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* Problems p. 320 #1,4.

* Projects 10-15 min presentation

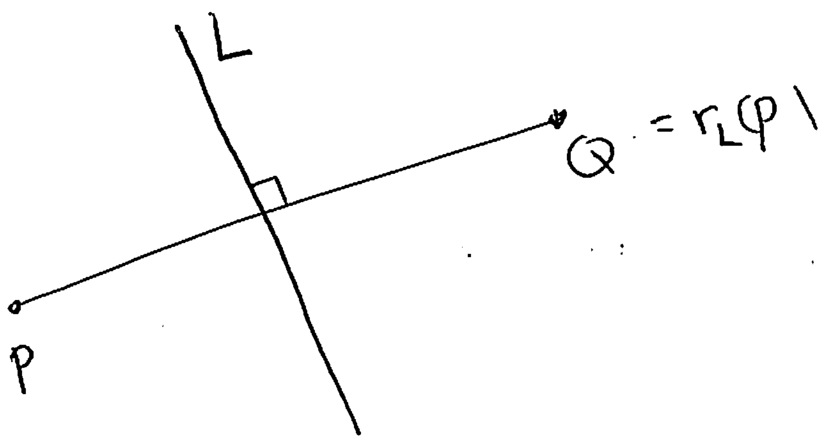
11-30 (w): (5)
12-5 (m): (5)
12-7 (w): (4)

Reflections

L = line in $\mathbb{R}^2$.

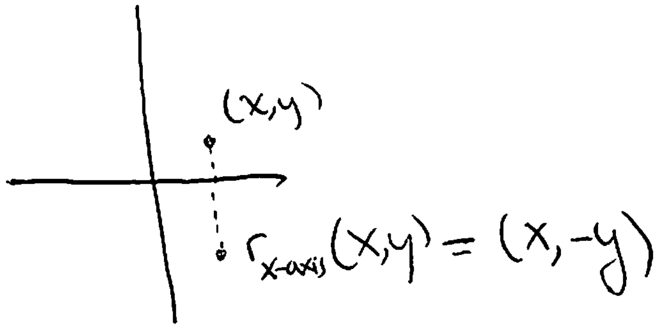
$r_L(P)$ = reflection of P across L

$= \begin{cases} \text{the unique point } Q \text{ s.t. } L \text{ is } \perp \text{ bisector of } PQ & \text{if } P \text{ not on } L \\ P & \text{if } P \text{ on } L. \end{cases}$



Three special cases: L = x-axis
y-axis
the line $y=x$

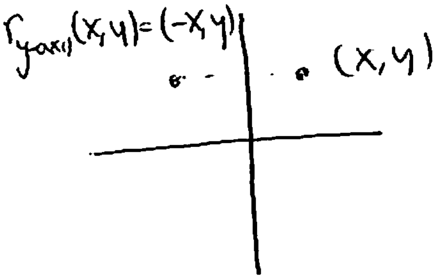
i)



$$r_{x\text{-axis}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

so the matrix of $r_{x\text{-axis}}$ is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

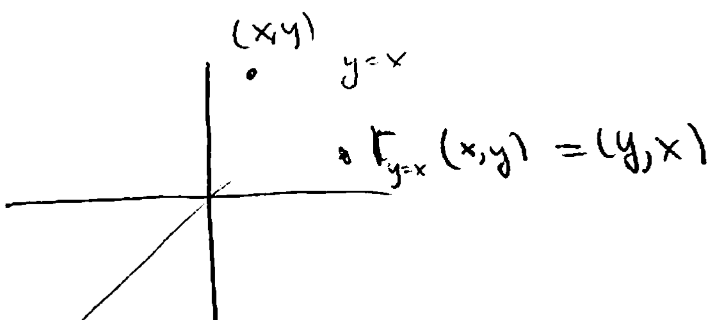
ii)



$$r_{y\text{-axis}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

so the matrix of $r_{y\text{-axis}}$ is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

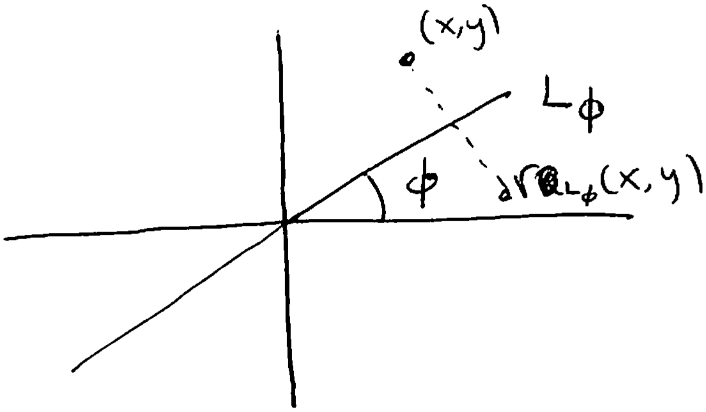
iii)



$$r_{x=y}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

so the matrix of $r_{x=y}$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

* Using these, we can figure out a formula (analytic) for r_{L_ϕ}, L_ϕ = the line through $(0,0)$ making angle ϕ w/ x-axis



