 Individual HW: 8, 10 

Last time: Problem Analysis

* A substance is 99% $H_2O$. Some $H_2O$ evaporates, leaving a substance that is 98% $H_2O$.
  How much $H_2O$ evaporated, as a percentage of original amount?

Sol: Sp& 100 units of mixture.

$$100 \text{ total} \begin{cases} 99 & H_2O \\ 1 & \text{stuff} \end{cases} \xrightarrow{H_2O \text{ evaps}} \begin{cases} x & H_2O \\ 1 & \text{stuff} \end{cases}$$

$$0.98 = \frac{x}{1+x} \quad \text{so} \quad 0.98(1+x) = x$$
$$0.98 + 0.98x = x$$
$$0.98 = x(1-0.98) = 0.02x$$
$$\Rightarrow x = 49$$

So $50 = 99-49$ units of $H_2O$ have evap'd.

or $\frac{50}{99} = 50.50\ldots\%$ of original amt.

SURPRISING!

Numerical Sol: Percent "stuff" goes from 1% to 2% DOUBLES so total $H_2O$ + stuff is HALVED

$\Rightarrow$ 50 units total, hence 49 units $H_2O$ are left.

* Want to EXPLAIN answer.

Graphical sol #1

$W$ = units $H_2O$ @ start

$W'$ = ______ @ end

$S$ = units stuff

$p$ = proportion $H_2O$ @ start

$p'$ = ______ @ end

$q$ = ______ that evaporated

$$p = \frac{W}{W+S}, \quad p' = \frac{W'}{W'+S}, \quad q = \frac{W-W'}{W} = 1 - \frac{W'}{W}$$

Want: Understand $q$ in terms of $p, p'$.

→ Solve for $W, W'$ in terms of $p, p'$:

$$p(W+S) = W \qquad pS = W(1-p)$$

$$pW + pS = W \qquad W = \frac{p}{1-p} S$$

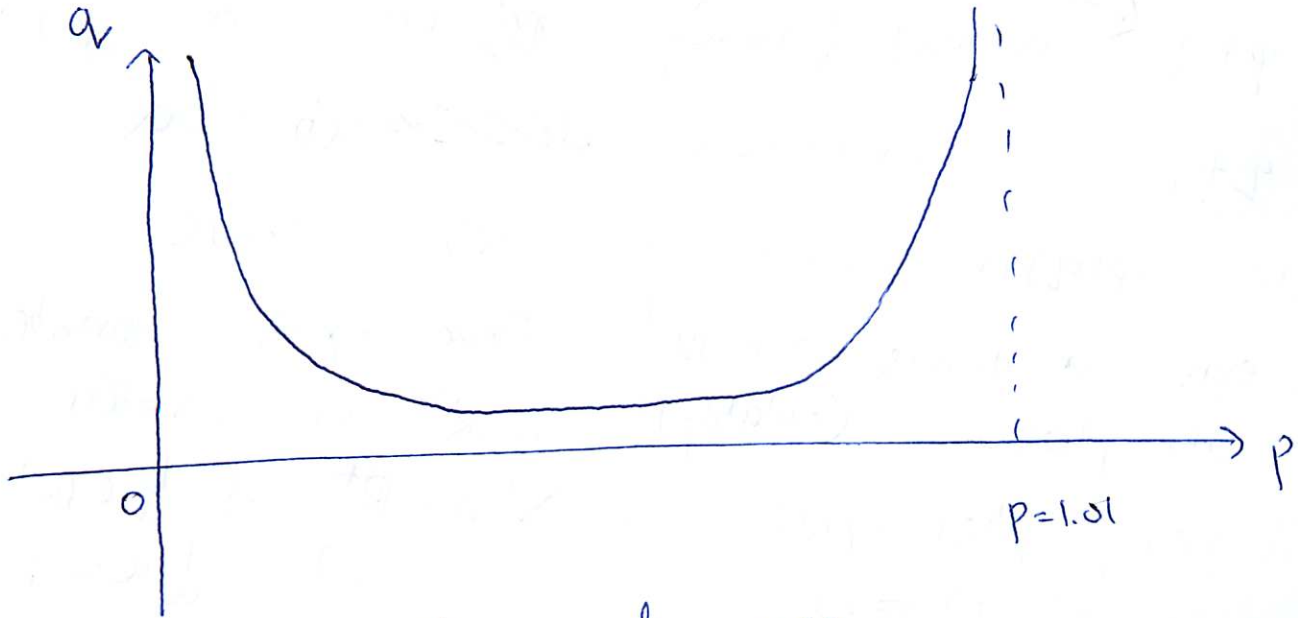
Similarly: $W' = \frac{p'}{1-p'} \cdot S$

