Correspondences, Integral Structures, and Compatibilities in p-adic Cohomology

by

Bryden R. Cais

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2007

Doctoral Committee:

Professor Brian Conrad, Chair
Professor Karen Smith
Professor Stephen Debacker
Professor Ralph Lydic

When the stars threw down their spears,
And watered heaven with their tears:
Did he smile his work to see?
Did he who made the Lamb make thee?

—WIlliam Blake, *The Tyger*

ACKNOWLEDGEMENTS

It is a pleasure to thank Robert Coleman, Johan de Jong, Kiran Kedlaya, Barry Mazur, Bill Messing, Martin Olsson, Karen Smith, and Jason Starr for many helpful exchanges and conversations. I heartily thank Karen Smith for several enlightening discussions, for her interest in my thesis, and for agreeing to read the final draft. I owe a particular debt of gratitude to Benedict Gross, both for his paper [22] that motivated this thesis, and for his encouragement over the course of my mathematical career. I would also like to thank Stephen Debacker and Ralph Lydic for agreeing to be on my thesis committe.

Many special thanks go to my family for their unwavering encouragement, and to Mark for his continued friendship and support.

It is impossible to measure the debt I owe to my advisor, Brian Conrad. Over the past five years he has been an indefatigable source of encouragement, wisdom, and inspiration, and his help and patient guidance have been invaluable in the writing of this thesis.

TABLE OF CONTENTS

LIST OF APPENDICES

Appendix

INSTRUCTIONS FOR THE READER

This thesis deals with two different problems. The first is that of equipping the Hodge filtration of a smooth curve over K with a canonical integral structure that is functorial in pullback and trace by finite K-morphisms of smooth curves. The second is that of showing that the various cohomology theories one can attach to a smooth curve over K are compatible with the action of correspondences on the curve and the Frobenius endomorphism (when it is defined). The first problem is dealt with in sections 2–3 and 7–9, while the second is the main subject of sections 4–6. Section 10 combines the results of these sections to resolve the unchecked compatibilities in Gross' work [22]. For the most part, §4–5 contain no new results, and exist as a reference for later applications. These sections are necessarily quite technical (and are supplemented by the two appendices). Section 6 contains some new results, most notably a "theory of correspondences in p-adic cohomology" (cf. the compatibility issue of [22, 16.6]). Section 2 is a review of facts about the Hodge filtration of a smooth curve over a field of characteristic zero. The bulk of our contribution is in sections 3 and 6–10. The reader primarily interested in the work we have done in this thesis should therefore skim §4–5 and the appendices as needed (or simply take the contents thereof on faith) and concentrate on reading sections 2–3 and 6–10 (in that order).

CHAPTER I

Introduction

Broadly speaking, this thesis is concerned with the cohomological study of smooth curves over a p-adic field K of characteristic zero. Our central object of study is the Hodge filtration of the first de Rham cohomology group (over K) of such a curve, and the endomorphisms of it induced by correspondences on the curve.

The Hodge filtration is an indispensable tool in the study of many questions of arithmetic interest, and in particular the study of modular forms. Due to the geometric description of modular forms as global sections of certain vector bundles with integrable connection on modular curves (or Shimura curves), the Hodge filtration (and its analogues with coefficients in a vector bundle with integrable connection) allows one to interpret modular forms as (de Rham) cohomology classes. Via comparison isomorphisms with other cohomology theories, one can bring to bear many powerful *cohomological* tools on the study of modular forms. For example, Deligne's construction associating certain Galois representations to modular forms rests on this line of thinking. Although many of our results undoubtedly generalize, we will stick to the case of "constant coefficients" in this thesis (which provides enough framework to study mod p modular forms of arbitrary weight and level).

The p-adic setting seems to combine aspects of the classical analytic "characteristic zero" theory over $\mathbf{C}$, and genuinely arithmetic "mod p" phenomena. Moreover, translating between these two very different worlds requires a theory at the "integral level" (i.e. over the ring of integers R of K). Essentially, such a trichotomy arises because any geometric object over K (like a smooth curve) has "incarnations" over R, as well as over the residue field k of R. These "incarnations" are certainly not unique, but often their cohomological properties are sufficiently independent of choices that one can extract deep arithmetic information about the original situation over K by combining the three prespectives. Accordingly, our investigations are tripartate. We will study three types of cohomology theories that can be attached to a smooth curve over K, and the relationships between them. We are interested in the compatibilities between these different theories, especially as regards endmorphisms induced by correspondences.

