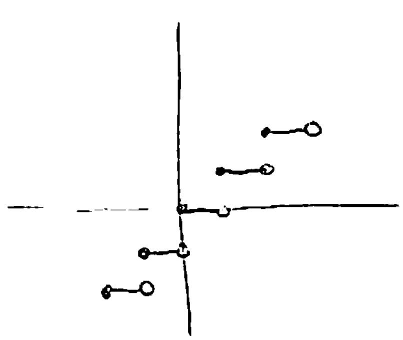
We say $f$ is strictly $\begin{Bmatrix} \text{increasing} \\ \text{decreasing} \end{Bmatrix}$ on $S$ iff

$$x<y \Rightarrow \begin{Bmatrix} f(x)<f(y) \\ f(x)>f(y) \end{Bmatrix} \quad \forall x,y\in S.$$

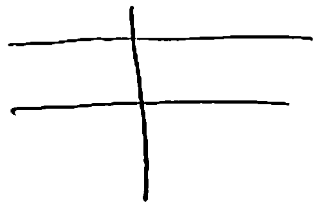
and that $f$ is strictly monotone if it is strictly inc or dec.

Examples:

1) $f(x) = \lfloor x \rfloor := \begin{Bmatrix} \text{the largest integer} \\ \leq x \end{Bmatrix}$ is monotone increasing, but not strictly increasing

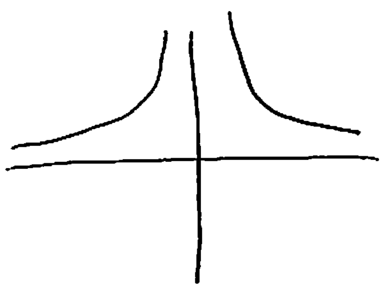


2) $f(x) = c$ (constant) is monotone increasing <u>and</u> monotone decreasing, but neither increasing nor decreasing (<u>not</u> strictly monotone)



3) $f(x) = \frac{1}{x^2}$ is strictly increasing on $\mathbb{R}_{<0}$ strictly decreasing on $\mathbb{R}_{>0}$

"witch hat".



<u>Thm:</u> If $f$ is strictly monotone on $S$, then $f$ is 1-1 on $S$.

<u>proof:</u> Let $x, y \in S$ and suppose $x \neq y$. Then either $x < y$ or $y < x$, and WLOG $x < y$.
Since $f$ is <u>strictly</u> monotone on $S$, this implies $f(x) < f(y)$ <u>or</u> $f(x) > f(y)$; in either case $f(x) \neq f(y)$ so $f$ is 1-1 on $S$.

<u>Cor:</u> If $f$ is strictly monotone on $S$, then $f|_S$ has an inverse.

Now suppose that $f$ is a <u>differentiable</u> function on $S \subseteq \mathbb{R}$.

<u>Thm:</u> - If $f'(x) > 0 \ \forall x \in S$ then $f$ is strictly increasing on $S$

- If $f'(x) < 0 \ \forall x \in S$ then $f$ is strictly decreasing on $S$.

<u>Pf.</u> Suppose that $f(x) = f(y)$ for $x, y \in S$ with $x \neq y$ (wlog $x<y$)

~~Then~~ By the mean value theorem, $\exists \ \alpha \in (x,y)$

s.t. $f'(\alpha) = \dfrac{f(y)-f(x)}{y-x} = 0$. Thus, if $f'(x) \neq 0 \ \forall x \in S$

then $f$ is strictly monotone on $S$.

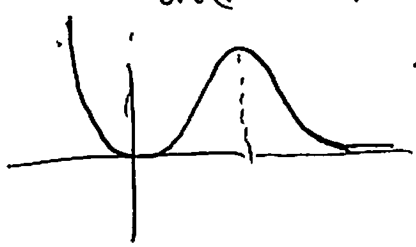
<u>Ex</u>: $f(x) = x^n e^{-x}$ is <u>not</u> monotone on $\mathbb{R}$ $\forall \ n > 0$.