<u>Trick</u>: Rotate coordinate axes!

$$R_{-\phi} \circ r_{L_\phi} = r_x \circ R_{-\phi}$$

Hence,

$$r_{L_\phi} = R_\phi \circ R_{-\phi} \circ r_{L_\phi} = R_\phi \circ r_x \circ R_{-\phi}$$

Yesterday, we computed that the matrix of R_ϕ is

$[R_\phi] = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$, so since composition corresponds to matrix multiplication, we get

$$[r_{L_\phi}] = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi - \sin^2\phi & 2\cos\phi\sin\phi \\ 2\cos\phi\sin\phi & -(\cos^2\phi - \sin^2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\phi) & -\sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [R_{2\phi}]\,[r_x]$$

In particular,

1. $r_{L_\phi}(x,y) = (x\cos 2\phi + y\sin 2\phi,\ x\sin 2\phi - y\cos 2\phi)$

2. $r_{L_\phi} \circ r_x = R_{2\phi} \circ r_x \circ r_x = R_{2\phi}$

any ~~rotation~~ reflection composed with itself is identity.

Corollary: The composition of two reflections across L_{ϕ_1} and L_{ϕ_2} is equal to ~~the~~ a rotation: =

$$r_{L_{\phi_2}} \circ r_{L_{\phi_1}} = R_{2(\phi_2 - \phi_1)}$$

In particular, this composite depends only on $\phi_2 - \phi_1$. [Pf. $r_{L_{\phi_2}} \circ r_{L_{\phi_1}} = r_{L_{\phi_2}} \circ r_x \circ r_x \circ r_{L_{\phi_1}} = r_{L_{\phi_2}} \circ r_x \circ (r_{L_{\phi_1}}^{-1} \circ r_x^{-1})$

$= r_{L_{\phi_2}} \circ r_x \circ (r_{L_{-\phi_1}} \circ r_x)$

$= R_{2\phi_2} \circ R_{-2\phi_1}$]

Complex picture

"Complex conjugation"

$$r_{\mathbb{R}}(z) = \bar{z} = a - bi, \text{ where } z = a + bi$$

Clearly, this corresponds to $r_{x\text{-axis}}$ on $\mathbb{R}^2$

We saw last time that R_ϕ corresponds to multiplication by $z_\phi = \cos\phi + i\sin\phi$, so we conclude that

since $r_{L_\phi} = r_{2\phi} \circ r_x$, get $r_{L_\phi}(z) = z_{2\phi} \cdot \bar{z}$

For a line not containing the origin, we get

$$r_{c,\phi}(z) = z_{2\phi}\overline{(z-c)} + c$$

where $r_{c,\phi}$ = reflection about line passing through c and making angle ϕ w/ positive ray of real axis.

Now suppose we compose two reflections across <u>parallel</u> lines:

$$r_{c_2,\phi} \circ r_{c_1,\phi}(z) = r_{c_2,\phi}\left(z_{2\phi}\overline{(z-c_1)} + c_1\right)$$

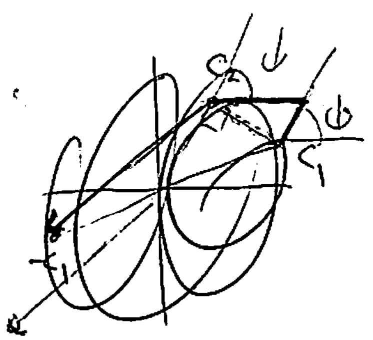
$$= z_{2\phi}\overline{\left(\left(z_{2\phi}\overline{(z-c_1)} + c_1\right) - c_2\right)} + c_2$$

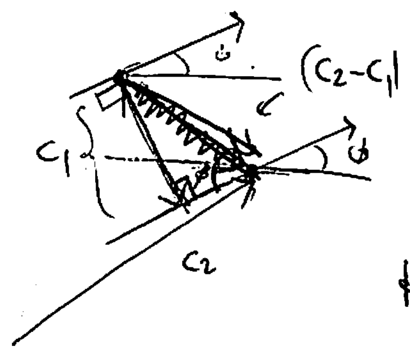
$$\cancel{z_{2\phi}\left(\overline{z_{2\phi}\overline{(z-c)}}\right) + c = z_{2\phi}\bar{z}_{2\phi}(z-c)} + c$$

$$= Z_{2\phi}\left(\overline{Z}_{2\phi}(z-c_1)+\overline{c}_1-\overline{c}_2\right)+c_2$$

$$= (z-c_1) + z_{2\phi}\overline{(c_1-c_2)} + c_2$$

$$= z + \underbrace{(c_2-c_1) - z_{2\phi}\overline{(c_2-c_1)}}$$





$\cancel{|c_2-c_1|}\sin(\phi)$

$c_1 + X = c_2$