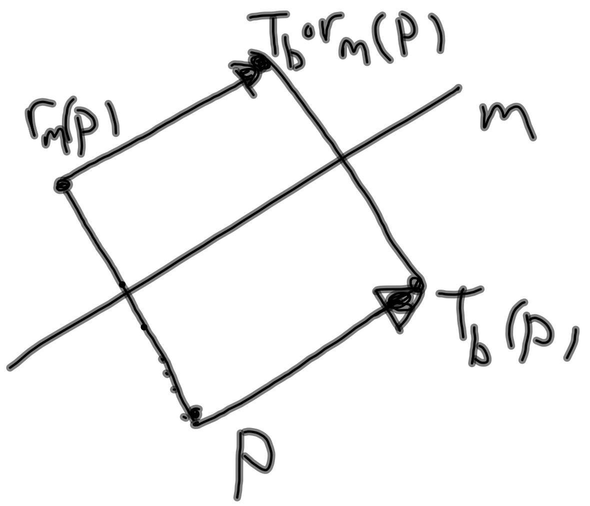
Problems: p328, #2.

m = line in plane, b = vector w/ same dir. as m.

Lemma: $r_m \circ T_b = T_b \circ r_m =: g_{m,b}$

This is called a glide reflection

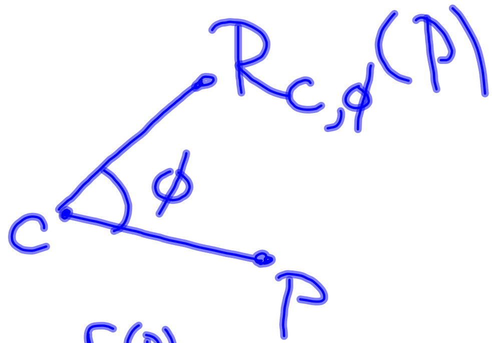
Pf.:



<u>Recap:</u> 1) Translations

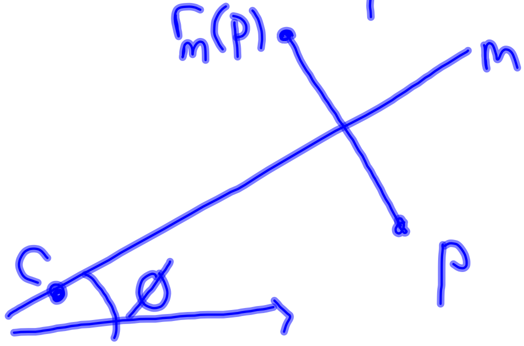
$$T_b(z) = z + b$$

2) Rotations:



