407
10/24

Problems: 2, 5.

1

Limit behavior of real fns

Let f be a real function $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$

For $c \in D$, we say <u>f diverges to ∞ as x approaches c from the right</u>, written

$\lim_{x \to c^+} f(x) = \infty$ iff: $\forall$ positive M, $\exists$ $d(M) \in D$ with
1. $d(M) > c$
2. $f(d(M)) > M$

We have similar defs for $\lim_{x \to c^+} f(x) = -\infty$ and $\lim_{x \to c^-} f(x) = \pm\infty$.

<u>Ex</u>: $f(x) = \frac{1}{x-c}$ diverges to $\begin{cases} \infty \\ -\infty \end{cases}$ as x approaches c from the $\begin{cases} \text{right} \\ \text{left} \end{cases}$.

<u>Def</u>: If f diverges to $\pm\infty$ as x approaches c from the left or right, we say <u>f has a vertical asymptote at $x=c$</u>, and ~~[illegible]~~ the line $x=c$ is called a <u>vertical asymptote</u> of f.

Ex: $f(x) = \frac{(x-1)}{(x-1)^3}\frac{(x-2)}{(x-2)}\cdot\frac{(x+4)^2}{(x+4)}\frac{(x-3)}{(x+5)}$

Then f is undefined at $x \in \{1, 2, -4, -5\}$.

Define

$$f^*(x) = \frac{(x+4)(x-3)}{(x-1)^2(x+5)}, \quad \text{so } f^*(x) = f(x) \text{ for } x \in \mathbb{R} - \{1, 2, -4, -5\}$$

In particular, $\boxed{f \text{ has zeroes at } x = 3,}$

$\boxed{f \text{ has vert. asymp. at } x = -5}$ ~~f has vert. asymptote at x = -5~~

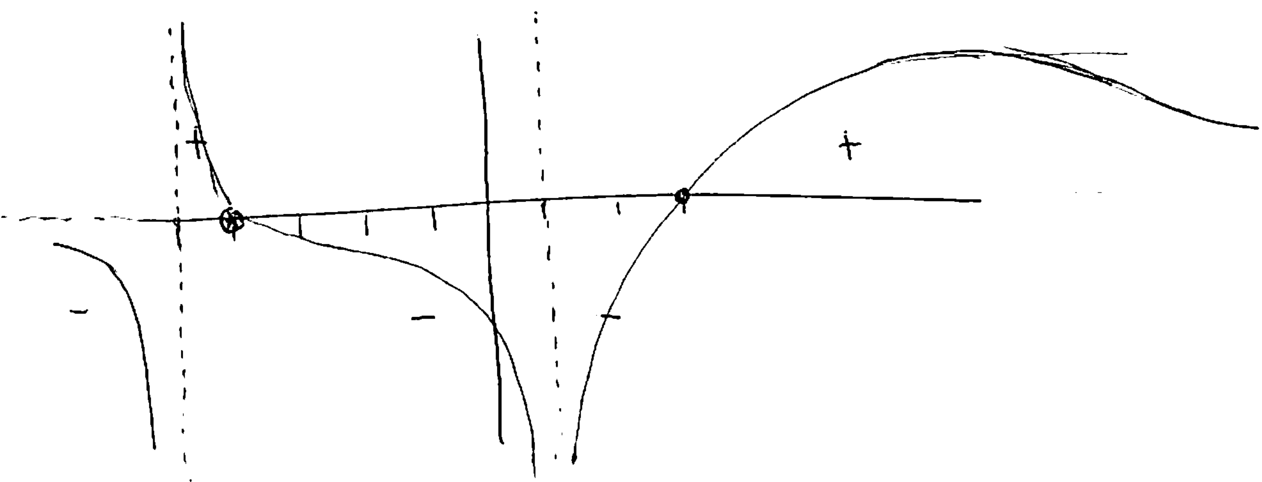
We compute, among possible asymptotes $\{1, 2, -4, -5\}$

$$\lim_{x \to -4} f(x) = \lim_{x \to -4} f^*(x) = 0$$

~~$\lim_{x \to -5^+} f(x) = \lim_{x \to -5} f^*(x) =$~~

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} f^*(x) = -\infty \quad \to f \text{ has vert. asymp. at } x = 1$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} f^*(x) = \frac{6 \cdot -1}{1^2 \cdot 7} = -\frac{6}{7}$$



Thm: $p(x)$, $q(x)$ cts functions on $D \subseteq \mathbb{R}$

Then $f(x) = \dfrac{p(x)}{q(x)}$ has a vertical asymptote

at $x = c \in D$ if $p(c) \neq 0$, $q(c) = 0$.

N.B. Converse is false: e.g. $\dfrac{x-c}{(x-c)^2}$ has vert. asymp. at $x = c$.

"End Behavior": Want to understand $\lim\limits_{x \to \pm\infty} f(x)$.

