11/16
407

Problems: p 485 #11.

Area of polygonal regions

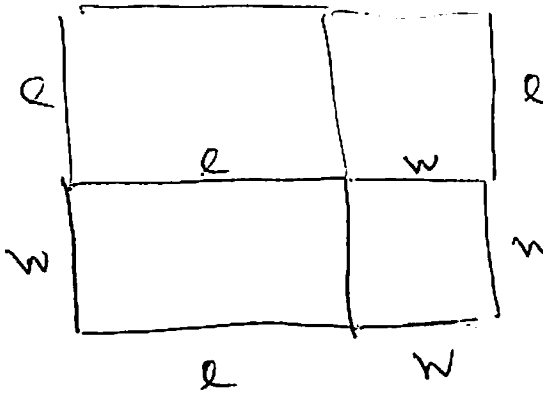
Def: Let $F \subseteq \mathbb{R}^2$ be a finite union of triangular regions (i.e. any polygon).
An area function α assigns to each such F a positive real number $\alpha(F)$ satisfying

1. If $F_1 \cong F_2$ then $\alpha(F_1) = \alpha(F_2)$
2. If $F_1 \cap F_2$ has empty interior then
   $\alpha(F_1 \cup F_2) = \alpha(F_1) + \alpha(F_2)$
3. If F is a square with side 1, then
   $\alpha(F) = 1$

* We'll develop the theory of area from these axioms.

Thm: A square of side x has area x^2.

Thm: Let R be a rectangle with side lengths l and w. Then $\alpha(R) = l \cdot w$.

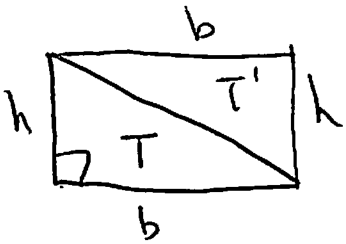


$$(l+w)^2 = l^2 + w^2 + 2(\alpha(R))$$

$$\Rightarrow \alpha(R) = lw. \quad \square$$

<u>Cor</u>: Let T be a right triangle, with legs of lengths b, h. Then $\alpha(T) = \frac{1}{2}bh$.

pf:



$$bh = \alpha(T \cup T') = \alpha(T) + \alpha(T') = 2\alpha(T)$$

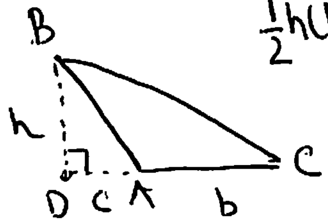
so $\quad \alpha(T) = \frac{1}{2}bh.$

<u>Thm</u>: ~~The~~ Let T be <u>any</u> triangle w/ base b and height h. Then $\alpha(T) = \frac{1}{2}bh$.

<u>pf</u>: A, B, C are the vertices.

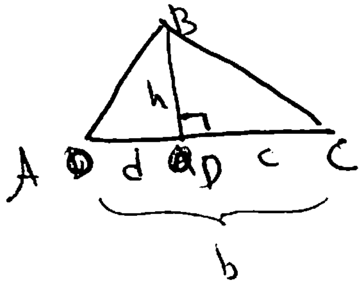
3 cases: i) $\angle BAC$ is a right angle
ii) $\angle BAC$ is obtuse
iii) $\angle BAC$ is acute.

i) ✓ $\qquad$ ii)



$$\frac{1}{2}h(b+c) = \alpha(\triangle BDC) = \alpha(\triangle BDA) + \alpha(\triangle BAC) = \frac{1}{2}hc + \alpha(\triangle BAC)$$

$$\Longrightarrow \alpha(\triangle BAC) = \frac{1}{2}hb. \quad \square$$

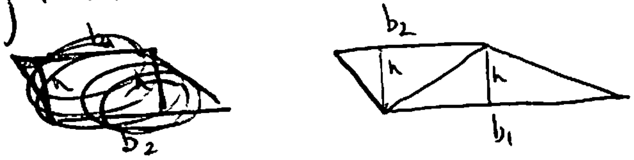
iii) 

