Q: When does $a + x = b$ have a sol? ($a, b \in S$ = a set with $+$)

solving for.

eqn	S: $\mathbb{R}$	$\mathbb{Z}$	$\mathbb{R}_{>0}$	even integers
$2+x=3$	✓	✓	✓	✗
$2+x=1$	✓	✓	✗	✗
$2(1-3x)=3(1-2x)$	✗	✗	✗	✗
$3x=2$	✓	✗	✓	✗
$2-3x=3$	✓	✗	✗	✗

Analogy $(\mathbb{R},+) \longleftrightarrow (\mathbb{R}_{>0},\times)$

$$a+b \longleftrightarrow a\cdot b$$

Via this correspondence

$$a+x=b \longleftrightarrow ax=b$$

So both eqns are of the form

$$a*x=b\ ,\quad *=+ \text{ or } \times$$

$+$	$\times$	$*$	props
$a+x=b$	$ax=b$	$a*x=b$	
$-a+(a+x)$ $=-a+b$	$\frac{1}{a}(ax)$ $=\frac{1}{a}b$	$a^{-1}*(a*x)$ $=a^{-1}*b$	existence of inverses
$(-a+a)+x$ $=-a+b$	$\left(\frac{a}{a}\right)x=$ $\frac{1}{a}b$	$(a^{-1}*a)*x$ $=a^{-1}*b$	assoc
$0+x$ $=-a+b$	$1\cdot x=$ $\frac{1}{a}b$	$I*x=a^{-1}*b$	prop of inverses
$x=-a+b$	$x=\frac{b}{a}$	$x=a^{-1}*b$ identity.	prop of idenity

If S is a set with a binary operation $*$, then $a*x=b$ has a unique solution in S provided:

1) S is closed under $*$ ($c*d \in S$ if $c,d \in S$)
2) $*$ is associative: $a*(b*c) = (a*b)*c$
3) $\exists\, I \in S$ such that $I*a = a*I = a$ for all $a \in S$
4) $\forall\, a \in S$, $\exists\, a^{-1} \in S$ s.t. $a*a^{-1} = a^{-1}*a = I$

Def: Any $(S,*)$ satisfying 1–4 is called a group.

Ex: $S=\mathbb{R}$, $*=+$

$S=\mathbb{Z}$, $*=+$

$S=\mathbb{R}_{>0}$, $*=\times$

$S=\mathbb{R}-\{0\}$, $*=\times$

$S=$ even integers, $*=+$

Non-ex

$S=\mathbb{N}=\{1,2,3,\ldots\}$

$*=\times$

$S=\mathbb{R}$, $*=\times$

0 has no inverse

$S=$ odd integers

$*=+$

Non-standard examples:

- $S = \{0\}$, $*$ is defined by
$$0 * 0 = 0$$

- $\mathbb{Z}/n\mathbb{Z} := \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-1}\}$

"$\mathbb{Z}$ mod $n\mathbb{Z}$"

$\bar{a} + \bar{b} := \overline{\{\text{The remainder when } a+b \text{ is divided by } n\}}$

Ex $\mathbb{Z}/3\mathbb{Z} := \{\bar{0}, \bar{1}, \bar{2}\}$

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

Ex $\mathbb{Z}/12\mathbb{Z}$ $=$ clock

$\mathbb{Z}/2\mathbb{Z} := \{\bar{0}, \bar{1}\}$

$+$	$\bar{0}$	$\bar{1}$
$\bar{0}$	$\bar{0}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	$\bar{0}$

$\bar{1} + \bar{1} = \bar{0}$

$1 + 1 = 0$

Linear eqns

$$a \cdot x + b = c$$

Q: What arithmetic properties do you need to solve this eqn?

- * 1-4 for $+$
- * 1-4 for $\times$

* distributivity of $\times$ over $+$ (not totally necessary)

However, need dist. for $ax+b=cx+d$

Def: A set F together with two binary ops $+, \times$ which satisfy

1) $(F,+)$ is a commutative group
2) $(F-0, \times)$ is a commutative group
3) $\times$ distributes over $+$

is called a field.

Ex: $\mathbb{R}, +, \times$

$\mathbb{Q}, +, \times$

$\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$

$+$	$\bar{0}$	$\bar{1}$
$\bar{0}$	$\bar{0}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	$\bar{0}$

$\times$	$\bar{0}$	$\bar{1}$
$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$

Non Ex: $\mathbb{Z}, +, \times$

Ex  F = a field

a group

$$GL_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a,b,c,d \in F \\ ad-bc \neq 0 \end{array} \right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} e & f \\ g & h \end{pmatrix} := \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$