9/21/11
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Problems: p. 176-177, # 4, 11

Problem A On a round trip, you average 30mph on the way out and 60mph on the way back. What is your average speed for the whole trip?

Problem B On a trip, I average 30 mph for a certain time, then you drive and average 60mph for the same time. What is the average speed for the total time we have driven?

Present Solution:

Possible Sols: A) d = total distance.

$$\text{ave Speed} = \frac{\text{distance}}{\text{time}} \quad \text{or} \quad \text{time} = \frac{\text{distance}}{\text{ave speed}}$$

So: time at 30mph $= \frac{d}{30}$ total distance $= 2d$

time at 60 mph $= \frac{d}{60}$ total time $= \frac{d}{30} + \frac{d}{60}$

$$\text{total average speed} = \frac{\text{total dist}}{\text{total time}} = \frac{2d}{\frac{d}{30} + \frac{d}{60}} = \frac{2}{\frac{1}{30} + \frac{1}{60}} = \frac{2 \cdot 60}{3}$$

$$= 40 \text{mph}$$

This makes sense: We spend twice as much time traveling at 30 mph than at 60, and

$$40 = \frac{2 \cdot 30 + 60}{3} \quad \text{(weighted average)}$$

B) ave speed = $\frac{\text{dist}}{\text{time}}$ Spse we each time for t-hours.

dist. traveled at 30mph = $30 \cdot t$

60mph = $60t$

total dist: $30t + 60t$ } ttl. ave speed = $\frac{\text{ttl dist}}{\text{ttl time}} = \frac{30t + 60t}{2t}$

total time: $2t$

$= \frac{30+60}{2}$

$= 45$ mph

Generalize problem A, B as follows.

A) Same problem, but with speeds v and w

B) ——— ———

Then, for A), we get (assuming distance d)

time at v mph = $\frac{d}{v}$ } ttl. time = $\frac{d}{v} + \frac{d}{w}$

time at w mph = $\frac{d}{w}$

ttl dist = $2d$.

total Ave speed = $\frac{\text{ttl dist}}{\text{ttl time}} = \frac{2d}{\frac{d}{v} + \frac{d}{w}} = \frac{2}{\frac{1}{v} + \frac{1}{w}}$

For B), we get total ave speed = $\frac{v+w}{2}$.

Def: The arithmetic mean of v, w is

$$A(v,w) := \frac{v+w}{2}$$

The harmonic mean of v, w is (terminology comes from music)

$$H(v,w) = \frac{2}{\frac{1}{v}+\frac{1}{w}}$$

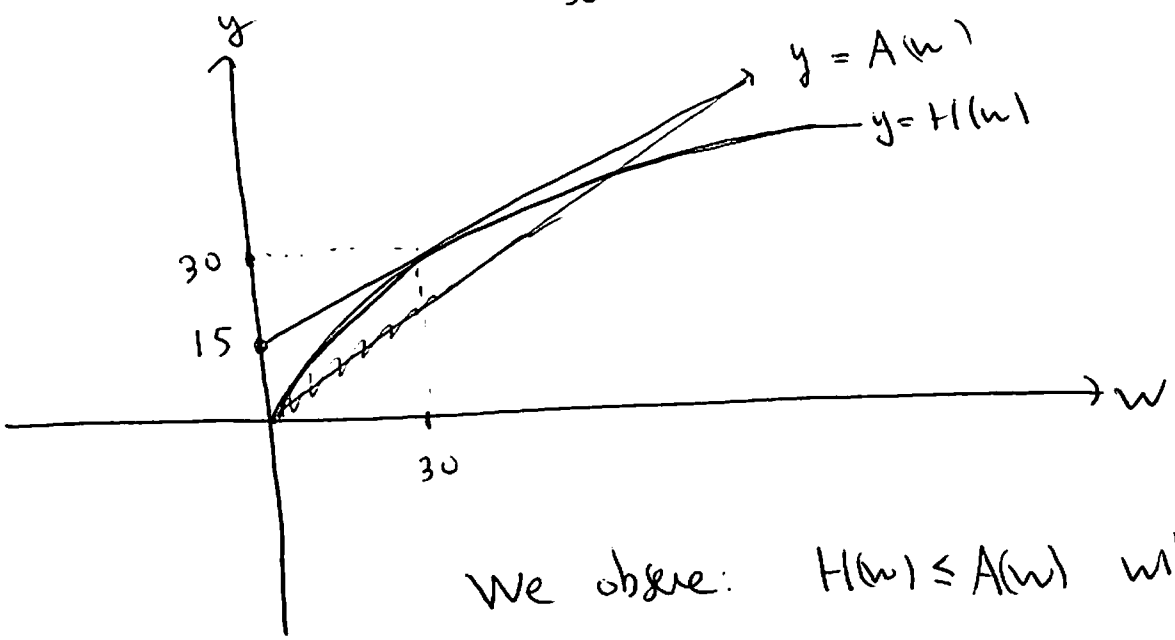
Observe: $$\frac{1}{H(v,w)} = \frac{\frac{1}{v}+\frac{1}{w}}{2} = A\left(\frac{1}{v},\frac{1}{w}\right)$$

Comparison of Harmonic & Arithmetic means

Fix $v=30$, say.

$$A(w) = A(v,w) = \frac{1}{2}(30+w) = 15+\frac{w}{2}$$

$$H(w) := H(v,w) = \frac{2}{\frac{1}{30}+\frac{1}{w}} = \frac{60w}{w+30}$$



We observe: $H(w) \le A(w)$ with $=$ iff $w=30$.

Thm: For any $v, w > 0$ with $v \neq w > 0$ we have

- $H(v,w) \leq$ [illegible] $2w$ [illegible] .

pf. Since $v, w > 0$ we get [illegible]

so $\frac{1}{v} + \frac{1}{w} \geq$ [illegible] $\frac{1}{w}$ [illegible]

Taking reciprocals,

$\frac{1}{\frac{1}{v}+\frac{1}{w}}$ [illegible] $\leq$ [illegible] w so [illegible] $H(v,w) = \frac{2}{\frac{1}{v}+\frac{1}{w}} \leq 2w$ [illegible]

- $H(v,w) \leq A(v,w)$ w/ $=$ iff $v = w$

pf: $(v-w)^2 \geq 0$ with $=$ iff $v = w$

$(v+w)^2 \geq 4vw$ with $=$ iff $v = w$

$A(v,w) = \frac{v+w}{2} \geq \frac{2vw}{v+w} = \frac{2}{\frac{1}{v}+\frac{1}{w}} = H(v,w)$. ∎

The Harmonic Series

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

Notice $H(1, \frac{1}{3}) = \frac{2}{1+3} = \frac{1}{2}$

$H(\frac{1}{2}, \frac{1}{4}) = \frac{2}{2+4} = \frac{1}{3}$

$H(\frac{1}{3}, \frac{1}{5}) = \frac{2}{3+5} = \frac{1}{4}$ etc.