Recall: We defined the area of any polygon which is a finite union of triangles:

1. α is invariant under congruence
2. If $F_1, F_2 \subseteq \mathbb{R}^2$ are two regions whose intersection has empty interior, then $\alpha(F_1 \cup F_2) = \alpha(F_1) + \alpha(F_2)$
3. If F is a square of side-length x, then $\alpha(F) = x^2$

Now we extend the definition to any "nice" closed region in the plane:

Def: $C \subseteq \mathbb{R}^2$, and for $i = 1, 2, \ldots,$ let

s_i (resp S_i) be a polygon which is a finite union of non-overlapping triangles s.t.

$$* \quad s_i \subseteq C \subseteq S_i$$

Suppose that the least upper bound of $\{\alpha(s_i)\}_i$ equals the greatest lower bound of $\{\alpha(S_i)\}_i$. Then

$$\alpha(C) \stackrel{\text{def}}{=} \text{this common bound.}$$

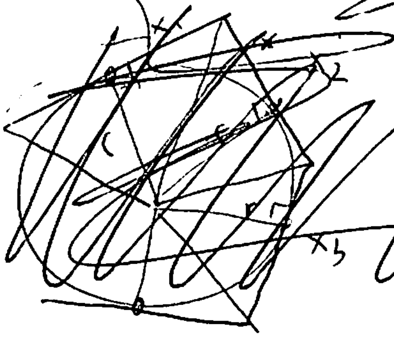
Thm: With this def of area,

$$\alpha(C) = \pi r^2$$

for C = circle of radius r.

Remember: $\pi \stackrel{\text{def}}{:=} \frac{P}{d}$ for P = perimeter, d = diameter

~~Recall: A polygon circumscribing a circle of radius r has area~~ $\cancel{\frac{1}{2} r p}$, ~~p = perimeter of polygon.~~



$\cancel{A = \frac{1}{2} r x_1 + \frac{1}{2} r x_2 + \cdots}$

$\cancel{= \frac{1}{2} r (x_1 + x_2 + \cdots) = \frac{1}{2} r p}$

~~As the number of sides of a regular circumscribing polygon tends to ∞, the polygon approaches the circle, as does~~

$$2^i \sin\left(\frac{\pi}{2^{i+1}}\right) \cos\left(\frac{\pi}{2^{i+1}}\right) r^2$$

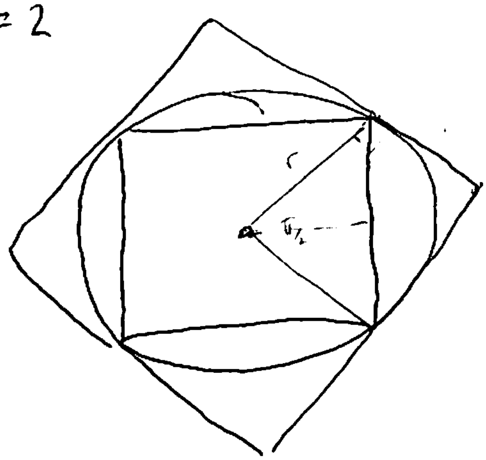
$$= 2^{i-1} \sin\left(\frac{\pi}{2^i}\right) r^2$$

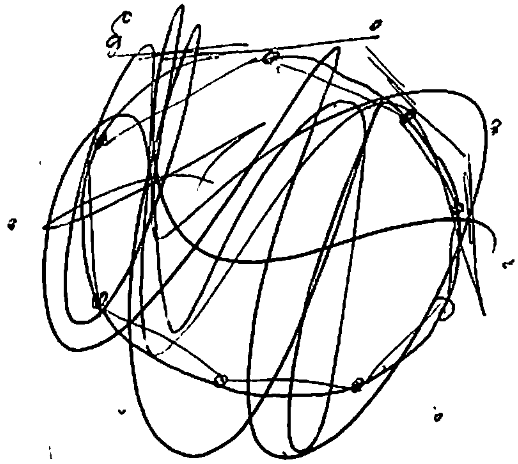
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

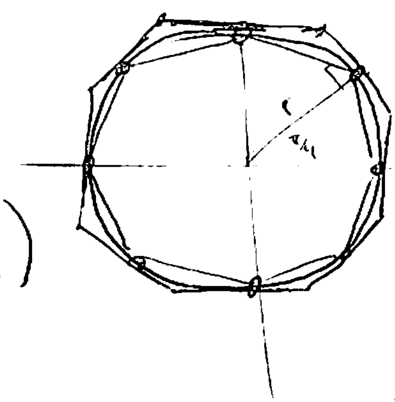
Let $\left.\begin{array}{l} s_i = \text{A } 2^i\text{-gon inscribed in } C \\ S_i = \text{A } 2^i\text{-gon circumscribed in } C \end{array}\right\} i \geq 2$