The single greatest impetus for this work comes from Gross' beautiful paper on Galois representations and companion forms [22], in which he employs the different theories we have alluded to in the case of modular curves to prove Serre's "modular criterion" for the splitting of the local Galois representation attached to a mod p modular form. In the introduction to [22], Gross writes:

> The proof of Serre's conjecture on companion forms uses p-adic techniques, and specifically the different p-adic cohomology theories (de Rham, crystalline, Washnitzer-Monsky) of modular curves and their Jacobians. Here we confess that we have occasionally used rather artificial methods for defining the action of Hecke operators on these cohomology groups, and have not always checked that the actions are compatible with isomorphisms between the theories. In particular, the assertions preceding (15.4), (15.7), and (16.7) depend on an unchecked compatibility.

As a consequence of our work, we resolve this "unchecked compatibility." In addition, we provide a reference for several results that play a key role in Gross' work that we have been unable to find in the literature (most notably the "integral" results of [22, §15]).

Throughout, our hypotheses and methods are kept as general as possible, so that our results will apply equally to modular curves as they will to other curves of arithmetic interest (e.g. Shimura curves).

We now give a detailed overview of the contents of this thesis. Fix a discrete valuation ring R with fraction field K and residue field k, and a proper smooth and geometrically connected curve X_K over K. Our main object of interest is the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$

$$0 \longrightarrow H^0(X_K, \Omega^1_{X_K/K}) \longrightarrow H^1_{\mathrm{dR}}(X_K/K) \longrightarrow H^1(X_K, \mathscr{O}_{X_K}) \longrightarrow 0\ , \tag{1.0.1}$$

and we begin in chapter 2 by recalling the basic facts about this exact sequence of K-vector spaces: its construction and functoriality (via pullback and trace by finite morphisms of curves), cup product pairing and duality.

By an *integral structure* on (1.0.1), we mean a short exact sequence of free R-modules that recovers (1.0.1) after extending scalars to K. The goal of chapter 3 is to equip (1.0.1) with a *canonical* integral structure, i.e. one that is functorial in both pullback and trace by finite K-morphisms of X_K, hence stable under any endomorphism of (1.0.1) that is induced by a correspondence on the curve X_K. When X_K has good reduction over R, the Hodge filtration of the de Rham cohomology of a smooth proper model over $S = \operatorname{Spec} R$ yields an integral structure, but it is not at all clear that it is functorial in K-morphisms of X_K, or that it is canonically independent of the chosen model over R. When X_K does not have good reduction, it is not even clear that one can define any integral structure geometrically, as the de Rham cohomology of a model will not be free over R. We study a broad class of curves X_K, namely those that have an *admissible model* over R (Definition 3.1.5). This class includes all smooth and proper curves that have a normal semistable model, so by the semistable reduction theorem, includes all smooth and proper curves if one allows extensions of K. Because we will ultimately study crystalline cohomology, which is highly sensitive to ramification (because of the need to have divided powers), we view such extension of scalars as undesirable, and so work with our more general models. We note that the class of curves over K having an admissible model over R includes (for suitable K) the modular curves studied in [22]; see Remark 3.1.12.

Given an admissible model X of X_K, we develop an "integral cohomology theory" that is a good replacement for algebraic de Rham cohomology when X is not smooth. Our theory rests on the construction of a subcomplex $\mathscr{O}_X \to \omega_{X/S}$ of the de Rham complex of X_K, where $\omega_{X/S}$ is the relative dualizing sheaf of X over S; see Theorem 3.4.1. Our main result in this chapter is Theorem 3.4.5, which states that if X_K has *some* admissible model over R, then there is a *canonical* integral structure on (1.0.1)—it is independent of the chosen admissible model and functorial via pullback and trace in finite K-morphisms of X_K. The proof of this theorem uses critically Grothendieck duality theory ([26], [15]) and the flattening techniques of Raynaud-Gruson [44]. As a nice application of this theory, one deduces that the Hecke operators act on "integral" cohomology—and hence have eigenvalues that are algebraic integers—without ever having to construct an integral theory of Hecke operators, which is a somewhat arduous and delicate task [13, Theorem 1.2.2].

In chapter 4, we review the definitions and functoriality properties of the three p-adic cohomology theories that we use: de Rham cohomology (of schemes, rigid spaces, and formal schemes), crystalline cohomology, and rigid cohomology. Let us note that we do not mention the cohomology of Monsky-Washnitzer, as it is basically superseded by rigid cohomology: indeed, a comparison isomorphism of Berthelot (Proposition 1.10 of [5]) shows that the two cohomology theories are functorially isomorphic when restricted to smooth affine schemes.

