Problems p. 305 #2, p. 313, # 11, 13

Recall: A congruence transformation

is: $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, s.t.

1.) f is 1-1

2.) f is "distance preserving" (isometry)

$$d(f(P), f(Q)) = d(P,Q)$$

$d=$ Euclidean dist. $d((x,y),(x',y')) = \sqrt{(x-x')^2+(y-y')^2}$

Today: Some examples of congruence transformations

a) synthetic b) analytic ($\mathbb{R}^2$) c) analytic ($\mathbb{C}$)

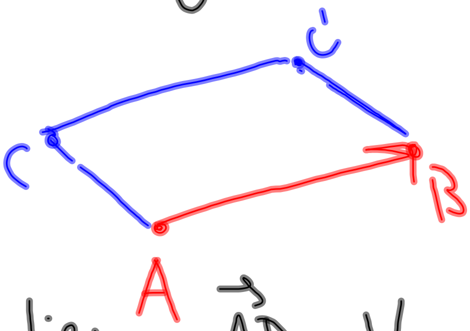
* Translations

a) Let $\overrightarrow{AB}$ be a directed line segment.

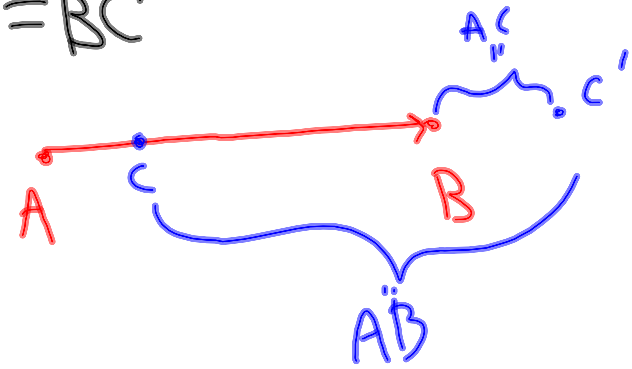
$T_{\overrightarrow{AB}}$ = translation along $\overrightarrow{AB}$

$T_{\overrightarrow{AB}}(C) \stackrel{\text{def}}{=} C'$, where

i) If C is not on $\overrightarrow{AB}$, then C' = the unique point s.t. $ABC'C$ is a parallelogram:



ii) If C lies on $\overrightarrow{AB}$, then C' = unique point s.t. $AB = CC'$ and $AC = BC'$



b) Let b = the vector in $\mathbb{R}^2$ rep'd by $\vec{AB} = (h,k)$

Translation along $\vec{AB} \longleftrightarrow$ addition of b

$$T_b(x,y) = (x+h, y+k)$$

$$(x,y) \xrightarrow{\quad b \quad} (x+h, y+k)$$

Rems: • $T_0(x,y) = (x,y), \quad \forall\, x,y$

so $T_0 = \mathrm{id}$ (identity transformation of $\mathbb{R}^2$)

• Can compose translations:

$$T_{b_2} \circ T_{b_1}(x,y) = T_{b_2}(x+h_1, y+k_1)$$

$b_2 = (h_2,k_2)$ $\quad$ $b_1 = (h_1,k_1)$

$$= (x+h_1+h_2, y+k_1+k_2)$$

$$= T_{b_1+b_2}(x,y) = T_{b_1} \circ T_{b_2}(x,y)$$

so $T_b \circ T_{-b} = T_0 = \mathrm{id}$

$\mathbb{T} = \left\{ \begin{array}{c} \text{translations of } \mathbb{R}^2 \\ T_b : \mathbb{R}^2 \to \mathbb{R}^2 \end{array} \right\}$

= a commutative group under composition of translations.

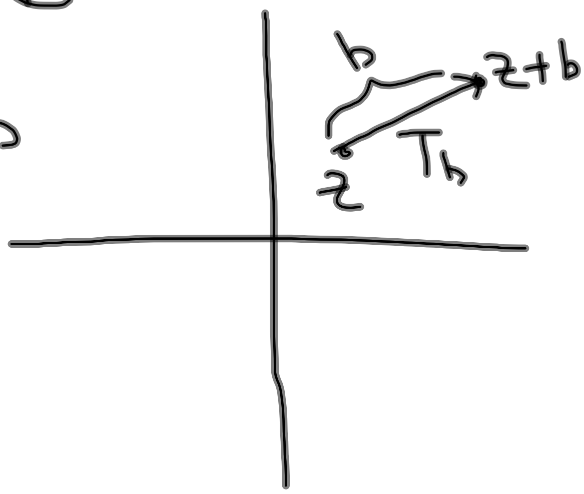
Moreover, the map

$$\begin{array}{ccc} (\mathbb{R}^2, +) & \longrightarrow & (\mathbb{T}, \underset{\text{composition}}{\circ}) \\ \rotatebox{90}{$\in$} & & \\ b & \longmapsto & T_b \end{array}$$

is an isomorphism of groups.

c) $b = h + ik \in \mathbb{C}$

$$T_b(\underset{\substack{\shortparallel \\ x+iy}}{z}) = z + b$$



<u>Thm</u>: For all $b \in \mathbb{C}$, T_b is an isometry.