$i = 2$





$i = 3$



<u>Clearly</u> $\alpha(s_i) < \alpha(S_i)$

In fact, $\alpha(s_i) = \begin{cases} \cancel{\dfrac{2^i r^2 \sin(\frac{\pi}{2^i}) \cos(\frac{\pi}{2^i})}{2} = 2^{i-2} \sin(\frac{\pi}{2^{i-1}}) r^2} \\ 2^i r^2 \sin(\frac{\pi}{2^i}) \cos(\frac{\pi}{2^i}) = 2^{i-1} \sin(\frac{\pi}{2^{i-1}}) r^2 \end{cases}$

$$\alpha(S_i) = 2^{i+1}\left(\frac{r \cdot r \tan(\frac{\pi}{2^i})}{2}\right) = 2^i \tan\left(\frac{\pi}{2^i}\right) r^2$$

~~Recall (Double angle formulae): $\sin(\frac{\theta}{2}) = \sqrt{\frac{1-\cos\theta}{2}}$~~

~~$= \sqrt{\frac{1-\sqrt{1-\sin^2\theta}}{2}}$~~

~~$\sin(2\theta) = 2\sin\theta\cos\theta$~~

~~$\sin(\theta) = 2\sin(\frac{\theta}{2})\sqrt{1-\sin^2(\frac{\theta}{2})}$~~

$$\tan\left(\frac{\theta}{2}\right) = \frac{\tan\theta}{1+\sqrt{1+\tan^2\theta}}$$

$$\cancel{\alpha(S_{i+1}) = 2^i \sin\left(\frac{\pi}{2^i}\right) r^2}$$

$$\cancel{= 2^i \left(\sqrt{\frac{1-\sqrt{1-\sin\left(\frac{\pi}{2^{i-1}}\right)^2}}{2}} \right) r^2}$$

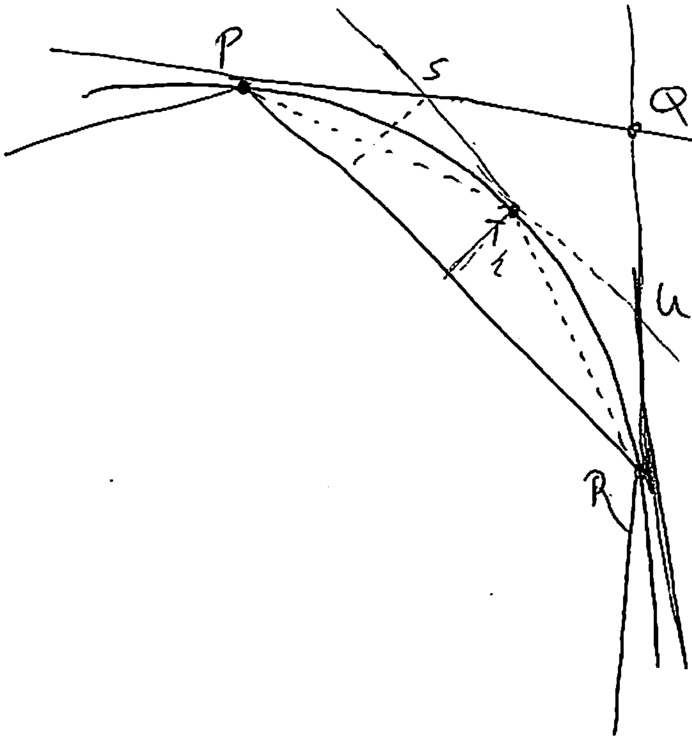
$$\cancel{= \sqrt{2^{2i-1} r^4 \left(1-\sqrt{1-\sin\left(\frac{\pi}{2^{i-1}}\right)^2}\right)}}$$

$$\cancel{= \left(2^i r^2 \left(2^{i-1} r^2 - \sqrt{2^{2i-2} r^4 - \left(2^{i-1} r^2 \sin\left(\frac{\pi}{2^{i-1}}\right)\right)^2} \right) \right)^{1/2}}$$

$$\cancel{= \left(2^{2i-1} r^4 - \sqrt{2^{2i-2} r^4 - s_i^2} \right)}$$

<u>Exercise</u>: Compute a table.

i	$\alpha(s_i)$ $2^{i-1}\sin\left(\frac{\pi}{2^{i-1}}\right)r^2$	$\alpha(S_i)$ $2^{i}\tan\left(\frac{\pi}{2^i}\right)r^2$	$\alpha(S_i)-\alpha(s_i)$
2	$2r^2$	$4r^2$	$2r^2$
3	$2.828\,r^2$	$3.313r^2$	$0.485\,r^2$
4	$\vdots$	$\vdots$	
5			
6			
7			
8			



