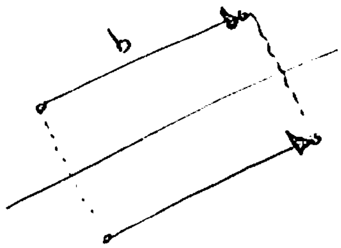
Problems. p328, #2

m = line in plane
b = vector parallel to m

Lemma $T_b \circ r_m = r_m \circ T_b$

pf:



Def: Any congruence transformation of the form

$$g_{m,b} := T_b \circ r_m = r_m \circ T_b \quad \text{for } b, m \text{ as above}$$

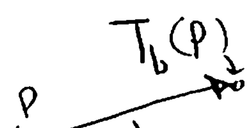
is called a glide reflection

In complex picture, if m contains c & has angle ϕ w/ x-axis,

$$g_{m,b}(z) = T_b(r_m(z)) = z_{2\phi}\,\overline{(z-c)} + c + b.$$

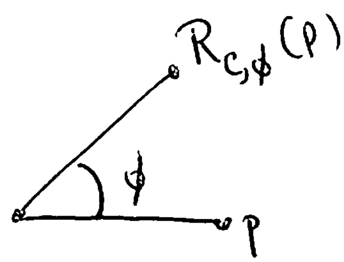
Recap of congruence transformations

1) Translations:



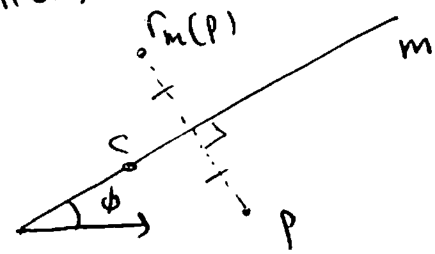
, $T_b(x,y) = (x+h, y+k)$ if $b = (h,k)$
$T_b(z) = z + b$

2) Rotations



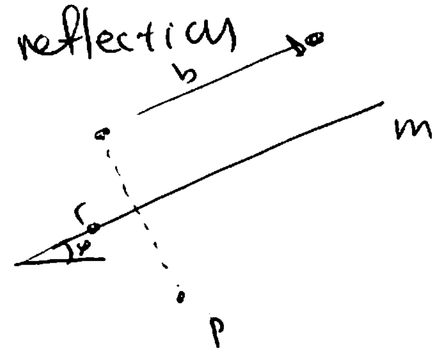
$$R_{c,\phi}(z) = z_\phi(z-c) + c$$

3) Reflections



$r_m(z) = z_{2\phi}\overline{(z-c)} + c$

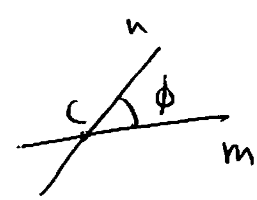
4) Glide reflections



$g_{m,b}(z) = z_{2\phi}\overline{(z-c)} + c + b$

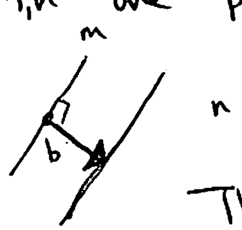
Thm. Let m, n be two lines.

i) m, n intersect at c and make angle ϕ.



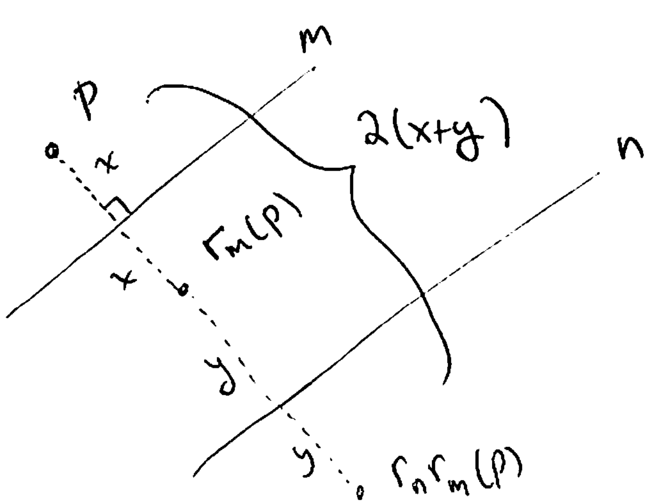
$r_n \circ r_m = R_{c,2\phi}$

ii) m, n are parallel. Let b = vector $\perp$ to m, n, in direction from m to n w/ magnitude the dist. b/w m, n



Then $r_n \circ r_m = T_{2b}$

pf



Cor: i) If m, n intesect @ C w/ angle ϕ, and m', n' are rotations of m, n by any angle θ, then

$$r_n \circ r_m = r_{n'} \circ r_{m'}$$

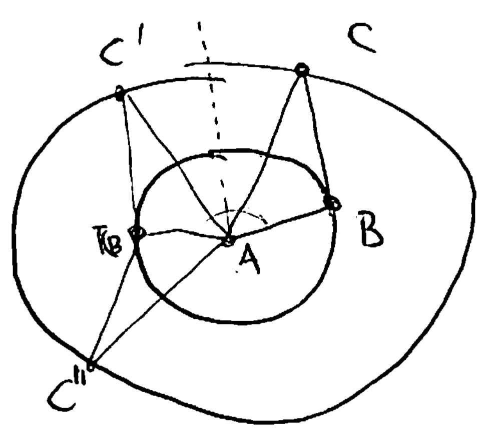
ii) If m, n are $\parallel$ w/ corr. vector b, and m', n' are translations of m, n by any vector v, then

$$r_n \circ r_m = r_{n'} \circ r_{m'}$$

pf: These composites depend only on C, ϕ in case i) and on b in case ii).