In chapter 5 we record the comparison isomorphisms between the cohomology theories that we use. Given a finite type scheme X over $S = \operatorname{Spec} R$, one can associate to X the scheme X_K over K (the generic fiber), the scheme X_0 over k (the closed fiber), the topologically finite type formal scheme $\widehat{X}$ over $\operatorname{Spf} R$ obtained by completing X along X_0, the rigid space (Raynaud generic fiber) $\widehat{X}^{\mathrm{rig}}$ over K, and the rigid space X_K^{an} (the rigid analytification of X_K) over K. To each of these geometric objects one can attach certain cohomology groups "of de Rham type", and we study the various maps between these groups. Some of these objects are not topological spaces in the usual sense (for example, a rigid space is not a topological space, nor is the crystalline site), and several of our comparison maps arise from morphisms of topoi (like sp) rather than morphisms of ringed spaces. This technical subtlety causes no problems, and is reviewed in Appendix A. Essential for later compatibility results is that crystalline cohomology and rigid cohomology are naturally isomorphic (Proposition 5.3.3), and we explain Berthelot's proof of this fact (see Proposition 1.9 of [5]).

In chapter 6, we turn to the problem of constructing a theory of correspondences in p-adic cohomology, as alluded to in [22], below (16.6). We begin by developing a theory of trace morphisms on differentials for finite flat maps of rigid spaces over K, and the corresponding morphisms in de Rham cohomology (Theorem 6.1.1). The construction is of local nature, but is rather more subtle than the corresponding construction for varieties (which is classical). We show how a construction of Monsky-Washnitzer can be adapted to our situation. We then show how this construction for rigid spaces yields a theory of trace morphisms in rigid cohomology. Since rigid cohomology is "of de Rham type" our task is almost straightforward. However, certain technical subtleties arise because of the direct limits involved in the definition of rigid cohomology; these issues are dealt with in Appendix B. We end this chapter by showing that the action of correspondences in p-adic cohomology that we define is compatible under canonical comparison isomorphisms with the action on the Hodge filtration of a curve over K; in particular, Theorem 6.2.2 settles the issue of the Hecke-compatibility of the map (16.6) in [22].

We will also study the first de Rham cohomology of the Jacobian J_K of a curve X_K over K and its corresponding Hodge filtration. Any correspondence on X_K gives an endomorphism of J_K via Albanese and Picard functoriality, and it is a classical fact (see, for example, [46]) that pullback along any Albanese morphism $X_K \to J_K$ identifies the Hodge filtrations of $H^1_{\mathrm{dR}}(J_K/K)$ and $H^1_{\mathrm{dR}}(X_K/K)$ compatibly with the actions of the correspondence. We review this fact in chapter 7.

In chapter 8 we explain how to provide the Hodge filtration of the first de Rham cohomology of an abelian variety over K with a canonical integral structure. Following [22, §15], we use the theory of the group functor $\mathscr{E}xtrig$ and canonical extensions of Néron models, as developed in [35, I,§5]. Applying these results to J_K and using the identification of the Hodge filtrations of $H^1_{\mathrm{dR}}(X_K/K)$ and $H^1_{\mathrm{dR}}(J_K/K)$ as discussed in chapter 7, we thus obtain *two* canonical integral structures, and we wish to compare them. Our main theorem here is Theorem 8.3.1, which states that when X_K has an admissible model with generically smooth closed fiber, these two integral structures are naturally identified. We remark that this hypothesis is in particular satisfied by the admissible models of modular curves used in [22].

We believe that our comparison result in Theorem 8.3.1, which we prove in chapter 9 and which we have been unable to find in the literature, holds in the full generality of the existence of an admissible model, but our methods do not work in this generality. One would like to extend the theory of Raynaud [43], which identifies $\mathrm{Pic}^0_{X/S}$ with the identity component of the Néron model of of J_K for any admissible model X of X_K, to canonical extensions. To do this would would require a functorial characterization of the canonical extension of a Néron model, a question posed (and as far as the author knows, as yet unanswered) in [35, I, §5].

In chapter 10, we prove several main compatibility theorems. We begin with Grothendieck's isomorphism. Given an abelian variety A over a perfect field k of characteristic p, this is an isomorphism between $H^1_{\mathrm{cris}}(A/W(k))$ and the Dieudonné module $\mathbf{D}(A[p^\infty])$ of the p-divisible group of A. The issue of whether or not this isomorphism is functorial in A is raised in [22] as a primary concern. However, Grothendieck's isomorphism *is* natural in A. Indeed, the work of Mazur-Messing [35] shows that Grothendieck's isomorphism is a canonical *isomorphism of functors.* When applied to the modular curve $X_1(Np)$ over $\mathbf{Q}_p(\zeta_p)$, our compatibility theorems yield the three Theorems in [22] that depended on "unchecked compatibilities." We explain more precisely how to deduce Gross' results from ours. One of the main issues is the Frobenius compatibility of the map (10.2.9), which is proved in Theorem 10.2.3. It is here that we use Berthelot's isomorphism between rigid and crystalline cohomology (Proposition 5.3.3). We remark that there is no comparison isomorphism between crystalline cohomology and Monsky-Washnitzer cohomology (as one theory is adapted to the case of smooth and proper schemes, while the other deals with smooth affine schemes), so the intervention of a third cohomology theory that "interpolates" these two is not only natural, but essential.

