9/19/11 Problems: p 166 #2 1

407

Often, you will recognize that an eqn has the form

$$h(f(x)) = h(g(x))$$

for functions f, g, h. It is tempting to solve such eqns by applying h^{-1} and then solving $f(x) = g(x)$. However, we must be careful!

Examples: (1) $\sqrt{3x-2} = -\sqrt{4x-3}$

Squaring both sides gives

$$3x-2 = 4x-3$$

$$\iff 3-2 = 4x-3x$$

$$\iff 1 = x$$

But $x=1$ is NOT a solution of (1) because $\sqrt{1} \neq -\sqrt{1}$. In fact, (1) has NO solutions!

(2) $\cos^{-1}(x) = \cos^{-1}(x) + \pi$

Applying cos to both sides:

$$\cos(\cos^{-1}(x)) = \cos(\cos^{-1}(x-2) + \pi)$$

$$= \cos(\cos^{-1}(x-2))\cos(\pi) - \sin(\cos^{-1}(x))\sin(\pi)$$

$$x = 2 - x$$

So $x = 1$

But $\cos^{-1}(1) \neq \cos^{-1}(-1) + \pi$

because $\cos^{-1}(1) = 0$, $\cos^{-1}(-1) = \pi$ and $0 \neq 2\pi$.

Sometimes, things work just fine:

(3) $\sqrt[3]{3x-2} = -\sqrt[3]{4x-3}$

$$(3x-2) = -(4x-3)$$

$$7x = 5$$

$$x = 5/7$$

check $\sqrt[3]{3(\frac{5}{7})-2} \stackrel{?}{=} -\sqrt[3]{4(\frac{5}{7})-3}$

$\sqrt[3]{\frac{15-14}{7}} \stackrel{?}{=} -\sqrt[3]{\frac{20-21}{7}} \leadsto \sqrt[3]{\frac{1}{7}} \stackrel{?}{=} -\sqrt[3]{-\frac{1}{7}}$

YES!

Q: What is going on?

→ Group discussion. Suggestions: • How are the graphs of $y = x^2$ and $y = x^3$ fundamentally different? • What are the domain and range of $y = \cos^{-1}(x)$?

Recall: A function $h(x)$ is said to be "1-1" (one-to-one) if:

$$\bullet \quad h(x) = h(y) \iff x = y$$

i.e. if h "passes the horizontal line test".

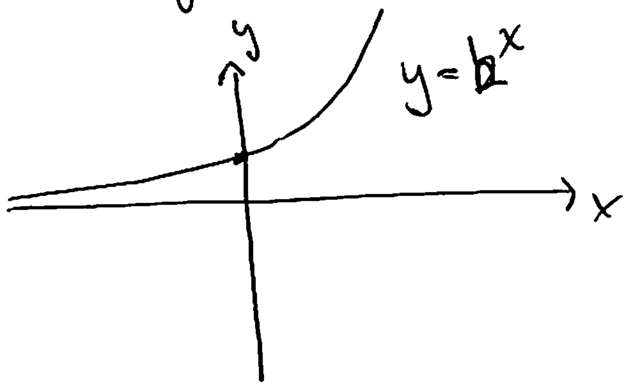
Thm: Let $h(x)$ be a 1-1 function. For all x in the domains of f and g for which $f(x)$, $g(x)$ are in the domain of h,

$$f(x) = g(x) \iff h(f(x)) = h(g(x))$$

proof: $\Rightarrow$ is easy, and just uses that h is a function

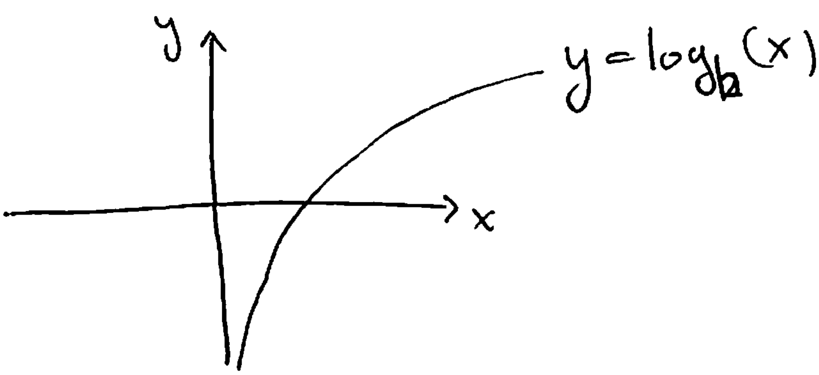
$\Leftarrow$: Since h is 1-1, $h(f(x)) = h(g(x)) \Rightarrow f(x) = g(x)$.

Two "classic" 1-1 functions are exponentials and logarithms. Let $b > 1$ be any real #



domain: $(-\infty, \infty)$

range: $(0, \infty)$



domain: $(0, \infty)$

range: $(-\infty, \infty)$

Moreover,

$$b^{\log_b x} = x \quad \text{for all } x \in (0, \infty)$$

$$\log_b(b^x) = x \quad \text{for all } x \in (-\infty, \infty)$$

<u>So</u> when solving eqns, we can safely apply logs & exponents wherever they make sense.

(we'll review logs & exponentials in §3.2 in just a few weeks)

Ex: Solve

(1) $2\log_2(x+3) = \log_2(10x+14)$

$$\longrightarrow \quad 2^{2\log_2(x+3)} = 2^{\log_2(10x+14)}$$

$$(x+3)^2 = 10x+14$$

$$x^2+6x+9 = 10x+14$$

$$x^2-4x-5 = 0$$

$$(x-5)(x+1) = 0$$

$$x = 5 \quad \text{or} \quad x = -1 \quad \checkmark$$

(2) Solve

$$10^{x^2-2x} = \frac{1}{10^x}$$

$$\longrightarrow \quad \log_{10}\left(10^{x^2-2x}\right) = \log_{10}\left(\frac{1}{10^x}\right)$$

$$x^2-2x = -x$$

$$x^2-x = 0$$

$$x(x-1) = 0$$

$$x=0 \quad \text{or} \quad x=1 \quad \checkmark$$

#5, p166:

$L :=$ linear functions on $\mathbb{R}$

$= \{ f(x) = ax+b \mid a, b \in \mathbb{R},\ a \neq 0 \}$

~~the composition is a binary~~

We can compose two linear functions:

$f(x) = ax+b$
$g(x) = cx+d$

$$f \circ g\,(x) = f(g(x)) = a(cx+d)+b = (ac)x + (ad+b)$$

it is again a linear function, so

- L is closed under $\circ$
- $\circ$ is associative, since function composition is
- $id(x) = x$ is the identity for $\circ$ because $id \circ f = f \circ id = f \quad \forall f \in L$
- The inverse of $f(x) = ax+b$ is $f^{-1}(x) = \frac{1}{a}x - \frac{b}{a}$ since

$f(f^{-1}(x)) = a\left(\frac{1}{a}x - \frac{b}{a}\right)+b = x$

$f^{-1}(f(x)) = \frac{1}{a}(ax+b) - b/a = x$.

Solve
$5x+14 = 4$

inverse of $5x+14$ is $\frac{1}{5}x - \frac{14}{5}$

so $x = \frac{1}{5}(4) - \frac{14}{5} = -\frac{10}{5} = -2$