Pf: $|T_b(z) - T_b(w)| = |(z+b) - (w+b)| = |z - w|$

Rotations:

a) C a point in the plane,

$\phi =$ a real number (radians)

$-\pi < \phi \leq \pi$

$R_{C,\phi} :=$ rotation through angle of ϕ with center C.

- $R_{C,\phi}(C) = C$

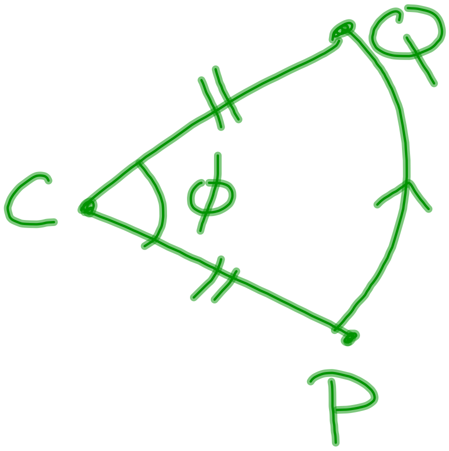
$P \neq C$ • $R_{C,\phi}(P) = Q$, for Q the unique point satisfying

i) $PC = QC$

ii) $m\angle PCQ = |\phi|$

- $\triangle PCQ$ oriented CCW if $\phi > 0$
- CW if $\phi < 0$

$* \; R_{C,0} = id$.



$(\phi > 0)$

* Can generalize to allow any ϕ.

If $\underbrace{\phi \equiv \phi' \bmod 2\pi\mathbb{Z}}_{\phi - \phi' = 2\pi \cdot n,\ n \in \mathbb{Z}}$, then $R_{C,\phi} = R_{C,\phi'}$

Can compose rotations

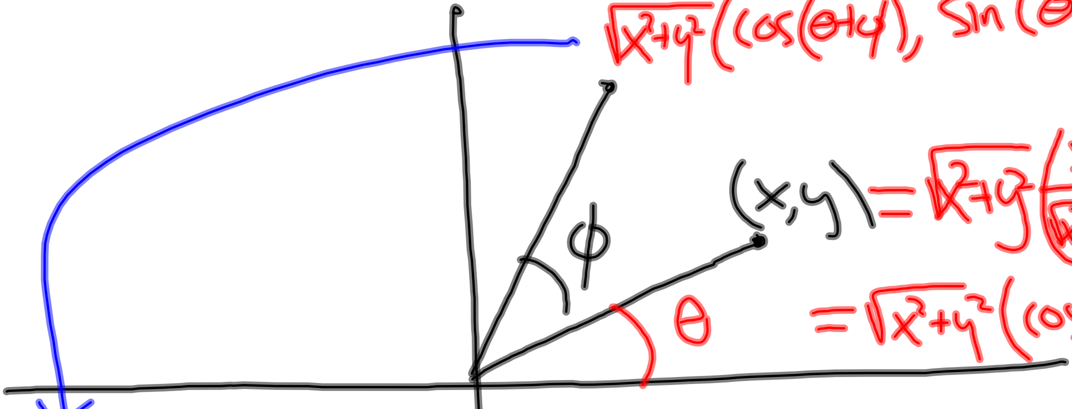
$$R_{c,\phi_2} \circ R_{c,\phi_1} = R_{c,\phi_1+\phi_2}$$

$R_{c,0} = \mathrm{id}$, so we get

$$\mathbb{R}/2\pi\mathbb{Z} = \left\{\begin{array}{l}\text{the group} \\ \text{of reals} \\ \text{mod } 2\pi\mathbb{Z} \\ \text{under } +\end{array}\right\} \xrightarrow{\phi \mapsto R_{c,\phi}} \left\{\begin{array}{l}\text{rotations} \\ \text{about } C, \\ \text{under } \circ\end{array}\right\}$$

is a group isomorphism.

b) Suppose $C = \text{origin}$, $R_\phi := R_{C,\phi}$



$$\sqrt{x^2+y^2}\left(\cos(\theta+\phi), \sin(\theta+\phi)\right)$$

$$(x,y) = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= \sqrt{x^2+y^2}\left(\cos\theta, \sin\theta\right)$$

$$\sqrt{x^2+y^2}\left(\cos\theta\cos\phi - \sin\theta\sin\phi, \sin\theta\cos\phi + \cos\theta\sin\phi\right)$$

$$= \left(x\cos\phi - y\sin\phi, y\cos\phi + x\sin\phi\right)$$

In matrix form

$$[R_\phi] = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

More generally

$$R_{c,\phi} = T_c \circ R_\phi \circ T_{-c}$$

( ) $R_\phi(x+iy) = (x\cos\phi - y\sin\phi)$
$\qquad + i(x\sin\phi + y\cos\phi)$

$$= (x+iy)\underbrace{(\cos\phi + i\sin\phi)}_{z_\phi = e^{i\phi}}$$

Hence

$$R_{c,\phi}(z) = z_\phi(z-c) + c$$

$$\left\{ z \in \mathbb{C} \,\middle|\, |z| = 1 \right\} \longrightarrow \left\{ \begin{array}{l} \text{rotations} \\ \text{about } c \end{array} \right\}$$
under mult. $\qquad$ under comp.

$$z \longmapsto R_{c,\phi}$$

is an isom. of gps.

<u>Ex</u> prove $R_{c,\phi}$ is an isometry.