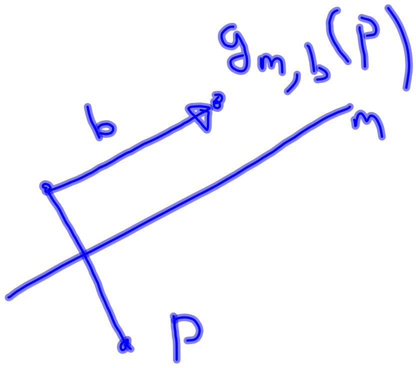
$$R_{c,\phi}(z) = z_\phi(z-c)+c$$

3) Reflections



$$r_m(z) = z_{2\phi}\overline{(z-c)} + c$$

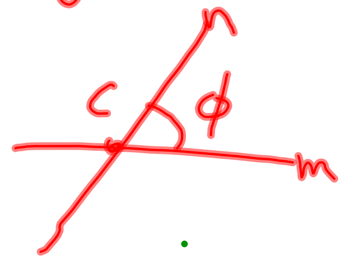
4) Glide reflections



Thm: Let m, n be two lines

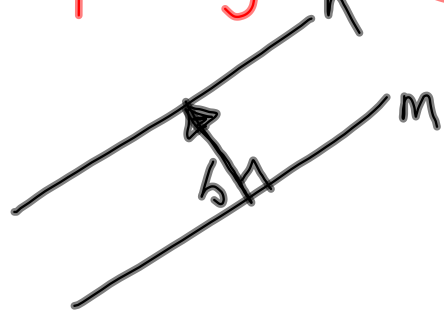
i) If m, n intersect at C w/ angle ϕ

then $r_n \circ r_m = R_{C, 2\phi}$



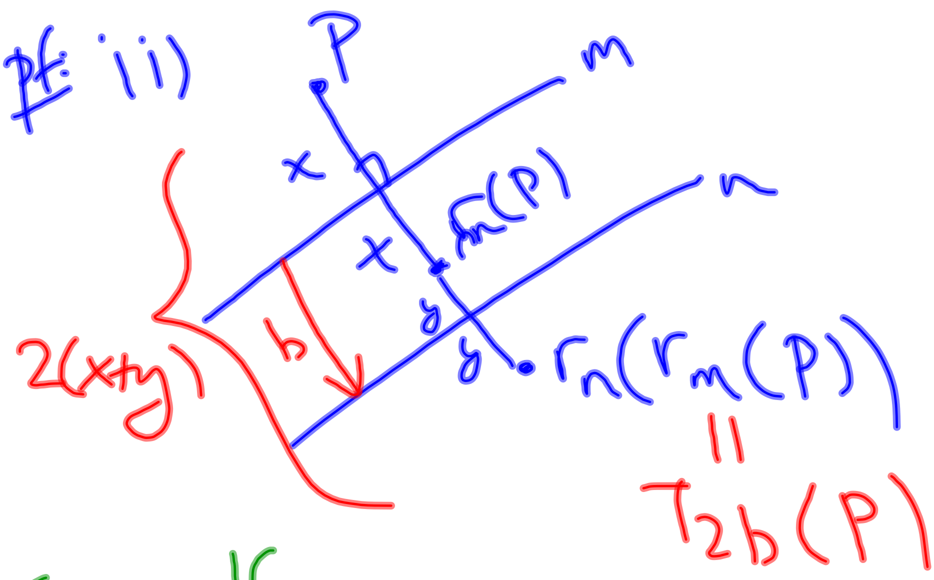
ii) If m, n are $\parallel$, and

b = vector w/ direction is $\perp$ to m, n, from m to n & w/ magnitude = dist. b/w m, n.



Then $r_n \circ r_m = T_{2b}$

Pf: ii)



<u>Cor:</u> • If m, n intersect at C w/ angle ϕ and m', n' are rotations of m, n about C thru an angle of θ, then

$$r_{n'} \circ r_{m'} = r_n \circ r_m$$

• If m, n are $\parallel$, with vector b and m', n' are translations of m, n by a vector v, then

$$r_{n'} \circ r_{m'} = r_n \circ r_m .$$

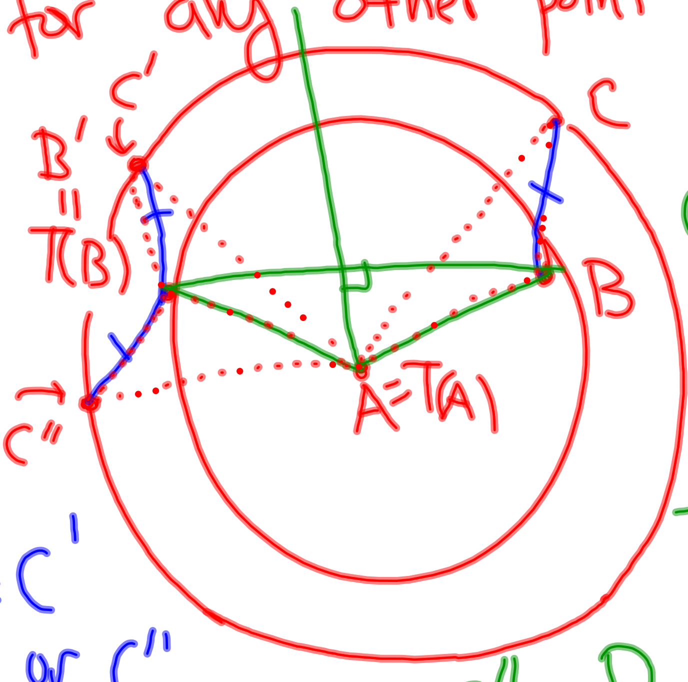
<u>Thm:</u> Every congruence transformation T is a composite of reflections.

