<u>Definitions</u> are important?

1) Give meaning to symbols/words
2) Single out ideas/concepts
3) Distinguish an idea from related ones
4) Provide "foundational pillars" on which a theory can be built.

<u>Ex</u>: A positive integer is <u>prime</u> if it is not 1, and its only positive divisors are 1 and itself.

↳ singles out "prime" as worth studying.
↳ distinguishes primes from composites...
↳ gives way to check primality

What constitutes a good definition?

* As general as possible.
* Be specific.
* Uniquely pin down idea / concept.
* Should provide interesting/useful foundation.
* Be clear/concise.

Ex: The def of "prime" that I wrote down is good because of:

Thm: Every integer $n > 1$ can be decomposed uniquely as a product

$$n = p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}, \quad k \geq 1, \quad r_i \geq 1$$

$$p_1 < p_2 < p_3 < \cdots < p_k, \text{ primes.}$$

Rem: If 1 is prime, this thm is false: e.g. $6 = 2^1 \cdot 3^1 = 1^{27} \cdot 2^1 \cdot 3^1$
$= 1^2 \cdot 2^1 \cdot 3^1$

Semantics

Every definition is an "if and only if" statement.

$\Longrightarrow$ define term

$\Longleftarrow$ sufficient condition.

Ex A real number is rational iff it is the quotient of two integers w/ nonzero denominator.

meaning

- $x \in \mathbb{R}$ is rational $\Longrightarrow x = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$
- $x \in \mathbb{R}$, $x = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$ $\Longrightarrow x$ is rational

sufficiency

The word "is"

A) A prime is a pos. integer

true, but: -

B) A prime is an integer > 1
w/ no pos. divisors except 1
& itself.

def

Alternative Defs

* Give different/valuable approaches
* More useful to develop certain aspects of theory
* Allow important generalizations

Ex: Def: A positive integer $p > 1$ is prime iff

$$p \mid (ab) \Longrightarrow p \mid a \text{ or } p \mid b$$

for all $a, b \in \mathbb{Z}$.

Fact: This def is equivalent to our earlier one. However, the two defs are generally not equivalent for "number systems" other than $\mathbb{Z}$.

Ex: $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$

$$2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$$

Congruence

Def (Euclid): Two figures are congruent if they coincide, i.e. if one can be moved to the other by a "rigid motion" (preserves size/shape).

Pros: * easy to understand
* captures intuition

Cons: * mirror image
* Bad in non-Euclidean geom!

<u>Def</u>: • Two line segments are <u>congruent</u> if they have equal <u>length</u>.

• Two angles are <u>congruent</u> if they have equal measures

• Two triangles are <u>congruent</u> if there is a bij. correspondence b/w their vertices s.t. corresponding sides and angles are congruent.

<u>Pros</u>
* precise
* Works in different geometries

<u>Cons</u>
* very limited
* Loses intuition

Def: A plane transformation is a 1-1 function $T : \mathbb{R}^2 \to \mathbb{R}^2$.

Def: A congruence transformation (isometry) is a plane transformation T such that $d(T(P), T(Q)) = d(P, Q)$ $\forall\, P, Q \in \mathbb{R}^2$, $d(\cdot, \cdot)$ = distance function

Def: Two figures $A, B \subseteq \mathbb{R}^2$ are congruent if $\exists$ a congruence transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(A) = B$.