9/12/11

407

Problems: p.159, # 4, 7

Last time: Solving linear equations.

$\rightsquigarrow$ Groups

Today: Solving quadratic eqns

$\rightsquigarrow$ Complex numbers.

* Find two numbers m, n whose sum is 10 and whose product is 12.

$$(1) \qquad \begin{cases} m+n=10 \\ mn = 12 \end{cases}$$

Sol 1: As $mn=12$, $n \neq 0$ so $m = \frac{12}{n}$.

Substituting: $\frac{12}{n} + n = 10$, so $12 + n^2 = 10n$ (again since $n \neq 0$)

or "standard form": $n^2 - 10n + 12 = 0$

* Observe: 1) -sum, product

2) Since eqn (1) is symmetric in m, n,

$m^2 - 10m + 12 = 0$ as well.

Q: A) How do we solve this eqn?
B) Why does our method work?

A A) - Quadratic equation
- Factoring
- Complete the square

B) - ??
- Division algorithm in $\mathbb{C}[x]$
- Algebraic manipulation

* Students answer A), B) on board

Solution: Since $m+n=10$, the average of m & n is 5. Let's write $m=5+x$, $n=5-x$.

Since $mn=12$, we get

$$(5+x)(5-x)=12$$
$$25-x^2=12 \qquad \text{so} \qquad x=\pm\sqrt{13}$$
$$x^2=13$$

Hence $m=5+\sqrt{13}$, $n=5-\sqrt{13}$ or vice-versa.

Thm: If m,n are solutions to

$$x^2+bx+c=0$$

Then $m+n=-b$ and $mn=c$
and conversely.

Proof: ($\Rightarrow$) Since m, n are solutions,

$$x^2 + bx + c \overset{\text{as quadratic polynomials}}{=} (x-m)(x-n) \quad \text{(WHY??)}$$

$$= x^2 - mx - nx + mn$$

$$= x^2 - (m+n)x + mn$$

$\Rightarrow$ $b = -(m+n)$, $c = mn$ [see p. 155 for alternate pf]

($\Leftarrow$): Left as exercise.

Let's use this to <u>solve</u> $x^2 + bx + c = 0$.

Let m, n be the solutions of $x^2 + bx + c = 0$,

so $\begin{cases} m+n = -b \\ mn = c \end{cases}$. Then the average

of m, n is $\dfrac{m+n}{2} = -\dfrac{b}{2}$, so

writing

$$m = -\frac{b}{2} + v \qquad \text{we get} \qquad mn = \left(-\frac{b}{2} + v\right)\left(-\frac{b}{2} - v\right)$$

$$n = -\frac{b}{2} - v \qquad\qquad\qquad = c$$

or

$$\frac{b^2}{4} - v^2 = c \quad \text{so} \quad v^2 = \frac{b^2}{4} - c$$

and $$v = \pm\sqrt{\frac{b^2}{4} - c}$$

<u>Then</u> $$m = -\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}$$ or vice versa $$n = -\frac{b}{2} - \sqrt{\frac{b^2}{4} - c}$$

<u>Equivalently</u> $$m = \frac{-b + \sqrt{b^2 - 4c}}{2}, \quad n = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

