Problems: Read synthetic derivation of Hero's formula (p.486-489)

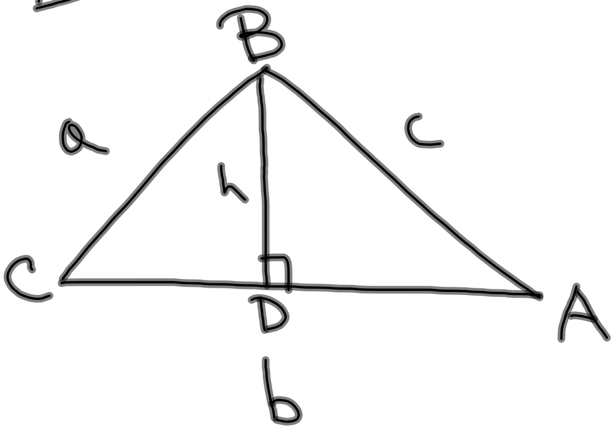
SSS

Thm: (Hero(n)'s formula): Let T be a triangle, with side lengths a, b, c, then

$$\alpha(T) = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{1}{2}p = \frac{a+b+c}{2}$$

pf 1 (standard) — Law of cosines



$$a^2 = b^2 + c^2 - 2bc\cos(A)$$
$$= b^2 + c^2 - 2b\cdot AD$$

So $AD = \dfrac{b^2+c^2-a^2}{2b}$

$$h^2 = c^2 - (AD)^2 = c^2 - \frac{(b^2+c^2-a^2)^2}{4b^2}$$

diff of 2-□'s $= \dfrac{4b^2c^2 - (b^2+c^2-a^2)^2}{4b^2}$

$$= \frac{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}{4b^2}$$

$$= \left((b+c)^2 - a^2\right)\left(a^2-(b-c)^2\right)/4b^2$$

$$= \frac{1}{4b^2}(b+c+a)(b+c-a)(a-b+c)(a+b-c)$$

$(s = a+b+c)$

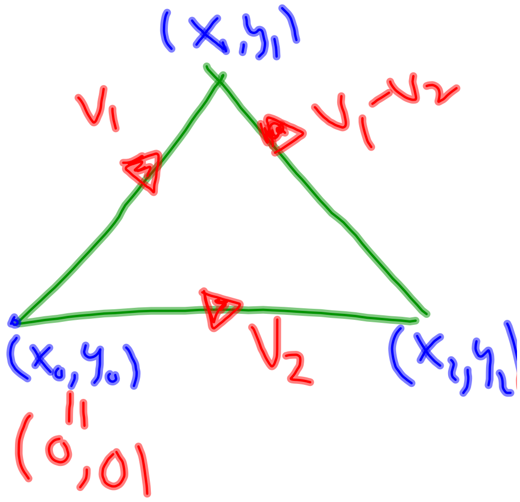
So

$$a(T)^2 = \frac{h^2b^2}{4} = \frac{1}{4^2}(2s)(2s-2a)(2s-2b)(2s-2c)$$

Pf 2: (my favorite)

Let T be a triangle with vertices $(x_0,y_0),(x_1,y_1),(x_2,y_2)$

Can assume $(x_0,y_0)=(0,0)$



$$\alpha(T)=\frac{1}{2}\det\underbrace{\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}}_{A}$$

$$\alpha(T)^2=\frac{1}{4}\det(A^tA)=\frac{1}{4}\det\begin{pmatrix} v_1\cdot v_1 & v_1\cdot v_2 \\ v_2\cdot v_1 & v_2\cdot v_2\end{pmatrix}$$

Since $v_i\cdot v_j=\frac{1}{2}\left(|v_i|^2+|v_j|^2-|v_i-v_j|^2\right)$

degree 2 homogeneous poly in side lengths a,b,c

$\Rightarrow$ $\alpha(T)^2$ is a degree 4 homogeneous polynomial in a,b,c.