$$\alpha(s_{i+1}) - \alpha(s_i) = \alpha(\Delta PTR)$$

$$\alpha(S_i) - \alpha(S_{i+1}) = \alpha(SQU)$$

So: $\alpha(s_i) < \alpha(s_{i+1}) < \alpha(S_{i+1}) < \alpha(S_i)$

That is, $[\alpha(s_{i+1}), \alpha(S_i)]$

$\subsetneq [\alpha(s_i), \alpha(S_{i+1})]$

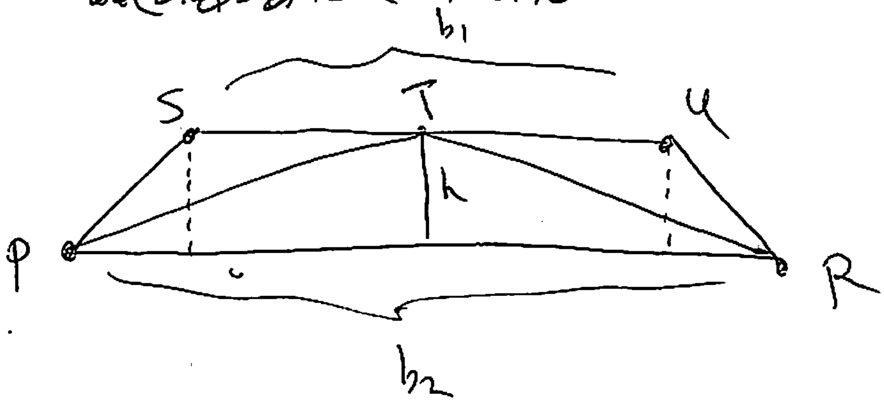
<u>More</u> precisely:

$$\alpha(S_{i+1}) - \alpha(s_{i+1}) = \alpha(\Delta PST) + \alpha(\Delta RUT)$$

$$\alpha(S_i) - \alpha(s_i) = \alpha(PQR)$$

<u>But</u>: $\alpha(PSUR) = h\left(\frac{b_1+b_2}{2}\right)$ while

~~$\alpha(\Delta PST) + \alpha(\Delta RUT)$~~



$$\alpha(PST) + \alpha(RUT) = \tfrac{1}{2} h b_1$$

$$\alpha(PQR) > \alpha(PSUR) = \tfrac{1}{2} h (b_1 + b_2) = \tfrac{1}{2} h b_1 + \tfrac{1}{2} h b_2$$

$$> 2\left(\tfrac{1}{2} h b_1\right)$$

We conclude:

$$\alpha(S_{i+1}) - \alpha(s_i) < \frac{1}{2}\left(\alpha(S_{i+1}) - \alpha(s_i)\right)$$

So this length approaches zero.

Recall: Area of circumscribed 2^i-gon is $\frac{1}{2} r P_i$, where P_i = perimeter of circumscribed 2^i-gon.

~~By def we have~~

$\cancel{\alpha(C_i) = \lim_{i \to \infty}}$

As $i \to \infty$, P_i approaches the perimeter of the circle, which is $\pi \cdot d = 2\pi r$,

So area approaches $\frac{1}{2} r p = \frac{1}{2} r \cdot 2\pi r = \pi r^2$.

Iterative algorithm for π:

$$\tan\left(\frac{\theta}{2}\right) = \frac{\tan\theta}{1 + \sqrt{1 + \tan^2\theta}} = \frac{\sin\theta}{1 + \cos\theta}$$

$$\alpha(S_{i+1}) = 2^{i+1} \cdot \tan\left(\frac{\pi}{2^{i+1}}\right)$$

$$= 2^{i+1} \frac{1}{\frac{1}{\tan\left(\frac{\pi}{2^i}\right)} + \sqrt{1 + \frac{1}{\tan\left(\frac{\pi}{2^i}\right)^2}}}$$

$$= 2\left(\frac{1}{\alpha(S_i)} + \sqrt{\frac{1}{2^{2i} r^4} + \frac{1}{\alpha(S_i)^2}}\right)^{-1}$$

$$\lim_{i \to \infty} 2^i \tan\left(\frac{\pi}{2^i}\right) = \pi$$

$$\alpha_i := 2^i \tan\left(\frac{\pi}{2^i}\right)$$

$$\boxed{\begin{aligned} \alpha_{i+1} &= 2^{i+1}\left(\frac{1}{\tan\left(\frac{\pi}{2^i}\right)} + \sqrt{1 + \frac{1}{\tan^2\left(\frac{\pi}{2^i}\right)}}\right)^{-1} \\ &= 2\left(\frac{1}{\alpha_i} + \sqrt{\frac{1}{2^{2i}} + \frac{1}{\alpha_i^2}}\right)^{-1} \end{aligned}}$$

$$\alpha_2 = 4$$

$$\alpha_3 = 2\left(\frac{1}{4} + \sqrt{\frac{1}{\cancel{8}16} + \frac{1}{4^2}}\right)^{-1}$$

$$= 2\left(\frac{1}{4} + \frac{1}{2\sqrt{2}}\right)^{-1} = 2\left(\frac{1+\sqrt{2}}{4}\right)^{-1}$$

$$= \frac{8}{1+\sqrt{2}}$$