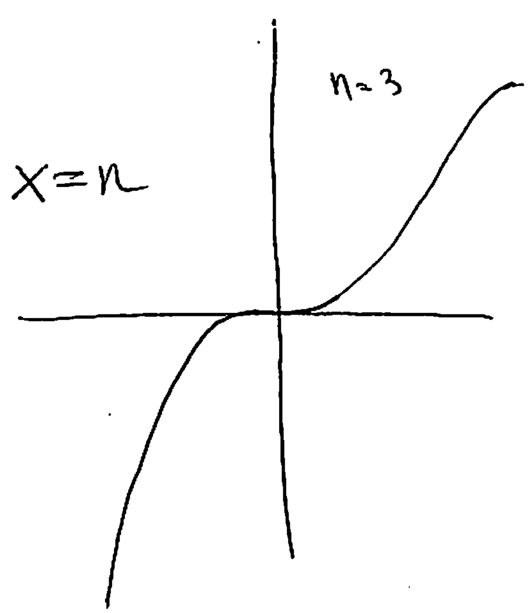
$$f'(x) = nx^{n-1}e^{-x} - x^n e^{-x}$$

$$= e^{-x}x^{n-1}(n - x) = 0 \quad \text{for} \quad x = n$$

and $f' > 0$ if $0 < x < n$

$< 0$ if $x > n$





The derivative condition may also be interpreted as follows:

$f$ differentiable on $S$, $x_0 \in S$, then

$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. Provided $f'(x_0) \neq 0$,

so $\dfrac{(f(x) - f(x_0))}{f'(x_0)} + x_0 \cong x$ and

$f^{-1}(x) \cong \dfrac{x - f(x_0)}{f'(x_0)} + x_0$

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Thm: If $f$ is strictly $\begin{Bmatrix} \text{incr.} \\ \text{decr} \end{Bmatrix}$ on $S$ then $f^{-1}$ is strictly $\begin{Bmatrix} \text{incr} \\ \text{decr} \end{Bmatrix}$ on $f(S)$.

