9/14/11 Problems: p163, #3

<u>407</u>

Q: What are some of the "rules" for solving eqns $f(x)=g(x)$? For each rule, give an example of it in use

Possible answers:

"exact" methods
- Isolate variable
- Combine like terms
- Add / subtract same expression from both sides
- Multiply / divide both sides by same expression
- Apply same fn to both sides

approximate methods
- Tables of values
- graphs
- Newton's method for finding roots.
- linear approx

Two Key Points: 1) The vast majority of equations "in life" can not be solved exactly, so we focus on ones that can.

2) The above methods don't always work, and must be applied with care!

Ex: Solve $\cos(x) = x$. (x in radians)

Try as we might, we can't isolate the var:

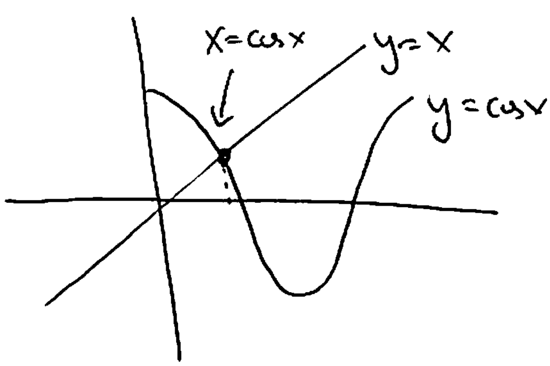
- $x = \cos^{-1}(x)$, $\left[-1 \le x \le 1 \text{ and } 0 \le x \le \pi\right]$

RHS only makes sense for $-1 \le x \le 1$!

- $x = \cos(x) = \cos(\cos(x)) = \cos(\cos(\cos(x)))\ldots$

To solve, we need to ~~either graph, or mo~~

- graph
- tables
- .....

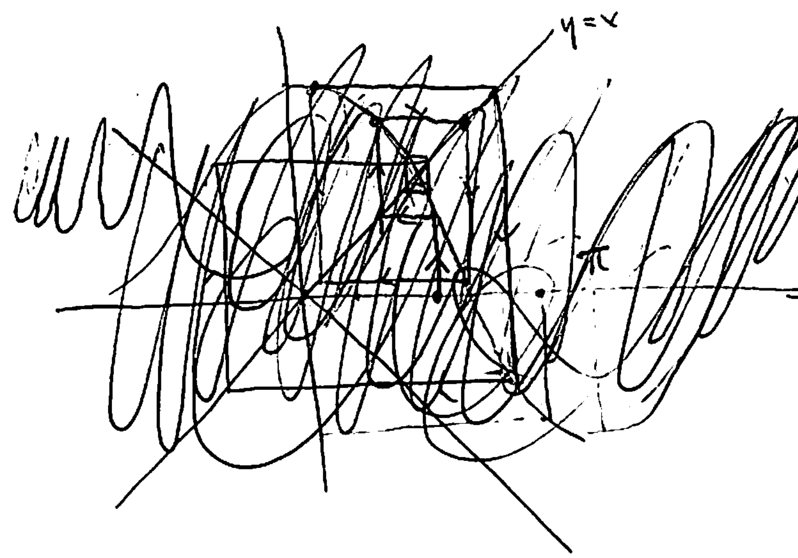


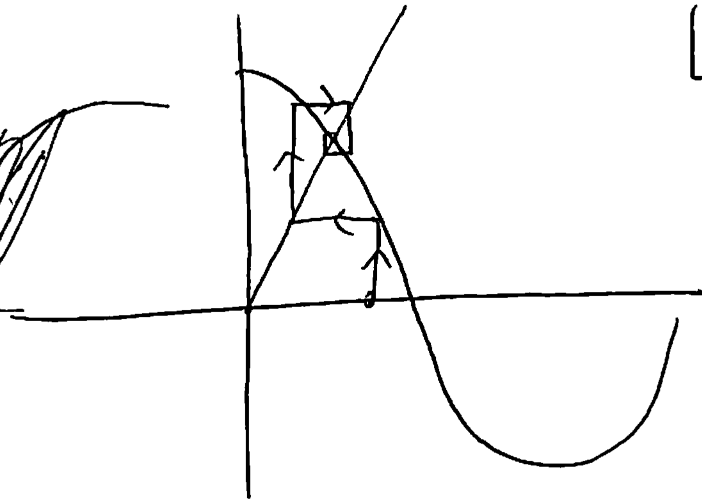
$x \approx 0.73908513321516$

Or. pick any $x \in (0,1)$. ~~Take $z = \cos(x)$~~

Then compute $\cos(\cos(\cos(\cos \ldots \cos(x))))$

Why does this work?





