Problems: p. 85 #2.

More on functions:

Recall: $f: A \to B$ is injective (1-1) iff each $b \in B$ is equal to $f(a)$ for at most one $a \in A$.

Ex: $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$ is not 1-1: $f(-1) = f(1) = 1$
$x \mapsto x^2$

Ex: $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is injective
$x \mapsto x^2$

(if $x_1^2 = x_2^2$ then $x_1 = \pm x_2$ so since $x_1, x_2 \geq 0$, must have $x_1 = x_2$)

Def: f is surjective (onto) iff each $b \in B$ is equal to $f(a)$ for at least one $a \in A$.

Def: f is bijective iff each $b \in B$ is $f(a)$ for exactly one $a \in A$.

Ex: $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto -x$
bij

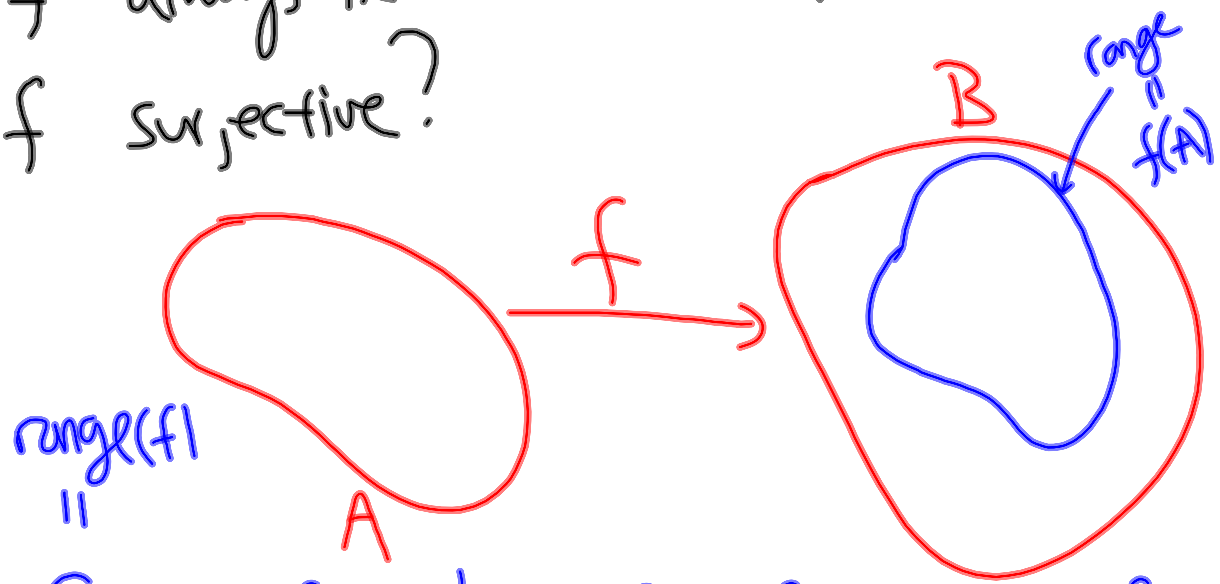
$f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$
neither inj. nor surj.

but $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$
$x \mapsto x^2$
is surj
not inj

Q: Can the domain of f be changed to make f surjective?

(Yes. $f: \mathbb{R} \cup \mathbb{R}\cdot i \longrightarrow \mathbb{R}$
$x \longmapsto x^2$
is surj.)

Q: Can the codomain of a function f always be restricted to make f surjective?



$$f(A) := \{ b \in B \mid b = f(a) \text{ for some } a \in A \}$$

$$f : A \to f(A) \subseteq B$$

is surjective!

Let A, B be sets, and $|A|$ = cardinality of A

$|B|$ = cardinality of B.

We say

$|A| \leq |B|$ iff: $\exists\, f: A \to B$ that is 1–1

$|A| \geq |B|$ iff: $\exists\, f: A \to B$ that is surjective

$|A| = |B|$ iff: $\exists\, f: A \to B$ that is bijective.

This is good def, but strange things can happen!

Ex: $|\mathbb{N}| = |\mathbb{Q}|$ $\qquad \mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b \neq 0 \right\}$