pf: If $f$ is strictly incr, then for $x, y \in f(S)$,

$$f^{-1}(x) < f^{-1}(y) \Rightarrow f(f^{-1}(x)) < f(f^{-1}(y))$$

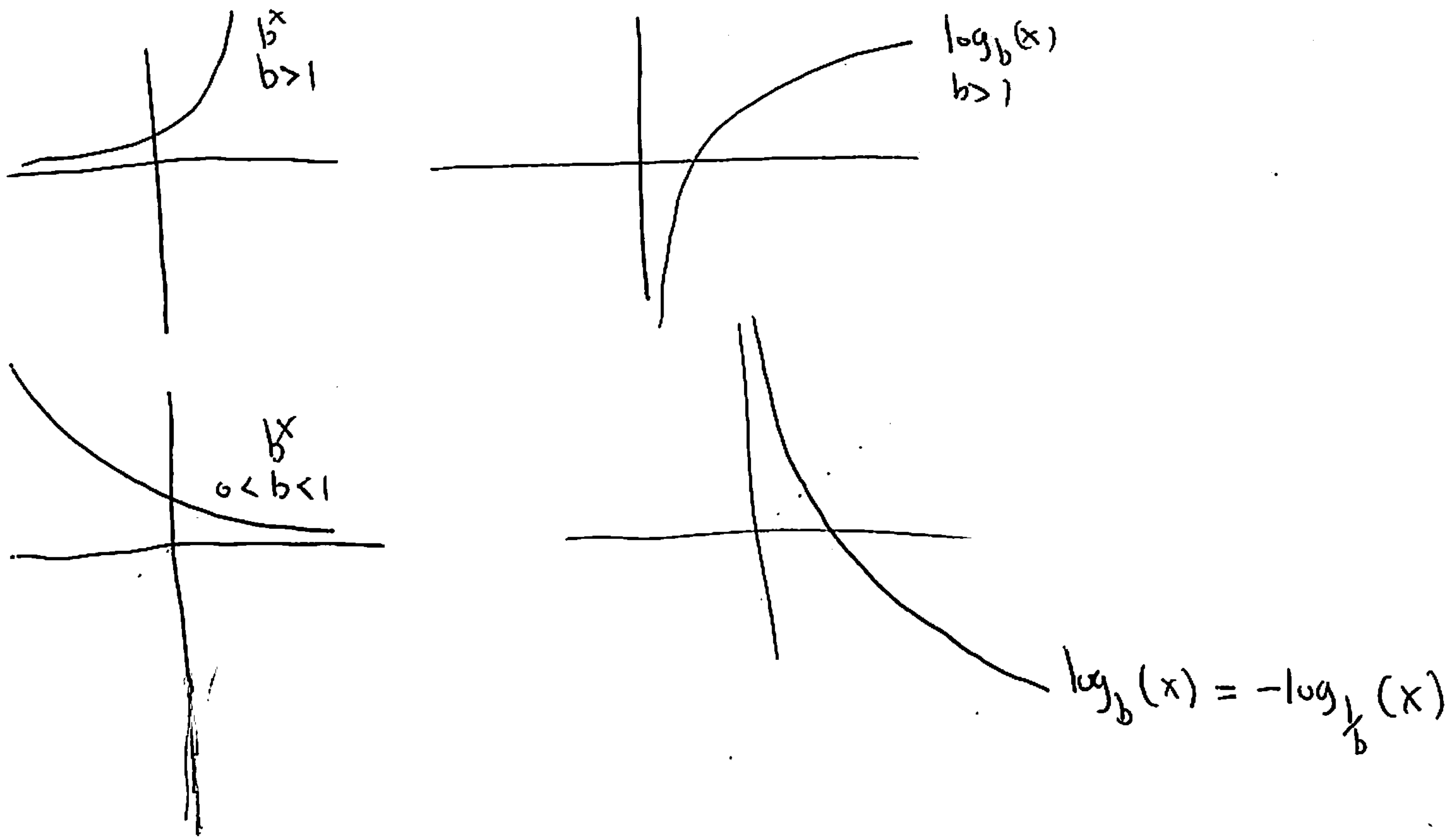
$$\Rightarrow x < y$$

Hence, $x \geq y \Rightarrow f^{-1}(x) \geq f^{-1}(y)$

Since $x = y$ iff $f^{-1}(x) = f^{-1}(y)$, we conclude $x > y \Rightarrow f^{-1}(x) > f^{-1}(y)$. Similar pf for str. dec.

Ex: $b \neq 1$. Then $y = f(x) := b^x$ is strictly $\begin{Bmatrix} \text{increasing} \\ \text{decreasing} \end{Bmatrix}$ if $\begin{Bmatrix} 1 < b \\ 0 < b < 1 \end{Bmatrix}$.

We have $f^{-1}(x) = \log_b(x)$. It is strictly $\begin{Bmatrix} \text{increasing} \\ \text{decreasing} \end{Bmatrix}$ if $\begin{Bmatrix} 1 < b \\ 0 < b < 1 \end{Bmatrix}$

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Sequences:

$\{s_n\}_{n \in \mathbb{Z}_{\geq k}}$ a sequence.

$\{s_n\}$ is $\begin{Bmatrix} \text{increasing} \\ \text{decreasing} \end{Bmatrix}$ iff $\begin{Bmatrix} s_n < s_{n+1} \\ s_n > s_{n+1} \end{Bmatrix}$ $\forall\ n \in \mathbb{Z}_{\geq k}$.

Ex: Let $S_n$ be defined by $S_0 = 1$,

$$S_{n+1} = \frac{1}{8}(3S_n + 6).$$

We compute:

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $S_n$ | 1 | $9/8$ | $\frac{75}{64}$ | ... | |

Claim: $S_n$ is ~~monotone~~ strictly increasing.

pf. ~~Induction.~~ $S_{n+1} = \frac{1}{8}(3S_n + 6) > S_n$

$$\iff 3S_n + 6 > 8S_n$$

$$\iff 6 > 5S_n$$

$$\iff S_n < \frac{6}{5} \quad \forall\, n.$$

We prove this by induction:

base case: $S_1 = 1 < 6/5$ ✓

If $S_k < \frac{6}{5}$ then
$$S_{k+1} = \frac{1}{8}(3S_k + 6) < \frac{1}{8}\left(3 \cdot \frac{6}{5} + 6\right) = \frac{1}{8}\left(\frac{48}{5}\right) = \frac{6}{5}. \quad ∎$$

Ex: Take any $n \in \mathbb{Z}_{\geq 1}$. Define $\{S_i(n)\}$ by

$$S_1(n) = n, \qquad S_{i+1}(n) = \begin{cases} 3S_i(n) + 1 & \text{if } S_i(n) \text{ is odd} \\ \frac{S_i(n)}{2} & \text{if } S_i(n) \text{ is even} \end{cases}$$