Lemma: A congruence transformation with a fixed point is either a reflection or a rotation.

pf: Let T be such a transformation & spse $T \neq id$, and $T(A) = A$. Let B be any point with $T(B) \neq B$. Then for any point $C \neq A, B$ we have



so $T(C) = \begin{cases} C' \\ C'' \end{cases}$

as T preserves distance.

In first case,

$T(C) = C' = \underbrace{\text{reflection of } C \text{ across } \perp\text{-bisector of } \overline{BT(B)}}_{\text{doesn't depend on } C!}$.

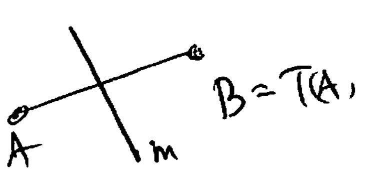
In second case,

$T(C) = C'' = $ rotation of C about A through an angle of $\underbrace{m\angle BAT(B)}_{\text{doesn't depend on } C!}$

∎

<u>Thm</u>: Every congruence transformation is a composite of reflections.

<u>pf</u>: Let T be a congruence transformation. If T has a fixed point, done, as every rotation is a composite of reflections. Else $T(A) = B$, $B \neq A$ for some A. Let m be the $\perp$ bisector of AB



so $r_m \circ T\,(A) = A$.

Hence, $r_m \circ T$ is a composite of refl's, whence so is $T = r_m \circ (r_m \circ T)$. ∎

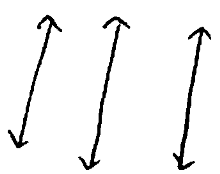
Thm Every congruence transformation is either a translation, a rotation, a reflection, or a glide reflection

pf. We analyze composites of reflections.

- one reflection ✓
- two reflections $\to$ translation, $\to$ rotation
- three reflections

$T = r_c \circ r_b \circ r_a$,

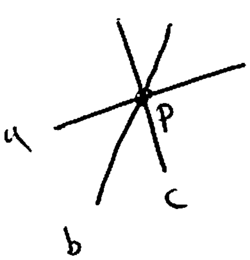
case 1 a, b, c have no intersection



translate a, b so that $c = b'$.

$$r_c \circ (r_b \circ r_a) = r_c \circ (r_{b'} \circ r_{a'}) = r_c \circ r_c \circ r_{a'}$$
$$= r_{a'}$$

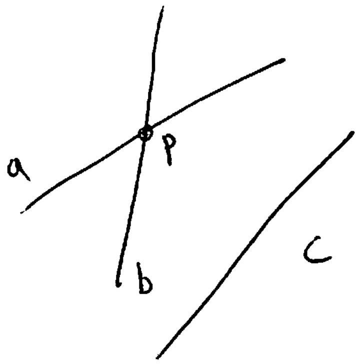
case 2 a, b, c have one point of intersection



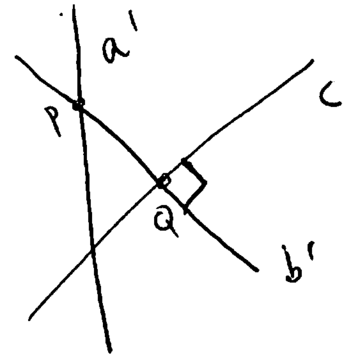
rotate a, b about P so $c = b'$

$$r_c \circ (r_b \circ r_a) = r_c \circ (r_{b'} \circ r_{a'}) = r_c \circ r_c \circ r_{a'} = r_{a'}.$$

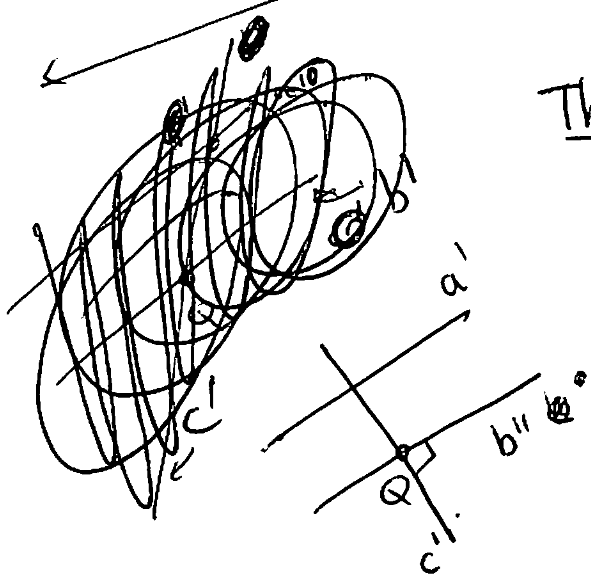
case 3 a, b, c have ≥ 2 points of intersection



rotate a, b about P so $b' \perp c$



rotate b', c around Q so $b'' \parallel a'$



<u>Then</u>: $r_c \circ r_b \circ r_a = r_c \circ (r_{b'} \circ r_{a'})$

$= (r_c \circ r_{b'}) \circ r_{a'}$

$= (r_{c'} \circ r_{b''}) \circ r_{a'}$

$= r_{c'} \circ (r_{b''} \circ r_{a'})$

$= r_{c'} \circ (\text{Translation in dir of } c')$

= glide reflection.

- <u>four reflections</u>

<u>Lemma</u>: Any composite of four reflections is also a composite of <u>two</u> reflections

pf.