$\alpha(\triangle BAC) = \alpha(\triangle BAD) + \alpha(BDC)$

$= \frac{1}{2}hd + \frac{1}{2}hc$

$= \frac{1}{2}h(d+c) = \frac{1}{2}hb.$

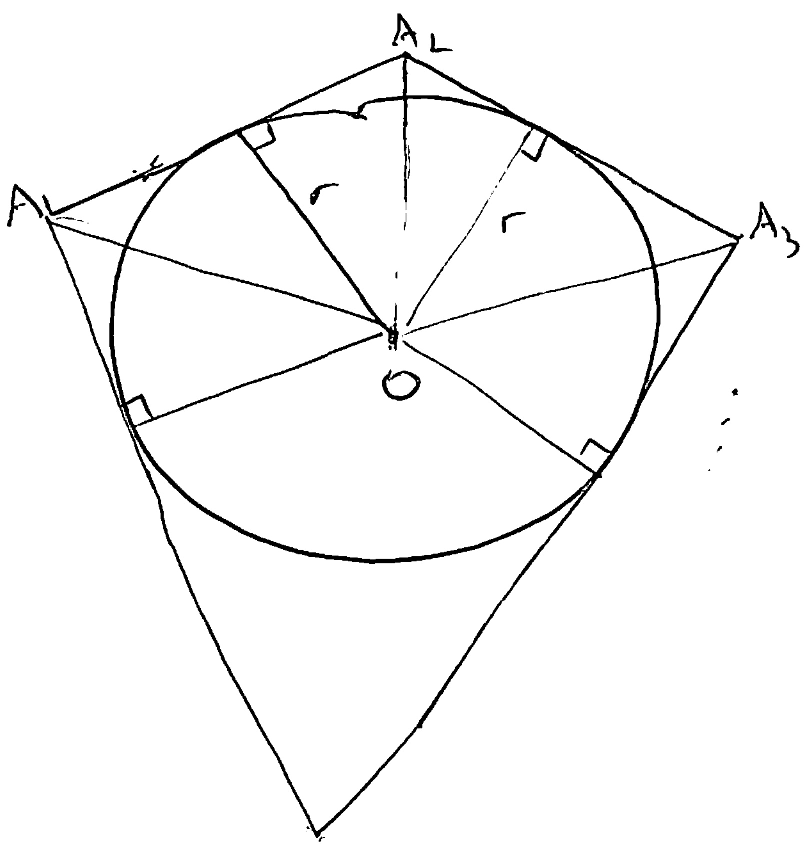
Cor: ~~There~~ Let Z be a trapezoid with bases b_1, b_2, ht. h. Then $\alpha(Z) = \frac{1}{2}h(b_1+b_2)$

Pf: 

. . . . . .

Thm: Let P be a polygon with perimeter p ~~inscribed in~~ circumscribing a circle of radius r.

Then $\alpha(P) = \frac{1}{2}rp$



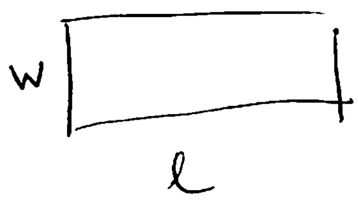
$\alpha(P) =$

$\frac{1}{2}rA_1A_2 + \frac{1}{2}rA_2A_3 + \cdots$

$= \frac{1}{2}rp.$

<u>Thm</u> of all rectangles with a given perimeter, the square has largest area.

<u>pf</u>:



$$P = 2(l+w) \implies l+w = \frac{P}{2} \text{ so}$$
$$A = lw \qquad 0 \le l \le P/2$$

So write $l = \frac{P}{4} + x$, $w = \frac{P}{4} - x$

$lw = \left(\frac{P}{4}+x\right)\left(\frac{P}{4}-x\right) = \frac{P^2}{16} - x^2$ is maximized

when $x=0$ and $l=w$. $\blacksquare$

<u>Thm</u> of all triangles with a given perimeter, the equilateral has greatest area.