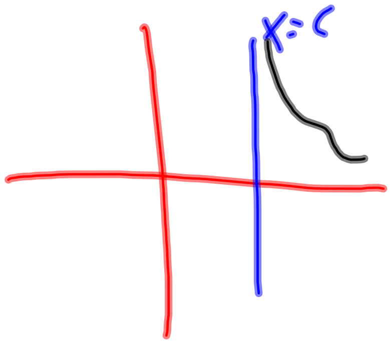
Problems: p. 112, #2, 5.

Limiting behavior of real functions

Let f be a function $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$.

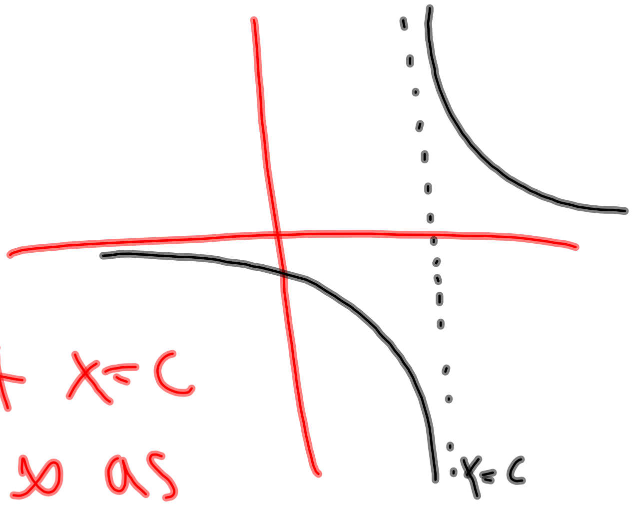
We say, for $c \in D$, that f diverges to ∞ as x approaches c from the right, $\lim_{x \to c^+} f(x) = \infty$

if: $\forall M > 0$, $\exists \alpha_M \in D$ with $\alpha_M > c$ and $f(\alpha_M) > M$.



Similar defs for div. to $-\infty$ and/or as x approaches c from left.

Ex: $f(x) = \dfrac{1}{x-c}$



Def: $f(x)$ has a vertical asymptote at $x=c$ if: f diverges to $\pm\infty$ as x approaches c from left/right.

Ex: $f(x) = \dfrac{(x-1)(x-2)(x+4)^2(x-3)}{(x-1)^3(x-2)(x+4)(x+5)}$

Q: What are the vert. asymptotes of f? How about zeroes?

What is the domain of f?

$\therefore D = \mathbb{R} - \{1, 2, -4, -5\}$

$$f^*(x) = \frac{(x+4)(x-3)}{(x-1)^2(x+5)}$$

$f(x) = f^*(x)$, for all $x \in D$.

What is the domain of f^*?

