Problems: p 485 # 11

Area of polygonal regions.

Def: Let $F \subseteq \mathbb{R}^2$ be a finite union of triangles (e.g. polygons)

An area function α assigns a positive real # $\alpha(F)$ to each such F such that:

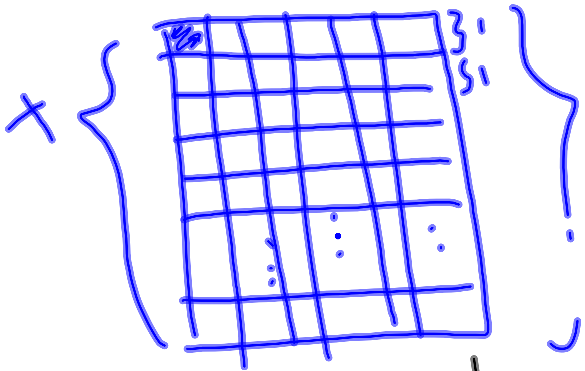
1. $\alpha(F_1) = \alpha(F_2)$ if $F_1 \cong F_2$
2. If $F_1 \cap F_2$ has empty interior, then $\alpha(F_1 \cup F_2) = \alpha(F_1) + \alpha(F_2)$ $\forall F_1, F_2$
3. $\alpha(S) = 1$ for S = square w/ side length 1.

not strong enough to handle squares w/ irrational base.

3.' $\alpha(Q) = x^2$ if Q is a square w/ side of length x.

Thm: A square Q of side length x has area x^2.

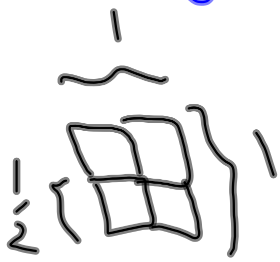
Pf: If x is an integer:



x^2 total 1 by 1 squares, so

$$\alpha(Q) = x^2 \cdot \alpha(S) = x^2.$$

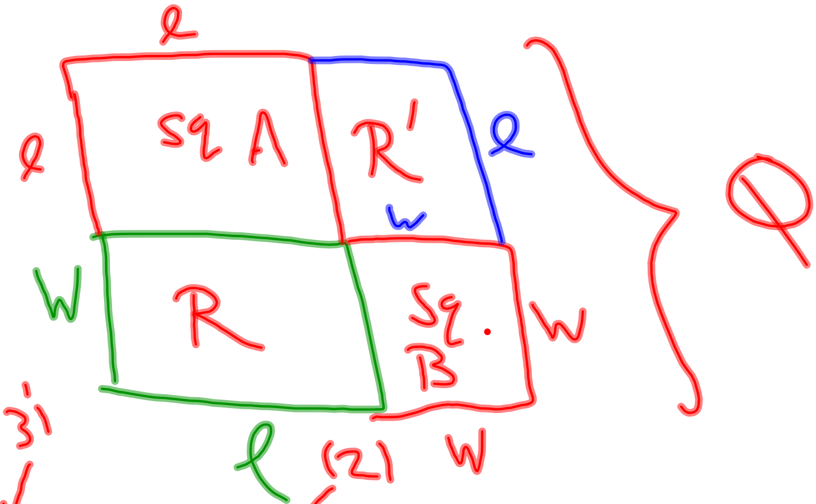
If $x = \frac{1}{2}$



$$4\alpha(S) = 1$$

Thm: Let R be a rectangle w/ sides of lengths ℓ & w. Then $\alpha(R) = \ell \cdot w$.

Pf.



$$(\ell+w)^2 \overset{(3')}{=} \alpha(Q) \overset{(2)}{=} \alpha(R)+\alpha(R')+\alpha(A)+\alpha(B)$$

$$\overset{(3')}{=} \alpha(R)+\alpha(R')+\ell^2+w^2$$

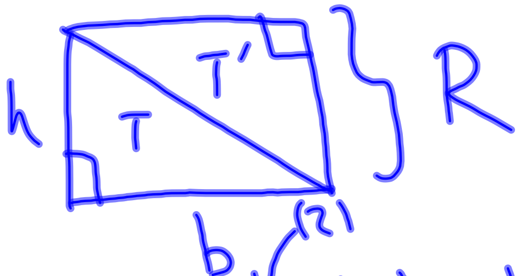
$$\overset{(1)}{=} 2\alpha(R)+\ell^2+w^2$$

$$(\ell+w)^2 = \ell^2+w^2+2\ell w$$

$$\Rightarrow \ell w = \alpha(R) \checkmark$$

Cor: Let T be a right triangle w/ legs of length h and b. Then $\alpha(T) = \frac{1}{2}bh$.