To solve $ax^2 + bx + c = 0$, $(a \neq 0)$, ~~we apply~~

we rewrite as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

giving

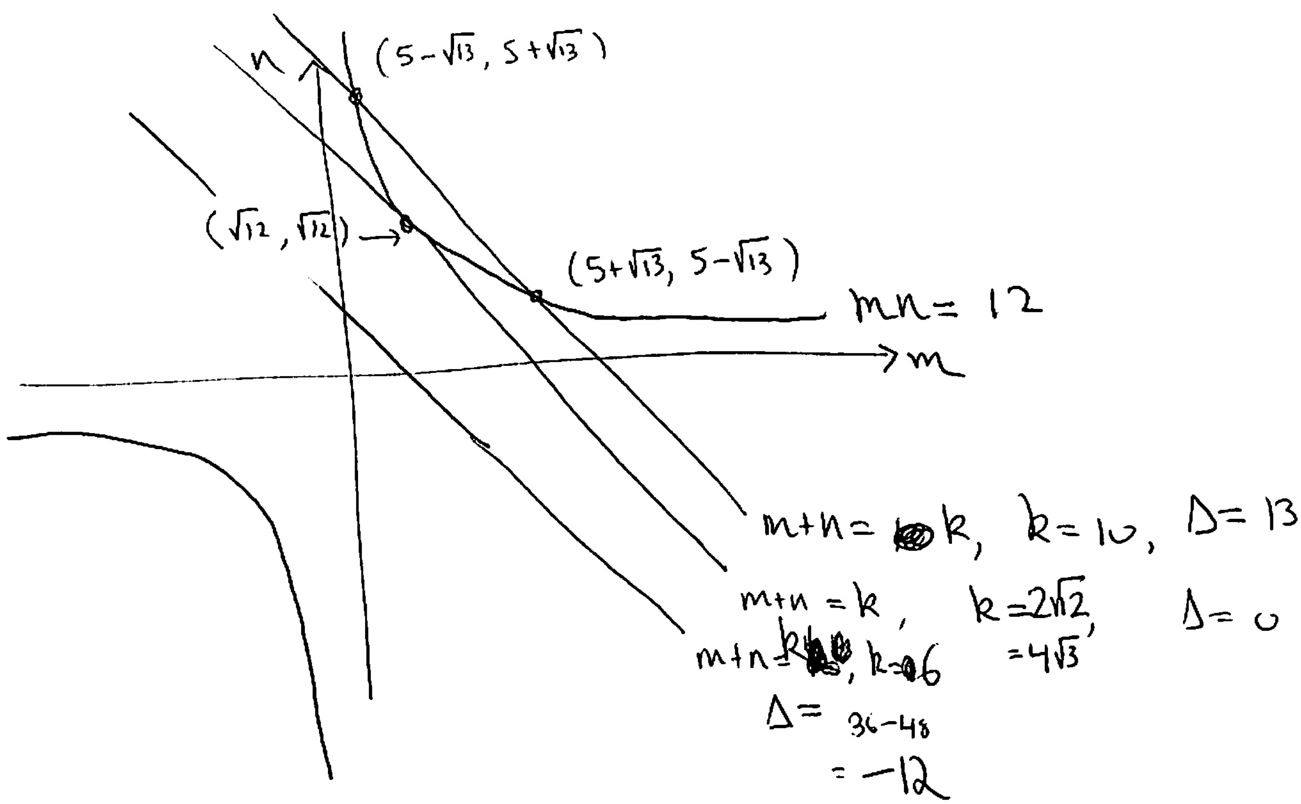
$$x = \frac{-\frac{b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - 4c}}{2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<u>Def:</u> $b^2 - 4ac$ is called the <u>Discriminant</u> of $ax^2 + bx + c = 0$.

<u>Ex.</u> $x^2 + 3x + 5 = 0$

~~Thm~~ **Cor.**: If a, b, c are real, then $ax^2+bx+c=0$ has $\begin{cases} 2 \\ 1 \\ 0 \end{cases}$ real solutions according to $\begin{cases} b^2-4ac > 0 \\ b^2-4ac = 0 \\ b^2-4ac < 0 \end{cases}$

We can <u>see</u> this graphically



<u>Fact</u>: The quadratic formula remains true if we work with complex numbers, provided that we are careful.

$\mathbb{C} = \{ a+bi \mid a,b \in \mathbb{R} \}$

$(a+bi)+(c+di) = (a+c)+(b+d)i$

we multiply as usual (FOIL), ~~simplify~~ simplifying where possible by using $i^2=-1$.

$$(a+bi)(c+di) = ac+bdi^2+adi+bci$$
$$= (ac-bd)+(ad+bc)i$$

* $\mathbb{C},+,\times$ is a <u>field</u>.

? We must be careful! $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is not ~~true~~ always true if we allow complex numbers: $1=\sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i \cdot i$
$= i^2$
$= -1$

<u>Thm</u> (<u>Fundamental Thm of Algebra</u>): Any polynomial in x with complex coefficients has at least one root in $\mathbb{C}$.

$\Longrightarrow$ a degree n-polynomial ~~can be factored~~ has n roots (counted w/ mult) in $\mathbb{C}$.

<u>Ex</u>: Find $u \in \mathbb{C}$ such that $u^4 = -1$.