1013
407 More on functions p85 #2 LL

Recall: $f: A \to B$ is 1-1 ("one to one") iff each $b \in B$ is equal to $f(a)$ for at most one value of a. Equiv: $f(a_1) = f(a_2) \implies a_1 = a_2$.

Ex: $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$, $x \mapsto x^2$ is not 1-1 because $f(-1) = f(1)$

$f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $x \mapsto x^2$ is 1-1: If $x_1^2 = x_2^2$ then $x_1 = \pm x_2$ and if $x_1, x_2 \geq 0$ then $x_1 = x_2$

$f: A \to B$ is surjective if each $b \in B$ is equal to $f(a)$ for at least one $a \in A$.

~~Eg~~: Ex: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto -x$ is surj, $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is not

$f: A \to B$ is bijective if it is surj + inj: $\forall b \in B$, $b = f(a)$ for exactly one $a \in A$.

Def: Let A, B be any two sets.

~~We say A and B have the cardinality~~

~~If A is less than B if $\exists f: A \to B$~~

~~fn~~

and let $|A|$, $|B|$ be the "cardinality" of A, B resp.

<u>We say</u>

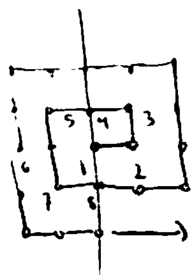
- $|A| \leq |B|$ if $\exists$ $f: A \to B$ ~~[illegible]~~ injective
- $|A| \geq |B|$ if $\exists$ $f: A \to B$ surjective.
- $|A| = |B|$ if $\exists$ $f: A \to B$ bijective.

  ($\Leftrightarrow$ $|A| \leq |B|$, and $|A| \geq |B|$)

<u>Fun examples</u>. 1) $|\mathbb{N}| = |\mathbb{Q}|$

<u>"pf"</u>: $|\mathbb{N}| \leq |\mathbb{Q}|$: $a \mapsto \frac{a}{1}$ is injective

$|\mathbb{N}| \geq |\mathbb{Q}|$.



etc.

$$\mathbb{N} \to \mathbb{Z}^2 \to \mathbb{Q}$$

$$(a,b) \mapsto \begin{cases} \frac{a}{b} & \text{if } b \neq 0 \\ 0 & \text{if } b = 0 \end{cases}$$

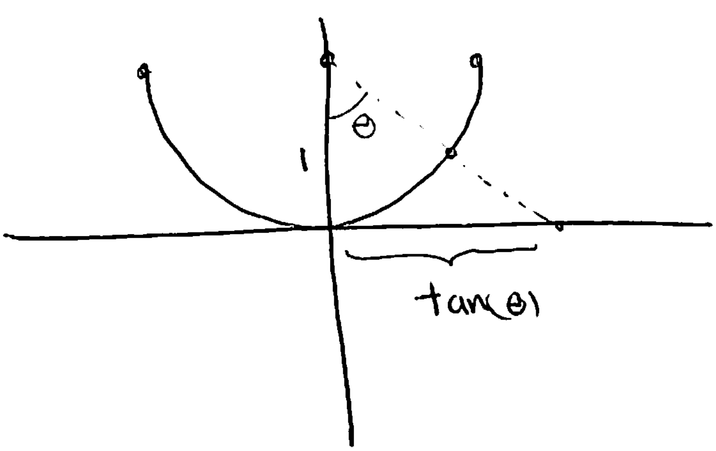
<u>Ex</u>. Construct a <u>bijective</u> fn $\mathbb{N} \to \mathbb{Q}$.

2) $|(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$

$f: (-\pi/2, \pi/2) \to \mathbb{R}$

$\theta \mapsto \tan(\theta)$

is 1-1, ~~bijective~~ surjective!



Last time: $f : A \to B$ may be thought of as

$$\begin{array}{l} \Gamma(f) \subseteq A \times B \\ \parallel \\ \{(a, f(a))\} \end{array}$$

Def: A **function** $f : A \to B$ is any subset S_f of $A \times B$ satisfying

* For each $a \in A$, we have $(a, b) \in S_f$ for at most one $b \in B$.

Equiv: If $(a, b_1), (a, b_2) \in S$ then $b_1 = b_2$.

