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10/31 Definitions

Defs are important because:

1) Gives meaning to term/phrase/symbol
2) Singles out and draws attention to idea/concept
3) Distinguishes an idea from related ones
4) Provides "foundation pillars" on which a theory can be built.

Ex: A positive integer is <u>prime</u> if it is not 1 and is divisible only by ± 1 and $\pm$ itself

↳ Singles out a fundamental concept
Distinguishes primes from composite numbers
Gives us concrete method for determining primality

What constitutes a <u>good</u> definition?

1) Accurately describes idea being defined
2) Includes only words/symbols which are
   - commonly understood
   - defined earlier
   - purposely left undefined
3) Includes no more information than necessary

* 4) Allows for a rich and interesting theory to be built upon it ("good pillar")

Ex: Our definition of <u>prime</u> is good, because it makes the following a true statement.

<u>Thm</u>: Every integer $n>1$ can be decomposed <u>uniquely</u> as a product $n = p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_k^{r_k}$

with $k \geq 1$, $p_1 < p_2 < \cdots < p_k$, primes and $r_i > 0 \ \forall i$

<u>Rem</u>: If we allowed 1 to be prime, then, for example $6 = 2^1 \cdot 3^1 = 1^2 \cdot 2^1 \cdot 3^1 = 1^{22} \cdot 2^1 \cdot 3^1$

and we lose uniqueness above.

* Making good defs is as important as it is difficult.

<u>Semantics</u>

<u>Every</u> definition is an "if and only if" statement,

$\Rightarrow$ * defines term

$\Leftarrow$ * gives sufficient condition

Ex: A real number is <u>rational</u> iff it is the quotient of two integers, with nonzero denominator.

meaning

$x \in \mathbb{R}$ is <u>rational</u> $\Longrightarrow$ $x = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$

$x \in \mathbb{R}$, $x = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0 \Longrightarrow$ x is rational

<u>sufficient</u> condition

<u>Rems</u>: • Often, we only say "if" in a definition, but we <u>always</u> mean <u>iff</u>.

• The word "is":

A) A prime is a positive integer

B) A prime is a positive integer, $\neq 1$, whose only positive divisors are 1 and itself

While A) is a true statement, it is <u>not</u> a definition (it is a necessary condition, but not a sufficient one).

B) is, of course, our definition

⟿ "is" is ambiguous, isn't it?

To address this issue, we <u>always</u> underline or italicize the term being defined, to signal that "is" is being used in a definitional, <u>iff</u> capacity.

Alternatives

Alternative defs ~~are~~ are often used because
1) Give different, useful approach to ideas
2) Are more convenient for developing certain aspects of the theory.
3) Allow generalizations which original def does not.

* Two defs are equivalent if they define same thing, exactly.

Ex: A positive integer $p>1$ is prime iff

$$p \mid (ab) \implies p \mid a \text{ or } p \mid b. \qquad (a, b \in \mathbb{Z}_{>0})$$

Fact: This def is equivalent to our previous one. However, the two defs are not equivalent for "number systems" more general than $\mathbb{Z}$ (e.g. $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

Our second def is the better one, because
a) it generalizes to a useful concept
b) it has wide theoretical applicability.

Extended example. Congruence

Def (Euclid): Two figures are congruent if they coincide, i.e. if one can be moved onto the other by a rigid motion (preserving size & shape)

Pros - Captures our intuition
- Widely applicable

Cons - Imprecise
- "bad" def in non-Euclidean geometry

Def (SMSG ~~[illegible]~~): • Two line segments are congruent iff they have same length

• Two angles are congruent iff they have same measure

• Two triangles are congruent iff ∃ bij correspondence b/w their vertices s.t. corresponding sides & angles are congruent

Pros - very precise

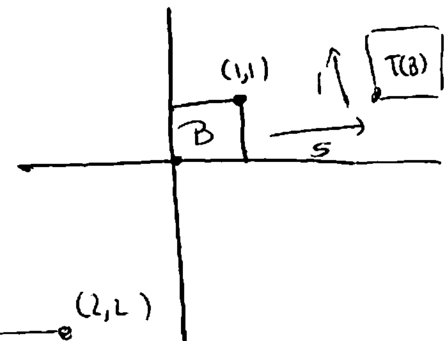
Cons - very specific
- loss of intuition

* ~~When~~ To capture Euclid's intuition ~~while~~ in a precise way, we need a new idea: distance

<u>Def</u>: A <u>plane transformation</u> is a 1-1 function $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

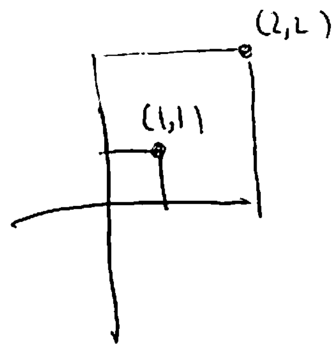
<u>Ex</u>: • $T(x,y) = (x+5, y+1)$

shifts right by 5 and up by 1



• $T(x,y) = (2x, 2y)$

is a "dilation" by 2



<u>Def</u> A <u>congruence transformation</u> or an <u>isometry</u> is a plane transformation T such that $d(P,Q) = d(T(P), T(Q))$ for <u>all</u> points $P, Q \in \mathbb{R}^2$. Here $d(\cdot,\cdot)$ is the Euclidean distance function.

<u>Def</u>: Two figures in the plane, A, B are <u>congruent</u> if $\exists$ a congruence transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(A) = T(B)$