$D^* = \mathbb{R} - \{1, -5\} \supseteq D$

Zeroes of f = zeroes of f^* on $\underline{D}$

since $f = f^*$ on D

$= \{3\}$

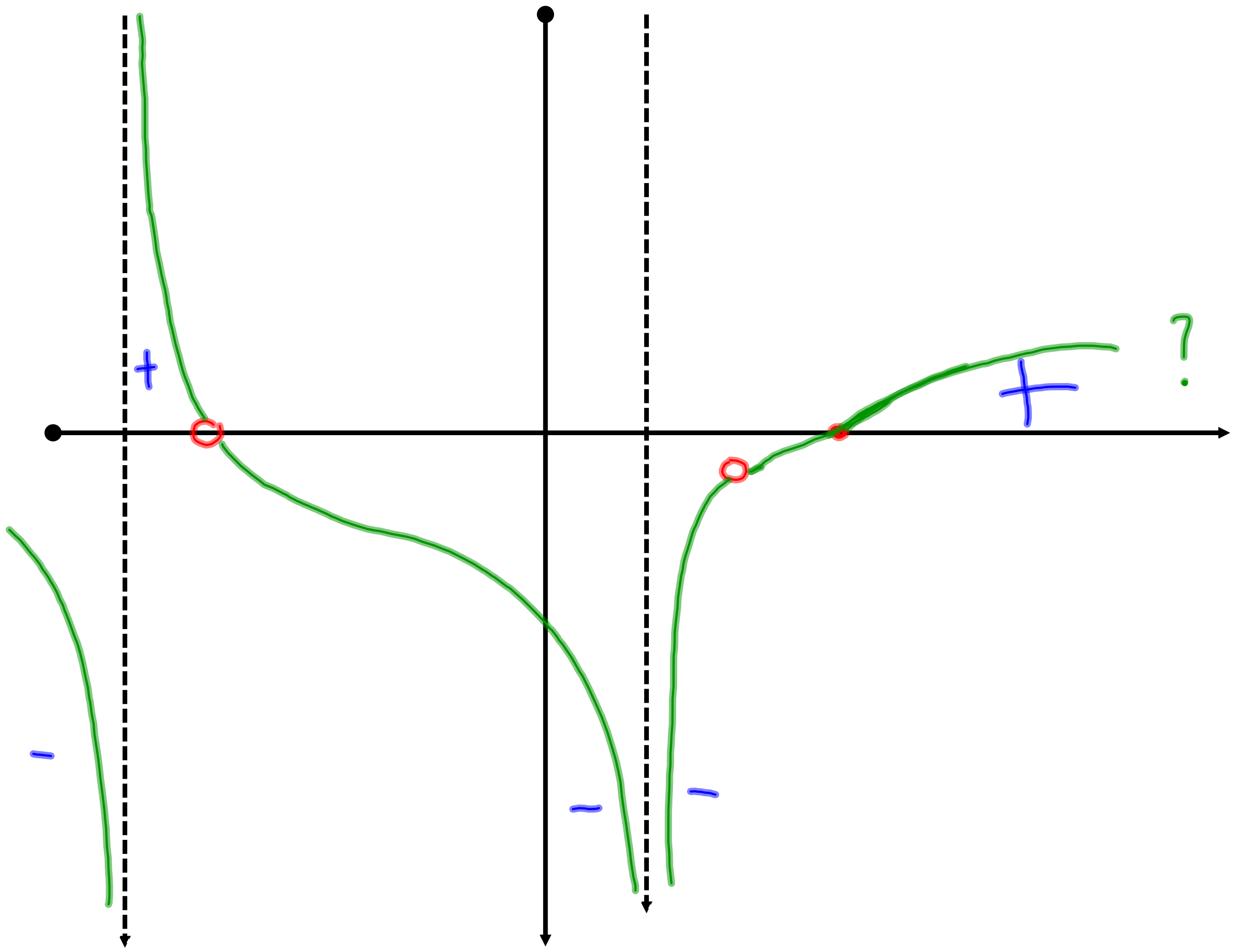
$\underline{\text{Asymptotes}}$ of f: -5 (because $f = \frac{p(x)}{q(x)}$, $p(-5) \neq 0$, $q(-5) = 0$)

Other possibilities: $1, 2, -4$.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} f^*(x) = f^*(2) = -\frac{6}{7}$$

$$\lim_{x \to -4} f(x) = \lim_{x \to -4} f^*(x) = f^*(-4) = 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} f^*(x) = -\infty$$ ← vertical asymptote.



We used the following: Let $p(x), q(x)$ be cts. real functions. Then

$f(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote at $x = c$ if $p(c) \neq 0$ and $q(c) = 0$

Warning: The converse is false: $p(x) = x - c$
$q(x) = (x-c)^2$

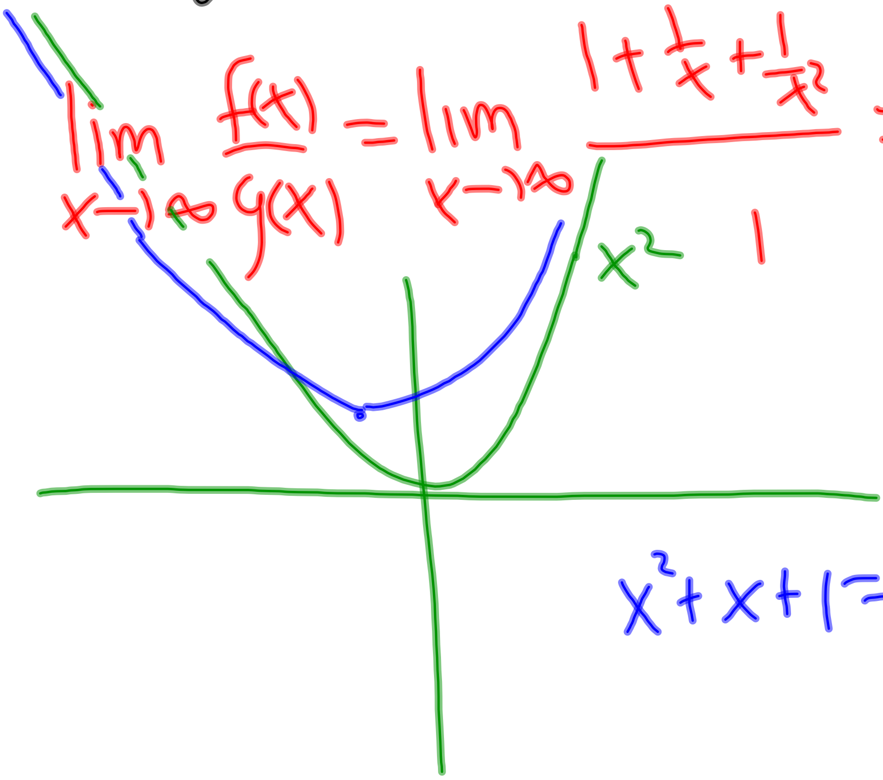
"End behavior"