So
$$q = 1 - \frac{W'}{W} = 1 - \frac{\frac{p'}{1-p'} S}{\frac{p}{1-p} S} = 1 - \frac{p'(1-p)}{p(1-p')}$$

$$= \frac{p-p'}{p(1-p')}$$

In our case, $p-p' = 0.01$, so

$$q = \frac{0.01}{p(1.01-p)}$$



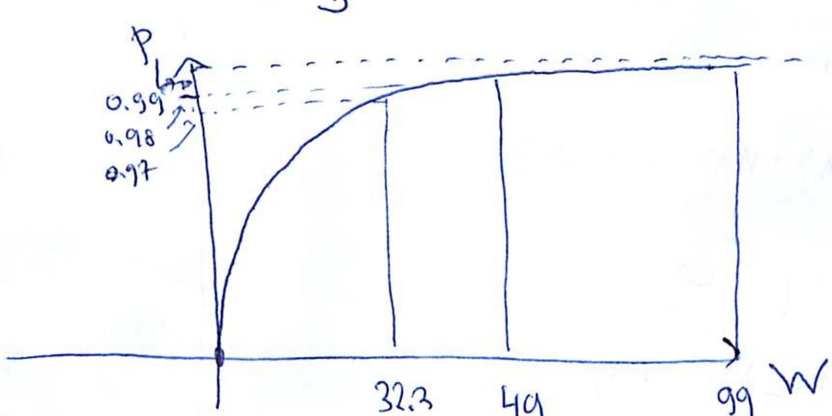

* Only $0<p<1$ makes sense.

Observe: For $p$ close to 1, a small change in $p$ corresponds to LARGE change in $q$!

So proportion of water that evaporates is very sensitive to initial $p$ if $p$ is near 1.

Graphical Sol #2. $p = \frac{W}{W+S}$. In our case, $S=1$

so $p = \frac{W}{W+1}$




Flatness of graph as $p \to 1$ explains this phenomenon!

# Mathematical Analogues

positive real #'s

| properties of $\mathbb{R}, +$ corresponds | properties " of $\mathbb{R}^+, \times$ ← multiplication |
|---|---|
| $p+q$ is real (closure) | ~~$p \cdot q$~~ $a \cdot b$ is real & pos (closure) |
| $p+q = q+p$ (commutative) | ~~$a+b=b+a$~~ $ab = ba$ |
| $p+(q+r) = (p+q)+r$ (assoc) | $a(bc) = (ab)c$ |
| There exists a number $0$ in $\mathbb{R}$ s.t. $0+p = p+0 = 0$ (identity) | There exists a number $1 \in \mathbb{R}^+$ s.t. $1 \cdot a = a \cdot 1 = a$ |
| For all $p \in \mathbb{R}$, there exists $-p \in \mathbb{R}$ s.t. $p+(-p) = 0$ | $\forall a \in \mathbb{R}^+, \exists \frac{1}{a} \in \mathbb{R}^+$ s.t. $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$ |
| | Ask students to fill in this column |
| $p+p = 2p$ | ~~$p \cdot p = p^2$~~ $a \cdot a = a^2$ |
| $p+p+p = 3p$ | ~~$p \cdot p$~~ $a \cdot a \cdot a = a^3$ |
| $\vdots$ | |
| $\underbrace{p+p+\dots+p}_{m} = mp$ | $\underbrace{a \cdot a \cdots a}_{m} = a^m$ |
| $mp + np = (m+n)p$ | $a^m \cdot a^n = a^{m+n}$ |
| $0 \cdot p = 0$ | $a^0 = 1$ |
| $m(p+q) = mp + mq$ | $(ab)^m = a^m b^m$ |
| $m(np) = (mn) \cdot p$ | $(a^n)^m = a^{mn}$ |

| $\mathbb{R}, +$ | $\mathbb{R}^+, \times$ |
|---|---|
| $(-1)\cdot p = -p$ | $a^{-1} = \frac{1}{a}$ |
| $-(-p) = p$ | $\frac{1}{\left(\frac{1}{a}\right)} = a$ |
| $p - q = p + (-q)$ | $a \div b = a \cdot \frac{1}{b}$ |
| $(p-q) + (r-s) = (p+r) - (q+s)$ | $\cancel{\frac{p}{q}\cdot\frac{r}{s} = \frac{pr}{qs}}$ $\quad \frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$ |
| $(p-q) - (r-s) = (p+s) - (q+r)$ | $\cancel{\frac{p}{q}\div\frac{r}{s} = \frac{ps}{qr}}$ $\quad \frac{a}{b}\div\frac{c}{d} = \frac{ad}{bc}$ |
| $\frac{1}{n}\cdot p =$ the number which, when added to itself $n$-times gives $p$ | $a^{\frac{1}{n}} =$ the number which, when multiplied by itself $n$-times gives $a$ |
| $\underbrace{\frac{1}{n}p + \frac{1}{n}p + \cdots + \frac{1}{n}p}_{n} = p$ | $\underbrace{(a^{\frac{1}{n}})\cdot(a^{\frac{1}{n}})\cdots(a^{\frac{1}{n}})}_{n} = a$ |
| $\vdots$ | $\vdots$ |

Underlying this correspondence is a function: Fix any $e > 1$ (e.g. $e = 2$)

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R}^+ \\ \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\ p & \longmapsto & e^p \end{array}$$

$$f(p+q) = e^{p+q} = e^p \cdot e^q = f(p) \cdot f(q)$$

$$f(p-q) = e^{p-q} = \frac{e^p}{e^q} = \frac{f(p)}{f(q)}$$

So the fact that props in right col hold are equivalent to fact that the ones in left col hold, via $f$!