<u>Pf:</u> <u>case 1:</u> T has a fixed point: $T(A)=A$

Then T is a rotation or a reflection.

Let B be any pt. with $T(B)\neq B$

Then for any other point C:



But, $C' = r_\ell(C)$

$\underbrace{\ell = \perp \text{ bisector of } BB'}_{\text{does not depend on } C\text{!}}$

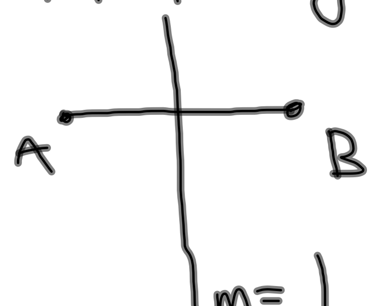
So $T(C) = C'$ or C''

$C'' = R_{A,\phi}(C)$

$\underbrace{\phi = m\angle BAB'}$ $\rightarrow$ indep. of C!

In general, T may not have a fixed point, so let A be any point

$T(A) = B \neq A$



$m = \perp$ bisector of AB

Consider $r_m \circ T$.

Then
$$r_m \circ T(A) = r_m(B) = A$$

So $r_m \circ T$ has a fixed point, so

$r_m \circ T =$ reflection
rotation = composition of two reflections

So $T = r_m \circ \underbrace{r_m \circ T}$ → composition of refls.

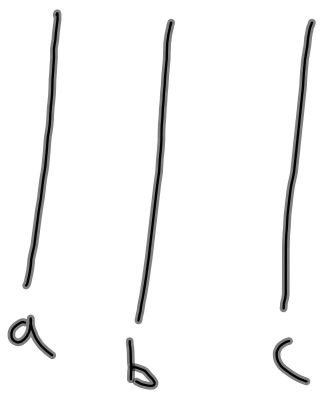
Thm: Every congruence transformation is either a translation, reflection, rotation, or glide reflection.

pf: Just study compositions of refl's (by prev. Thm).

- 1-refl: reflection
- 2-refl: translation or rotation
- 3-refl?

$r_c \circ r_b \circ r_a$

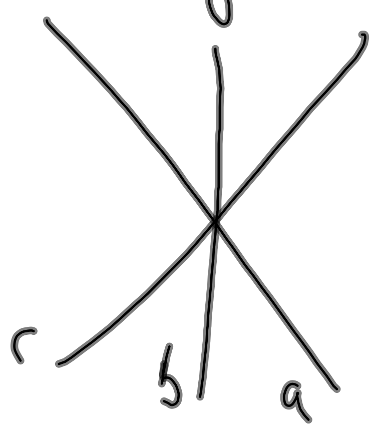
<u>case i</u>: a, b, c intersect at 0 points:



<u>translate</u> a, b to a', b', with $b' = c$.

$$\begin{aligned} r_c \circ r_b \circ r_a &= r_c \circ r_{b'} \circ r_{a'} \\ &= r_c \circ r_c \circ r_{a'} \\ &= r_{a'} \end{aligned}$$

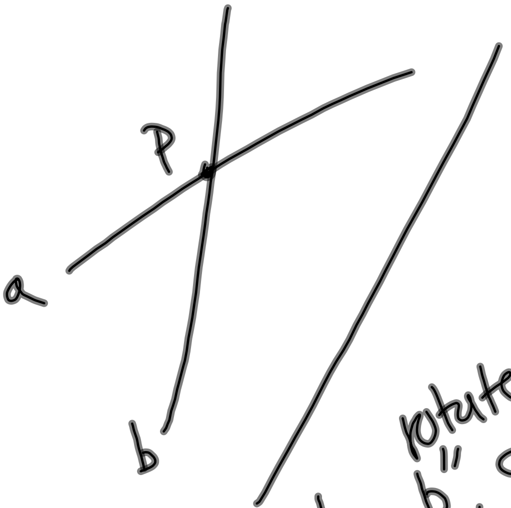
<u>case ii</u> a, b, c intersect at exactly 1 point



