<u>Recall:</u> P = polygon which is a finite union of non-overlapping Δ's.

1. $\alpha(P_1) = \alpha(P_2)$
   if $P_1 \cong P_2$
2. If P_1 & P_2 have intersection w/ empty interior, then
   $\alpha(P_1 \cup P_2) = \alpha(P_1) + \alpha(P_2)$
3. $\alpha(P) = x^2$ if $P = \underset{x}{\square}\, x$

Def: $C \subseteq \mathbb{R}^2$.

For $i = 1, 2, 3, \ldots$, spse given polygons s_i, S_i which are finite unions of Δ's, and $s_i \subseteq C \subseteq S_i$

Spse: G = greatest lower bound of $\{\alpha(S_i)\}$ exists

and L = least upper bound of $\{\alpha(s_i)\}$ exists

and $G = L$.

Then $\alpha(C) \overset{\text{def}}{=} G = L$

What is π?

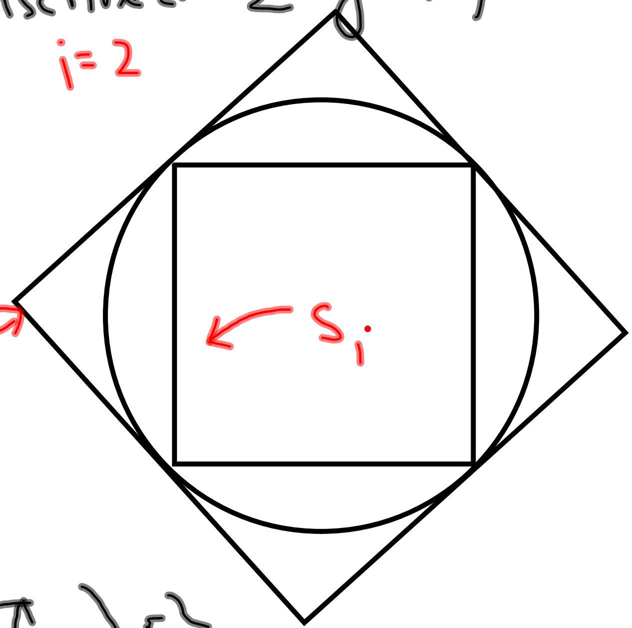
Def: $\pi = \frac{P}{d}$, P = perimeter of circle w/ diameter d

Today we'll prove:

Thm: C = circle of radius r, then
$$\alpha(C) = \pi r^2.$$

Let $$\left.\begin{aligned} s_i &= \text{inscribed } 2^i\text{-gon} \\ S_i &= \text{circumscribed } 2^i\text{-gon} \end{aligned}\right\} i \geq 2$$

$i = 2$



$$\alpha(s_i) < \alpha(C) < \alpha(S_i)$$

<u>In fact:</u>

$$\alpha(s_i) = 2^{i-1} \cdot \sin\left(\frac{\pi}{2^{i-1}}\right) r^2$$

$$\alpha(S_i) = 2^i \cdot \tan\left(\frac{\pi}{2^i}\right) r^2$$

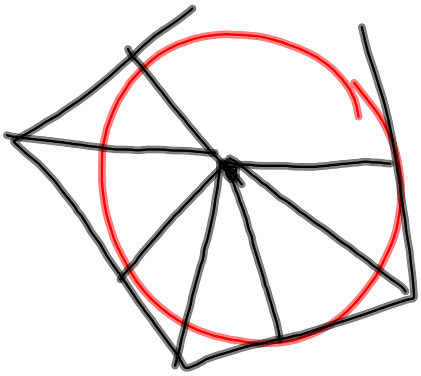
<u>Ex</u>: make a table of these values.

i	$\alpha(s_i)$	$\alpha(S_i)$	$\alpha(S_i) - \alpha(s_i)$
2	$2r^2$	$4r^2$	$2r^2$
3	$2.82r^2$	$3.314r^2$	$.486r^2$
4	$3.061r^2$	$3.183r^2$	$.122r^2$
5	$3.121r^2$	$3.151r^2$	$.030r^2$
6	$3.136r^2$	$3.144r^2$	$.008r^2$
7	$3.140r^2$	$3.142r^2$	$.002r^2$
8	$3.141r^2$	$3.142r^2$	$.001r^2$
9	$3.1415r^2$	$3.1416r^2$	$.0001r^2$
10	$3.14157r^2$	$3.14160r^2$	$.00003r^2$

We notice:

$$\alpha(s_i) < \alpha(s_{i+1}) < \alpha(S_{i+1}) < \alpha(S_i)$$

Recall: Area of circumscribed 2^i-gon is $\frac{1}{2} r P_i$, P_i = perimeter



So, as $i \to \infty$, $\alpha(S_i)$ approaches $\alpha(s_i)$, so by def $\alpha(C)$ = this common limit

$$= \frac{1}{2} r p$$

p = perimeter of C,

$$= \frac{1}{2} r (2r \cdot \pi)$$

$$= \pi r^2.$$

Algonthm for π:

Let α_i = area of circumscribed 2^i-gon w/ radius 1

$$= 2^i \tan\left(\frac{\pi}{2^i}\right)$$

$$\alpha_{i+1} = 2^{i+1}\tan\left(\frac{\pi}{2^{i+1}}\right) = 2^{i+1}\left(\frac{1}{\frac{1}{\tan\left(\frac{\pi}{2^i}\right)} + \sqrt{\frac{1}{\tan^2\left(\frac{\pi}{2^i}\right)} + 1}}\right)$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{\tan\theta}{1+\sqrt{1+\tan^2\theta}}$$

$$= 2\left(\frac{1}{\alpha_i} + \sqrt{\frac{1}{2^{2i}} + \frac{1}{\alpha_i^2}}\right)^{-1}$$

$$\alpha_2 = 4$$

$$\alpha_3 = 2\left(\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{1}{4^2}}\right)^{-1}$$

$$= 2\left(\frac{1}{4} + \frac{\sqrt{2}}{4}\right)^{-1} = 8(\sqrt{2} - 1$$

$$\alpha_4 = \ldots.$$