9/28
407

Functions

Q: What is a function?

- Each group comes up with a def. (5 mins)
- examples and non-examples for each def.

Def (modern standard): A function is a rule which assigns to each element of a set A exactly one element of a set B.

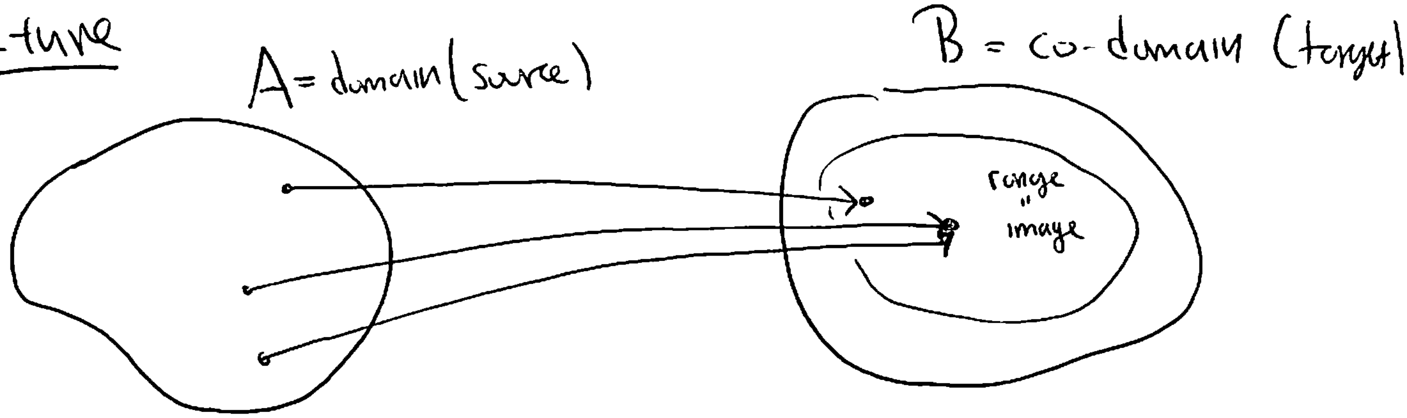
- We'll often use the letter "f" for function, and write

$$f : A \longrightarrow B$$

to signify that "f is a function from A to B"

- A = domain (~~[illegible]~~, source ...)
- B = codomain (target ...)
- The image of f, or the range of f is the subset of B consisting of elements ~~[illegible]~~ of B which are "outputs" of f.
- We write $f(a)$ "f of a" for $a \in A$ to denote the single element of B corresponding to a under f

Picture



- We will sometimes call a function f a "map" or "mapping", and when $f(a)=b$ will say "f maps a to b" and write $f: a \mapsto b$, or just $a \mapsto b$

- Def: f is injective or one-to-one (into or) (1-1) if $f(a)=f(a') \Rightarrow a=a'$

- Def: f is surjective or onto if the co-domain of f and the range of f are equal. That is, for every $b \in B$, $\exists\, a \in A$ s.t. $f(a)=b$.

- Def: f is bijective if it is injective and surjective.

Examples

- Functions can be given by <u>formulas</u>
  - $f(x) = x^2$
  - $f(x) = \dfrac{x^2}{x^3 - x + 1}$
  - $f(x) = \sin(e^x)$

    $\uparrow$ "independent variable"
    "input variable"

- Functions can be given by <u>descriptions</u>
  - $f(n) =$ the n^{th} prime number $(n \in \mathbb{N})$
  - $f(n) =$ the ~~[illegible]~~ average yearly rainfall in Tucson, AZ in year $2000 + n$, $n \in \mathbb{N}$.

- Functions can be given by <u>tables</u>:

A = things I have tasted
$B = \{\text{yum}, \text{yuck}\}$

x	$f(x)$
1	7
2	-3
3	24
4	1

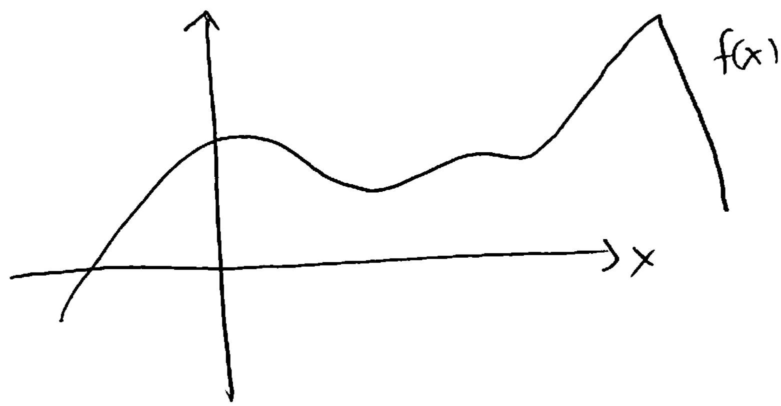
$a \in A$	$f(a) \in B$
apple	yum
pear	yum
orange	yum
bacon	yum
brocolli	yuck
dirt	yuck