N.B.: By "abuse of notation" we sometimes write f in place of S_f.

Def: Let $f : A \to B$ be a 1-1 function with range $f(A) \subseteq B$. The **inverse of f** is the function $f^{-1} : f(A) \to A$ defined by

$$f^{-1} = \{ (y, x) \in f(A) \times A \mid (x, y) \in f \subseteq A \times B \}$$

Equivalently: $f^{-1}(y) =$ the unique $x \in A$ satisfying $f(x) = y$.

Rem: f^{-1} is a function since if $(y, x_1), (y, x_2) \in f(A) \times A$

Then $(x_1, y), (x_2, y) \in f$ by def so $f(x_1) = y = f(x_2)$. Since f is 1-1, we conclude $x_1 = x_2$, and f^{-1} is a function.

* By construction, f^{-1} ~~is characterized by~~ satisfies

$$f \circ f^{-1} : f(A) \to f(A) \quad \underline{\text{is}} \quad id_{f(A)}$$
$$f^{-1} \circ f : A \to A \quad \underline{\text{is}} \quad id_A$$

These properties <u>characterize</u> f.

<u>Th</u>: Let $f: A \to B$ be a function. Then $\exists\ g: f(A) \to A$ s.t.

$$g \circ f = id_A \quad \text{and} \quad f \circ g = id_{f(A)}$$

iff f is 1-1. In this case, $g = f^{-1}$!

<u>pf</u>: ($\Rightarrow$) If f is 1-1, then we may take $g = f^{-1}$!

($\Leftarrow$) Spse $\exists$ g satisfying $g \circ f = id_A$

If $f(a_1) = f(a_2)$ then

$$a_1 = id_A(a_1) = g \circ f(a_1) = g \circ f(a_2) = id_A(a_2) = a_2$$

and f is 1-1. Hence f^{-1} exists and

$$\underset{\underset{id_A \circ g \,=\, g.}{\shortparallel}}{(f^{-1} \circ f) \circ g} = f^{-1} \circ id_{f(A)} = f^{-1}$$

Thus, $g = f^{-1}$ on $f(A)$.

Warning: Given $f: A \to B$ and $g: f(A) \to A$ satisfying $g \circ f = id_A$, it DOES NOT FOLLOW that $f \circ g = id_{f(A)}$

Ex: $A = \{\text{integrable functions } \psi : [0,1] \to \mathbb{R}\}$
$B = \{\text{differentiable functions } \varphi : [0,1] \to \mathbb{R}\}$

$$f: A \to B \qquad\qquad g: B \to A$$
$$\psi \mapsto \int_0^x \psi(t)\,dt \qquad\qquad \varphi \mapsto \frac{d}{dx}\varphi$$

FTC: $g \circ f(\psi) = \frac{d}{dx}\int_0^x \psi(t)\,dt = \psi(x) = id_A$

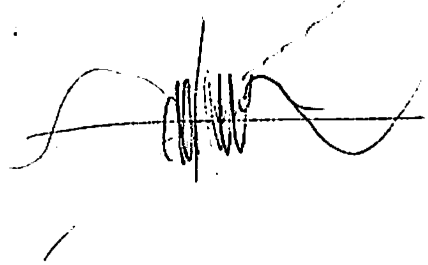
But $f \circ g(\varphi) = \int_0^x \frac{d}{dx}\varphi\, dx = \varphi(x) - \varphi(0)$
$\neq \varphi(x)$ if $\varphi(0) \neq 0$.

(However, can ignore warning provided A, B satisfy appropriate "finiteness" conditions... e.g. if A, B are finite sets)

<u>Moral</u>: - Functions are everywhere!

<u>Ex</u>: A "binary operation on a set S" is just a function $f: S \times S \longrightarrow S$

<u>Q</u>. Given $f: \mathbb{R} \longrightarrow \mathbb{R}$ continuous, (smooth...) does there exist an interval $[a,b] \subseteq \mathbb{R}$ on which f is 1-1?

<u>Ans</u>: <u>No</u>. <u>Ex</u>: $x \sin\left(\frac{1}{x}\right)$ 

$\frac{\sin x}{x} \to 1$ ✓ as $x \to 0$