Pf:



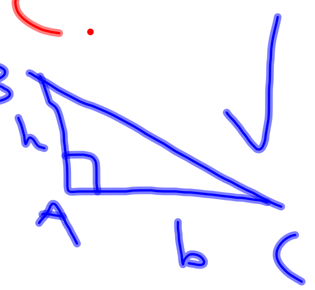
$$bh \overset{\text{thm}}{=} \alpha(R) \overset{(2)}{=} \alpha(T) + \alpha(T') \overset{(1)}{=} 2\alpha(T)$$

$$\Rightarrow \frac{1}{2}bh = \alpha(T).$$

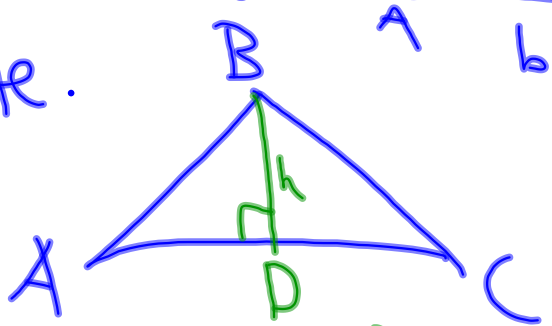
Thm: Let T be *any* triangle w/ base b & ht. h. Then $\alpha(T) = \frac{1}{2}bh$.

Pf: T has vertices A, B, C.

i) $\angle BAC$ is a right angle



ii) $\angle BAC$ is acute.



$$\alpha(T) \overset{(2)}{=} \alpha(\Delta BAD) + \alpha(\Delta BDC)$$
$$= \frac{1}{2}h(AD) + \frac{1}{2}h(DC)$$
$$= \frac{1}{2}h(AD + DC) = \frac{1}{2}hb.$$

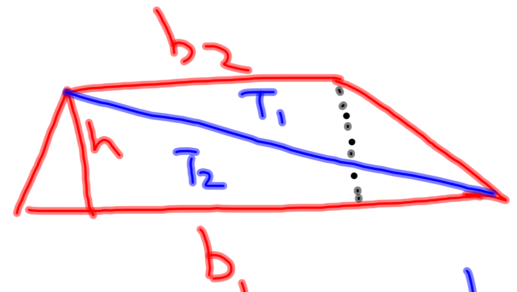
iii) $\angle BAC$ is obtuse.

$$\alpha(\Delta BDC) = \alpha(\Delta BDA) + \alpha(T)$$
$$\alpha(\Delta BDC) = \frac{1}{2}h(DA + AC)$$
$$\alpha(\Delta BDA) = \frac{1}{2}h(DA)$$

$$\Rightarrow \alpha(T) = \frac{1}{2}hb. \qquad \blacksquare$$

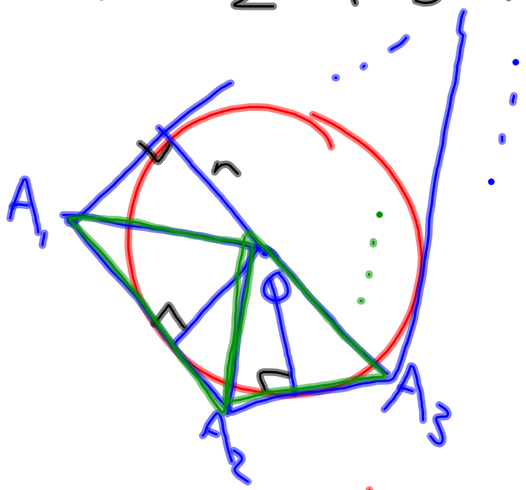
Thm: Let Z be a trapezoid with ht h & parallel sides b_1, b_2. Then $\alpha(Z) = \frac{1}{2}h(b_1+b_2)$

pf:



$$\alpha(Z) = \alpha(T_1) + \alpha(T_2) = \frac{1}{2}b_2h + \frac{1}{2}b_1h = \frac{1}{2}h(b_1+b_2)$$

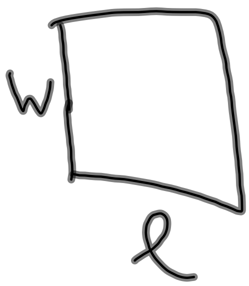
Thm: Let P be a polygon circumscribing a circle of radius r. Then $\alpha(P) = \frac{1}{2} r p$, p = perimeter of P.



$$\alpha(P) = \alpha(\Delta O A_1 A_2) + \alpha(\Delta O A_2 A_3) + \cdots$$
$$= \frac{1}{2} r (A_1A_2) + \frac{1}{2} r (A_2 A_3) + \cdots$$
$$= \frac{1}{2} r (A_1A_2 + A_2A_3 + A_3A_4 + \cdots) = \frac{1}{2} r p.$$

Thm: Of all rectangles w/ a given perimeter p, the square has largest area.

Pf:



$$p = 2(\ell + w)$$

$$\ell + w = \frac{p}{2}$$

Let $\ell = \frac{p}{4} + x$, then $w = \frac{p}{4} - x$

Then $$\ell w = \left(\frac{p}{4} + x\right)\left(\frac{p}{4} - x\right)$$

$$= \underbrace{\frac{p^2}{16} - x^2}_{\text{largest when } x = 0} \qquad \le 0$$

i.e. when $\ell = w = \frac{p}{4}$.

Q: Of all triangles w/ a given perimeter, which one(s) has maximum area?

Q: Of all n-gons w/ fixed perimeter, does the regular one always have max area?