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Problems: Read Synth. derivation of Hero's formula in book (486-489)

~~Thm: Let T be a triangle with side~~

By SSS (resp SAS resp ASA) congruence, any two triangles with 3 equal sides (resp 2 equal sides & equal included angle resp 2 equal angles & equal included side) have the same area, so:

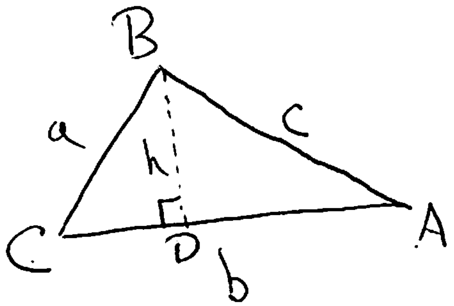
* Should be formula for area in terms of 3 sides (resp 2 sides & incl. angle resp 2 angles & incl side)

SSS:

Thm "Hero's formula": Let T be a triangle with side lens a, b, c and semiperimeter $s := \frac{p}{2} = \frac{a+b+c}{2}$. Then

$$\alpha(T) = \sqrt{s(s-a)(s-b)(s-c)}$$

pf 1 (analytic) By law of cosines,



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= b^2 + c^2 - 2b \cdot AD$$

So $\quad AD = \dfrac{b^2+c^2-a^2}{2b}$

and $\quad h^2 = c^2 - AD^2 = c^2 - \dfrac{(b^2+c^2-a^2)^2}{4b^2}$

$$= \frac{4b^2c^2 - (b^2+c^2-a^2)^2}{4b^2}$$

$$= \frac{1}{4b^2}(2bc + b^2+c^2-a^2)(2bc-b^2-c^2+a^2)$$

$$= \frac{1}{4b^2}\left((b+c)^2 - a^2\right)\left(a^2 - (b-c)^2\right)$$

$$= \frac{1}{4b^2}(\underbrace{b+c+a}_{2s})(\underbrace{b+c-a}_{2(s-a)})(\underbrace{a+b-c}_{2(s-c)})(\underbrace{a-b+c}_{2(s-b)})$$

So

$$h^2 = \frac{4}{b^2}\, s(s-a)(s-b)(s-c)$$

$$\Longrightarrow A^2 = \left(\frac{hb}{2}\right)^2 = \frac{h^2b^2}{4} = s(s-a)(s-b)(s-c).$$

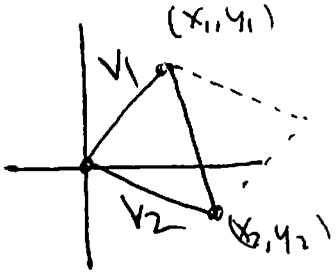
∎

<u>pf 2</u> (my favorite)

Let T be the triangle in the plane w/ vertices (x_i, y_i) for $i = 1, 2, 3$;

~~Then $\alpha(T) = \frac{1}{2}\det(A)$, A~~

can assume that $(x_0, y_0) = (0,0)$



Then $\alpha(T) = \frac{1}{2} \det \underbrace{\begin{array}{cc} v_1 & v_2 \\ \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \end{array}}_{A}$

So $\alpha(T)^2 = \frac{1}{4} \det(\underbrace{A^t A}_{\text{~~has entries of the form v~~}})$

$$= \frac{1}{4} \det \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{pmatrix}$$

<u>Since</u> $v_i \cdot v_j = \frac{1}{2} \left(|v_i|^2 + |v_j|^2 - |v_i - v_j|^2 \right)$

we see that $\alpha(T)^2$ is a <u>polynomial</u> in the squares of the edge lengths $|v_i|$ & $|v_i - v_j|$.

~~Since $\det(A^tA)$ is homogeneous deg 2 in ...,~~
~~we conclude $\alpha(T)^2$ is a degree 4 homogeneous poly in~~

Thus, $v_i \cdot v_j$ is a homogeneous deg 2 poly in edge lengths, so $\alpha(T)^2$ is homogeneous of degree 4 in edge lengths.

Let the edge lengths be a, b, c, and set $f(a,b,c) := \alpha(T)^2$.

Observation 1: The area of T is preserved by rotations & reflections of T $\Longrightarrow$ The order of a, b, c does not matter. $\Rightarrow f(a,b,c)$ is symmetric in a, b, c.

Observation 2 By triangle inequality, $a+b \geq c$ with $=$ iff T degenerates to a straight line. That is, $f(a,b,c) = 0$ if $a+b-c=0$.

$\Rightarrow$ $a+b-c$ is a factor of $f(a,b,c)$

By symmetry, $a+c-b$, $b+c-a$ are factors as well.

Since $f(a,b,c)$ is homogeneous of deg 4,

$$f(a,b,c) = (a+b-c)(a+c-b)(b+c-a)\, h(a,b,c)$$

with $h(a,b,c)$ symmetric & homogeneous of degree 1.

$\Rightarrow$ $h(a,b,c) = (a+b+c)k$ for a constant k.

Thus $\alpha(T)^2 = k(a+b+c)(a+b-c)(a+c-b)(b+c-a)$

for k independent of a, b, c.

Plugging in $(a,b,c) = (3,4,5)$ with $\alpha(T) = 6$,

we find $36 = k \cdot 12 \cdot 2 \cdot 4 \cdot 6 \Rightarrow k = \frac{1}{16}$

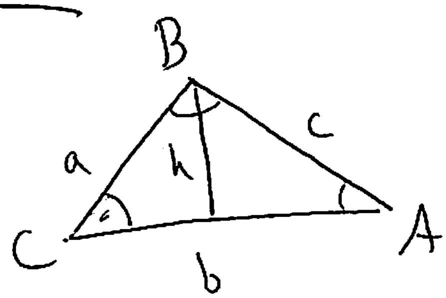
and this gives

$$\alpha(T) = \sqrt{\frac{a+b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{a+c-b}{2} \cdot \frac{b+c-a}{2}}$$

Which is Heron's formula.

$\square$

SAS



$$\alpha(T) = \frac{1}{2} bh$$
$$= \frac{1}{2} b(a \sin C)$$
$$= \frac{1}{2} ab \sin C$$

ASA

$$\alpha(T) = \cancel{\frac{1}{2} ab \sin C = \frac{1}{2}}$$

$$\frac{1}{2} ac \sin(B) = \frac{1}{2} ac \sin(B) \frac{\frac{1}{2} ab \sin C}{\frac{1}{2} bc \sin(A)}$$

$$= \frac{1}{2} a^2 \frac{\sin(B)\sin(C)}{\sin(A)}$$

Since $B+C = \cancel{180}\ \pi - A$

We have $\sin(B+C) = \sin(\pi)\cos(A) - \cos(\pi)\sin(A)$

$$= \sin(A)$$

<u>so</u> $\alpha(T) = \dfrac{1}{2} a^2 \dfrac{\sin(B)\sin(C)}{\sin(B+C)}$