Def: We say that two real fns. $\underbrace{f,\ g}$ have the same end behavior if at ∞

$$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right)=1$$

Ex $f(x)=x^2+x+1$, $g(x)=x^2$ } have same end behavior @ ∞, $-\infty$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{1+\frac{1}{x}+\frac{1}{x^2}}{1}=1$$



$$x^2+x+1=\left(x+\tfrac{1}{2}\right)^2+\tfrac{3}{4}$$

<u>More generally</u>: If $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots$

$g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots$

Then $\frac{f(x)}{g(x)}$ has the same end behavior @ $\infty, -\infty$ as

$$\frac{a_m}{b_n} x^{m-n}$$

<u>Def</u>: f has: - <u>horizontal asymptote</u> <u>$y = L$</u>

if $f(x)$ has same end behavior @ ∞ or $-\infty$ as $g(x) = L$

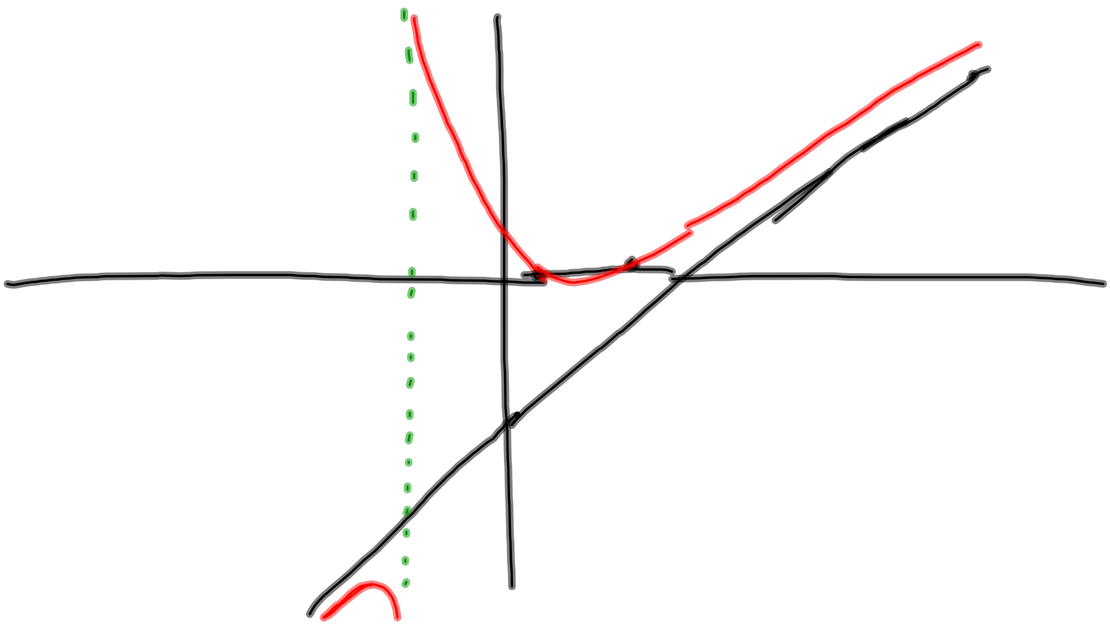
- <u>oblique asymptote</u>

$y = ax + b, a \neq 0$, if $f(x)$ has same end behavior as $g(x) = ax + b$

Ex: $f(x) = \dfrac{x^2-3x+1}{x+1} = x-4 + \dfrac{5}{x+1}$

$$\begin{array}{r|l} & x - 4 \\ \hline x+1 & x^2-3x+1 \\ & \underline{x^2+x} \\ & \quad -4x+1 \\ & \quad \underline{-4x-4} \\ & \qquad\quad 5 \end{array}$$

So $f(x)$ has the same end behavior @ $\infty, -\infty$ as $x-4$



Def: f, g real functions.

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \begin{cases} \pm\infty & f \text{ has higher growth rate} \\ 0 & f \text{ has lower growth rate} \\ L \neq 0 & f \text{ has the same growth rate as } g \end{cases}$$

Ex: Rank the following functions in terms of their growth rates

1) $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ $n \geq 2$, $a_n \neq 0$

2) $\log_b(x)$ $b > 1$

3) $x^{\frac{1}{n}}$ $n \geq 1$

4) b^x $b > 1$

5) $\log(\log(x))$

If f, g are diff and $\lim_{x\to\infty} f = \lim_{x\to\infty} g = \infty$

then $\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{f'}{g'}$

provided this exists.