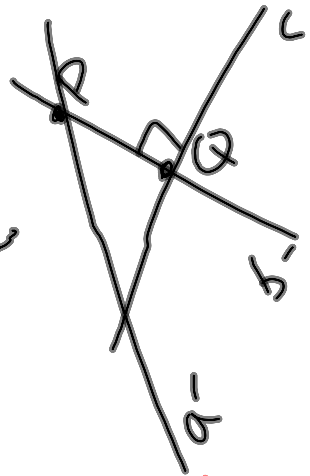
<u>rotate</u> a, b to a', b' with $b' = c$

$$\begin{aligned} r_c \circ r_b \circ r_a &= r_c \circ r_{b'} \circ r_{a'} \\ &= r_c \circ r_c \circ r_{a'} = r_{a'} \end{aligned}$$

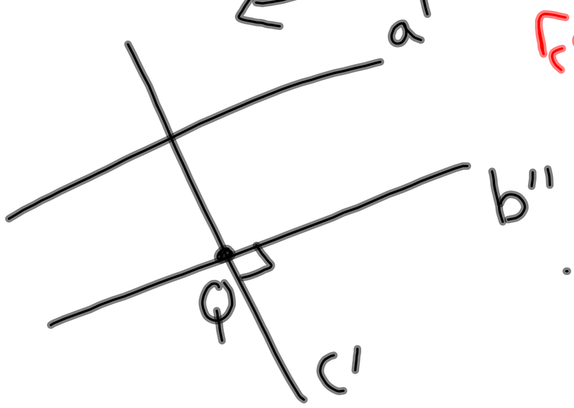
<u>case iii)</u> 2 or more intersection pts.



rotate a,b to a', b' with b' ⊥ c



rotate b', c to b'', c' with b'' || a'

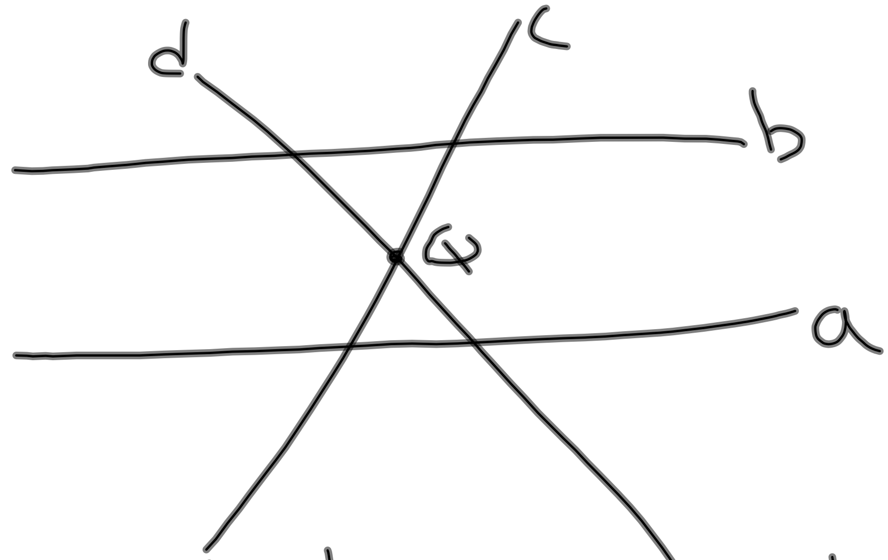


$$r_c \circ r_b \circ r_a = r_c \circ (r_{b'} \circ r_{a'})$$
$$= (r_c \circ r_{b'}) \circ r_{a'}$$
$$= r_{c'} \circ r_{b''} \circ r_{a'}$$
$$= r_{c'} \circ \text{Translation in direction of } c'$$
$$= \text{glide translation.}$$

- four reflections

$$\underbrace{r_d \circ r_c}\circ \underbrace{r_b \circ r_a}$$

- If both these are translations, then we get a translation.
- If these are rotation, translation



rotate c, d about Q to get c', d' with c' ∥ b

$$r_d \circ r_c \circ r_b \circ r_a = r_{d'} \circ \underbrace{r_{c'} \circ r_b \circ r_a}_{3\ \parallel\ \text{lines}} = r_{d'} \circ r_{a'}$$

Lemma: Any composite of 4 refls. is = to a composite of two refls.