Ex: Find all real solutions to

$$x + \log(3-x) = 4 + \log(3-x)$$

Sol:

$$\begin{aligned} -\log(3-x) \quad & \quad -\log(3-x) \\ \hline x \quad = & \quad 4 \end{aligned}$$

But this is not a solution to original eqn, as $\log(-1)$ is not well defined!

Q: How to modify the "addition property of =" to be a correct statement?

Thm: Let f, g, h be functions. For all x in the intersection of the domains of f, g, h,

$$f(x) = g(x) \iff f(x) + h(x) = g(x) + h(x)$$

* "Solutions" can be gained or lost if we don't pay attention to domains:

LOST		GAINED	
one sol	$x = 4$	($x=-2$) one sol	$x^2 + \log(1-x) = 4 + \log(1-x)$
zero sol	$x + \log(3-x) = 4 + \log(3-x)$	two sol ($x = \pm 2$)	$x^2 = 4$

Ex: Solve

$$* \quad \frac{6t-2}{t^2-1} + \frac{t-3}{t-1} = 0$$

LOGICAL EQUIV is CRUCIAL!

$$\overset{\cancel{*}}{\Longleftarrow} (6t-2) + (t+1)(t-3) = 0$$

$$\Longleftrightarrow 6t-2+t^2-2t-3 = 0$$

$$\Longleftrightarrow t^2+4t-5 = 0$$

$$\Longleftrightarrow (t+5)(t-1) = 0$$

$$\Longleftrightarrow t=-5 \text{ or } t=1$$

But $t=1$ is NOT a solution to *!

<u>Thm</u> (mult. prop of equality): Let f, g, h be functions.

For all x in the intersection of the domains of f, g, h, if $h(x) \neq 0$ then

$$f(x) = g(x) \iff f(x)h(x) = g(x)h(x).$$

In <u>Ex</u> above, $f(x) = \dfrac{6t-2}{t^2-1} + \dfrac{t-3}{t-1}$, $g(x) = 0$

$$h(x) = t^2 - 1$$

$$\begin{aligned}
\text{Domain}(f) &= (-\infty, -1) \cup (-1, 1) \cup (1, \infty) \\
\text{Domain}(g) &= (-\infty, \infty) \\
\text{Domain}(h) &= (-\infty, -\infty)
\end{aligned}$$

Intersection = all real x, except $x = \pm 1$.

<u>Example</u> If $1 = 2$ then $a = b$ for all real a, b.

<u>pf.</u>

$$\begin{aligned}
1 = 2 &\iff 0 = 1 \\
&\implies a \cdot 0 = a \cdot 1 \quad \forall a \\
&\implies 0 = a \quad \forall a.
\end{aligned}$$

Similarly, $0 = b \quad \forall b$. Hence $b = 0 = a \quad \forall a, b$.

<u>Solving eqns as motivation for number systems</u>

$$\mathbb{N} = \{1, 2, 3 \ldots\} \xrightarrow{x+n=1} \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

$$\xrightarrow{ax=b} \mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, b \neq 0,\ a, b \in \mathbb{Z} \right\}$$

$$\xrightarrow{x^2=2,\ \text{etc.}} \mathbb{R} \xrightarrow{x^2=-1} \mathbb{C}.$$

The Fundamental Thm of Alg says "the buck stops here".

<u>Thm</u> (FTA) Every ~~polynom~~ non-constant polynomial in x with coeffs in $\mathbb{C}$ has a root in $\mathbb{C}$.

$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$

<u>Ex</u>: Find $z \in \mathbb{C}$ st $z^4 = -1$.

<u>Sol</u>: $z^4 = -1 \iff z^4 + 1 = 0$

$\iff (z^2)^2 - (-1) = 0$

$\iff (z^2+i)(z^2-i) = 0$

$\iff z^2 = i$ or $z^2 = -i$

We seek to solve, for $z = a+bi$, $a,b \in \mathbb{R}$,

$(a+bi)^2 = i$

$a^2 + 2abi - b^2 = i \implies a = b, \quad 2ab = 1$

So $2a^2 = 1$, $a = \pm\sqrt{\tfrac{1}{2}} = b$

$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i, \quad -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$ or

$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i, \quad -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$