Indeed:
- $|\mathbb{N}| \leq |\mathbb{Q}|$
- $|\mathbb{N}| \geq |\mathbb{Q}|$

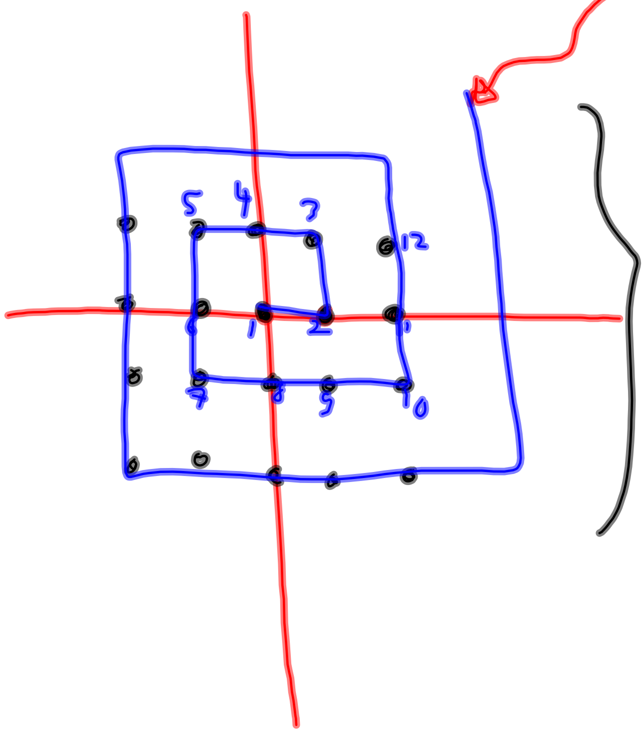
obvious: $f: \mathbb{N} \to \mathbb{Q}$, $a \mapsto \frac{a}{1}$ is injective.

I'll make $g: \mathbb{N} \to \mathbb{Q}$ that is **surjective!**

$$\begin{tikzcd} \mathbb{N} \arrow[r] \arrow[rr, bend left, "g"] & \mathbb{Z} \times \mathbb{Z} \arrow[r] & \mathbb{Q} \end{tikzcd}$$

$$(a,b) \longmapsto \begin{cases} \frac{a}{b} & \text{if } b \neq 0 \\ 0 & \text{if } b = 0 \end{cases}$$

SURJECTIVE



$\mathbb{Z} \times \mathbb{Z}$

Ex: $\left|\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right| = |\mathbb{R}|$

Fact: $|\mathbb{R}| > |\mathbb{N}|$

Last time: A function $f: A \to B$ gives rise to

$$\Gamma(f) := \{(a, f(a)) \mid a \in A\} \subseteq A \times B$$

Def: A function $f: A \to B$ is any subset $S_f \subseteq A \times B$ such that

$*$ For each $a \in A$, we have $(a,b) \in S_f$ for exactly one $b \in B$.

Abuse of notation: $S_f = f$

Def: Let $f: A \longrightarrow B$ be a 1-1 function with range $f(A)$. The inverse of f is the function $f^{-1}: f(A) \longrightarrow A$ defined by .

$$f^{-1} = \left\{ (y,x) \in f(A) \times A \;\middle|\; (x,y) \in \underset{\substack{\text{in} \\ A \times B}}{f} \right\}$$

i.e: $f^{-1}(y) \overset{\text{def}}{=}$ the unique $x \in A$ s.t. $f(x) = y$.

Rem: f^{-1} is actually a function

because if $(y, x_1) \in f^{-1}$ and $(y, x_2) \in f^{-1}$

then, by def $(x_1, y) \in f$
$(x_2, y) \in f$

So $f(x_1) = y = f(x_2)$

But f is 1-1, so $x_1 = x_2$.

* By construction, f^{-1} satisfies

$$\left.\begin{array}{l} f^{-1} \circ f = id_A \\ f \circ f^{-1} = id_{f(A)} \end{array}\right\}$$

for S = any set
$id_S : S \longrightarrow S$ is the
function $id_S(x) = x$

Theorem: Let $f: A \to B$ be a function.

Then $\exists\, g: f(A) \to A$ s.t

$$g \circ f = id_A$$

and $f \circ g = id_{f(A)}$

iff f is 1-1. In this case, $g = f^{-1}$.

pf: Exercise.

Warning: Given $f: A \to B$ and $g: f(A) \to A$ satisfying $g \circ f = id_A$

it <u>DOES NOT</u> follow that

$f \circ g = id_{f(A)}$.

Ex: $A = \{\text{integrable functions } \psi: [0,1] \to \mathbb{R}\}$

$B = \{\text{differentiable functions } \varphi: [0,1] \to \mathbb{R}\}$

$f: A \to B$ is $f(\psi) = \int_0^x \psi(t)\,dt$

$g: \underset{\overset{\|}{B}}{f(A)} \to A$ is $g(\varphi) = \frac{d}{dx}\varphi$

Notice $g \circ f(\psi) = \frac{d}{dx}\int_0^x \psi(t)\,dt \underset{FTC}{=} \psi(x)$

BUT

$$f \circ g(\varphi) = \int_0^x \frac{d}{dt}\varphi(t)\,dt$$

$$\varphi(t)\Big|_{t=0}^{t=x} = \varphi(x) - \varphi(0) \neq \varphi(x) \text{ if } \varphi(0) \neq 0.$$