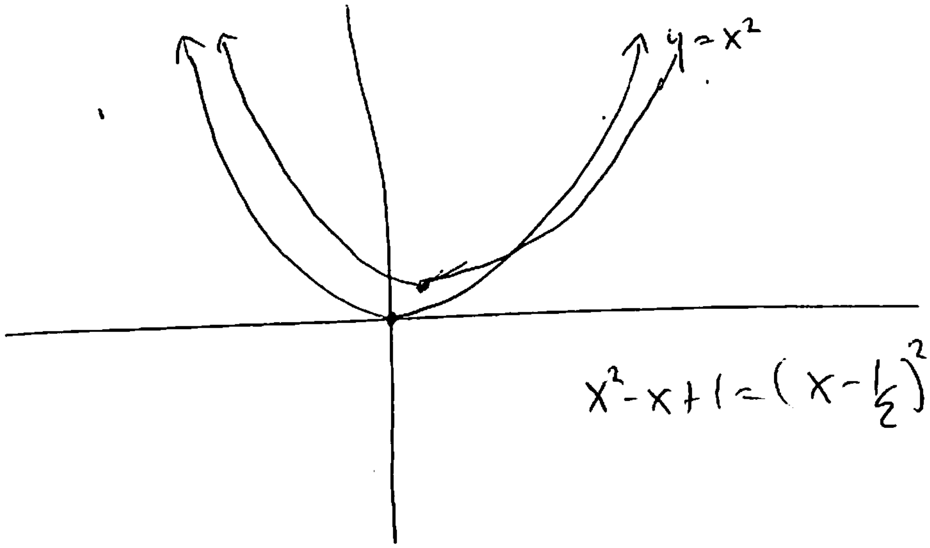
~~(we call the pair $\{\lim_{x\to-\infty} f(x), \lim_{x\to\infty} f(x)\}$ the end behavior of f)~~

Def: We say that two real functions f, g have the same end behavior @ $\begin{Bmatrix} \infty \\ -\infty \end{Bmatrix}$ if

$$\begin{Bmatrix} \lim\limits_{x\to\infty} \\ \lim\limits_{x\to-\infty} \end{Bmatrix} \frac{f(x)}{g(x)} = 1.$$

Ex: $f(x) = x^2 - x + 1$, $g(x) = x^2$ have the same end behavior @ $\pm\infty$: because

$$\lim_{x\to\pm\infty} \frac{f}{g} = \lim_{x\to\pm\infty} \frac{x^2 - x + 1}{x^2} = \lim_{x\to\pm\infty} \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{1} = 1.$$



$$x^2 - x + 1 = (x - \tfrac{1}{2})^2 + \tfrac{3}{4}$$

<u>Thm</u>: Let $f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots$

$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots$

Then $\frac{f(x)}{g(x)}$ has the same end behavior @ $\pm\infty$

as $\frac{a_m}{b_n} x^{m-n}$.

pf: see book

<u>Def</u>: Let f be a real function.

We say • f has a <u>horizontal asymptote</u> <u>$y = L$</u>

if $f(x)$ has the same end behavior as $g(x) = L$ @ ∞ <u>or</u> $-\infty$.

• f has an <u>oblique asymptote</u> <u>$y = ax + b$</u>

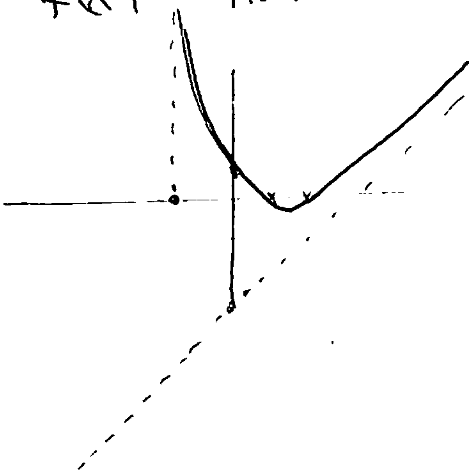
if $f(x)$ has the same end behavior as $g(x) = ax + b$, $a \neq 0$.

Ex: $f(x) = \dfrac{x^2-3x+1}{x+1}$

$$\begin{array}{r|l} & x-4 \\ \hline x+1 & x^2-3x+1 \\ & \underline{x^2+x} \\ & -4x+1 \\ & \underline{-4x-4} \\ & 5 \end{array}$$

$$f(x) = \frac{(x-4)(x+1)+5}{x+1} = x-4+\frac{5}{x+1}$$

So $f(x)$ has an oblique asymptote @ $y = x-4$



L'Hopital's Rule: Let f, g be differentiable real functions with $\lim_{x\to\infty} f = \lim_{x\to\infty} g = \infty$ (or $-\infty$ or 0) then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$, provided the latter exists.

Ex: $$\lim_{x\to\infty} \frac{\log(\log(x))}{\log(x)} = \lim_{x\to\infty} \frac{\frac{1}{\log(x)}\cdot\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\log(x)} = 0$$

<u>Def:</u> Let f, g be real functions.

We say

- f, g have the <u>same</u> order of growth if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0$
- f has a greater / higher order of growth
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = +\infty$$
- f has a lower / smaller ord of growth if
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

<u>Ex:</u> • $\log \log(x)$ has a lower order of growth than $\log(x)$

<u>Activity:</u> Rank (with proof) the following (classes of) functions by order of growth

1) polynomial functions of degree n
2) logarithms w/ base $b > 1$
3) n-th roots: $x^{1/n}$
4) exponentials w/ base $b > 1$.
5) Factorials.
6) Iterates of $\log(x)$ (n-times)