- Functions can be given by "piecewise" defn

  - $f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$ $(|x|)$

  - $f(x) = x - \lfloor x \rfloor$, $\lfloor x \rfloor$ = the largest integer $\leq x$

- Functions can be specified by "graphs"



Def: Let $f: A \to B$ be a function. The <u>graph</u> of f, denoted $\Gamma(f)$, is the subset

$$\Gamma(f) := \{(a, f(a)) : a \in A\} \subseteq A \times B$$

where $A \times B$ = cartesian prod = set of all <u>ordered</u> pairs (a,b) where $a \in A$, $b \in B$

Q: Why is this called the "graph" of f?

~~Def (Alternate):~~

Notice that the map

$$pr_A : A\times B \longrightarrow A$$
$$(a, b) \longmapsto a$$

is <u>bijective</u> on $\Gamma(f)$, i.e. that for each $a \in A$, there is <u>exactly one</u> element of $\Gamma(f)$ with first coordinate a.

<u>Def</u> (Alternate): A <u>function</u> $f : A \longrightarrow B$ is a subset S of $A\times B$ satisfying

$*$ The restriction of $pr_A : A\times B \longrightarrow A$ to S is bijective.

<u>Ex</u>: $A =$ set of all circles in $\mathbb{R}^2$
$B =$ set of all points in $\mathbb{R}^2$

$$S_1 = \{ (C, P) \in A\times B \mid P \text{ is the center of } C \}$$

This is a function, because every circle has a unique center! In fact $S_1 \longleftrightarrow f$, $f(C) =$ center of C
$f :$ Circles $\longrightarrow$ points.

$$S_2 = \{ (p, C) \in B \times A \mid p \text{ is the center of } C \}$$

NOT a function, because many circles have the same center! Ex $((0,0),\ x^2+y^2=1)$

$((0,0),\ x^2+y^2=9)$

$\downarrow pr_A \qquad \swarrow pr_A$

$(0,0)$

(pr_A not injective!)

Some wild examples.

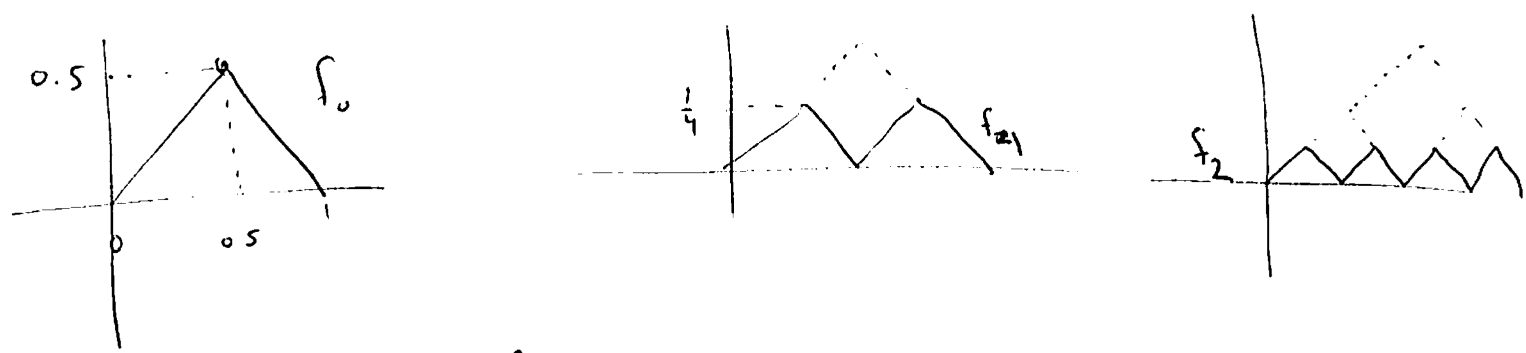
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$ is nowhere continuous!

Define $f_k : [0,1] \to [0,1]$

$$f_0(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

$$f_k(x) = \frac{1}{2^k} f_0(2^k x)$$



$$f(x) = \sum_{k=0}^{\infty} f_k(x)$$ is cts but nowhere differentiable!