Define $f(a,b,c) = \alpha(T)^2$

$= $ homog. degree 4 poly in a,b,c.

$\ast$) $f(a,b,c)$ is <u>symmetric</u> in a,b,c.

$\ast$) $a+b \geq c$, and if $a+b=c$ then $f(a,b,c)=0$

i.e. $a+b-c$ is a factor of $f(a,b,c)$

By symmetry,

$$f(a,b,c) = (a+b-c)(a+c-b)(b+c-a)\,h(a,b,c)$$

$h(a,b,c)$ is symmetric, homogeneous, degree 1

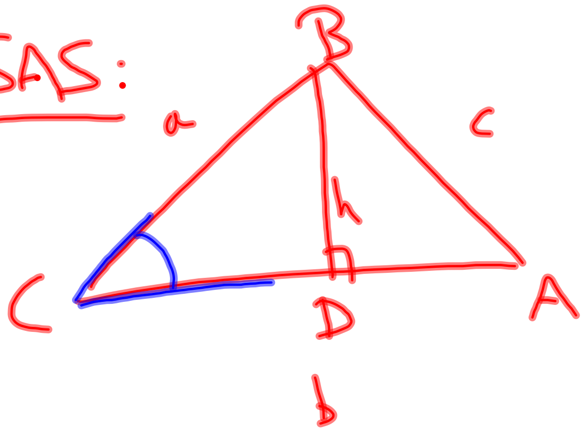
$$\Longrightarrow h(a,b,c) = (a+b+c)k$$

k ← constant indep of a,b,c.

$$\alpha(T)^2 = k\,2s(2s-2a)(2s-2b)(2s-2c)$$

Now use the fact that $\alpha(T)=6$ for T the $(3,4,5)$ right triangle to get $k=\frac{1}{16}$.

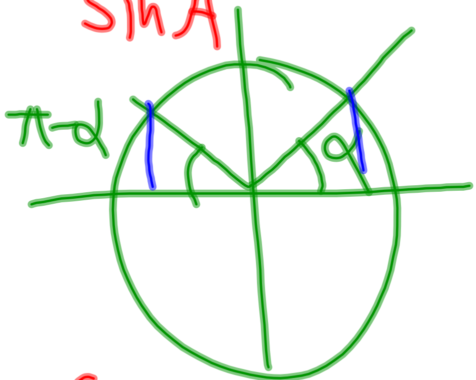
SAS:



$$\alpha(T) = \frac{1}{2} b h = \frac{1}{2} b (a \sin C) = \frac{1}{2} ab \sin C$$

ASA: $\alpha(T) = \frac{1}{2} ac \sin B$

$$= \frac{1}{2} ac \sin B \left(\frac{\frac{1}{2} ab \sin C}{\frac{1}{2} bc \sin A} \right) = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin A}$$



Since $A = \pi - (B + C)$

$\sin(A) = \sin(B + C)$

$$\alpha(T) = \frac{1}{2} a^2 \frac{\sin B \cdot \sin C}{\sin(B+C)}$$

Thm: Among all triangles w/ a fixed perimeter, the equilateral has largest area.

pf. Want to maximize

$$f(a,b,c) = s(s-a)(s-b)(s-c)$$

Subject to $a+b+c=2s=$ constant.

$$c = 2s-a-b$$
$$s-c = a+b-s$$

$$g(a,b) = s(s-a)(s-b)(a+b-s)$$

$$\left(\frac{\partial g}{\partial a}, \frac{\partial g}{\partial b}\right) = \left(s(s-b)(2s-2a-b), s(s-a)(2s-2b-a)\right) \stackrel{?}{=} (0,0)$$

$$s(s-b)(-1(a+b-s)+(s-a))$$
$$= s(s-b)(2s-2a-b)$$

iff

$$2s = 2a+b$$
$$2s = 2b+a$$
$$\Rightarrow a = b.$$

So $3b = 2s =$ perimeter

$$= a+b+c$$
$$= 2b+c$$

$$\Rightarrow c = b = a.$$