CHAPTER II

The Hodge filtration of a curve

Fix a field K of characteristic zero. We will later be especially interested in the case that K is a nonarchimedean field, but for now we do not require this. In this chapter, will briefly review some basic facts about our main object of study in this thesis: the Hodge filtration of the first de Rham cohomology of a curve over K.

2.1 The filtration

Let X be a smooth proper and geometrically connected curve X over K, and denote by $\Omega^{\bullet}_{X/K}$ the (algebraic) de Rham complex of X over K (cf. §4.1). We define the *de Rham cohomology* of X over K to be the hypercohomology of the de Rham complex:

$$H^{\bullet}_{\mathrm{dR}}(X/K) := \mathbf{H}^{\bullet}(X, \Omega^{\bullet}_{X/K}).$$

Since X is smooth and proper, for each integer i the abelian groups $H^i_{\mathrm{dR}}(X/K)$ are finite dimensional K-vector spaces, and since X is one-dimensional, we have $H^i_{\mathrm{dR}}(X/K) = 0$ for $i > 2$. We can non-algebraically determine the dimension of each $H^i_{\mathrm{dR}}(X/K)$ for $0 \leq i \leq 2$ as follows (an algebraic method will be explained in Remark 2.1.2). Since X is of finite type over K, we may assume that K is finitely generated over $\mathbf{Q}$. Since hypercohomology commutes with flat base change, corresponding to a choice of embedding of K into the complex numbers $\mathbf{C}$ we have a (non canonical) isomorphism of $\mathbf{C}$-vector spaces

$$H^i_{\mathrm{dR}}(X/K)\otimes_K \mathbf{C} \simeq H^i_{\mathrm{dR}}(X_{\mathbf{C}}/\mathbf{C}),$$

where $X_{\mathbf{C}}$ denotes the base change of X to $\mathbf{C}$ (via our choice of embedding $K \hookrightarrow \mathbf{C}$). Denoting by $X^{\mathrm{an}}_{\mathbf{C}}$ the complex analytic space attached to $X_{\mathbf{C}}$, the canonical morphism of ringed spaces

$$X^{\mathrm{an}}_{\mathbf{C}} \to X_{\mathbf{C}}$$

induces a functorial map

$$H^i_{\mathrm{dR}}(X_{\mathbf{C}}/\mathbf{C}) \to H^i_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbf{C}}/\mathbf{C}). \tag{2.1.1}$$

Using the Hodge to de Rham spectral sequence (which we will review in a moment) and GAGA ($X_{\mathbf{C}}$ is proper), it is not hard to show that (2.1.1) is an isomorphism. In fact, as Grothendieck showed, (2.1.1) is an isomorphism even when X is not proper [23, Theorem 1′], though we will not need this.

Now the Poincaré lemma ensures that the canonical map

$$\mathbf{C} \to \Omega^{\bullet}_{X^{\mathrm{an}}_{\mathbf{C}}/\mathbf{C}}$$

is a *resolution* of the constant sheaf $\mathbf{C}$, and so we have a (non canonical) functorial isomorphism

$$H^i_{\mathrm{dR}}(X/K)\otimes_K \mathbf{C} \simeq H^i(X^{\mathrm{an}}_{\mathbf{C}}, \mathbf{C}), \tag{2.1.2}$$

from which we conclude that the dimension of the K-vector space $H^i_{\mathrm{dR}}(X/K)$ is that of the usual complex cohomology $H^i(X^{\mathrm{an}}_{\mathbf{C}}, \mathbf{C})$.

The de Rham complex of X over K has an evident filtration by subcomplexes $\tau_{\geq n}\left(\Omega^{\bullet}_{X/K}\right)$ (“la filtration bête” [17]). Associated to this filtration is a first quadrant spectral sequence (the Hodge to de Rham spectral sequence) that "computes" the de Rham cohomology of X over K:

$$E_1^{p,q} = H^p(X, \Omega^q_{X/K}) \implies \mathbf{H}^{p+q}(X, \Omega^\bullet_{X/K}) = H^{p+q}_{\mathrm{dR}}(X/K). \tag{2.1.3}$$

Theorem 2.1.1. *Let X be a smooth proper and geometrically connected curve over a field K of characteristic zero. The spectral sequence* (2.1.3) *degenerates at the E_1-stage, and there is a short exact sequence of finite-dimensional K-vector spaces*

$$0 \longrightarrow H^0(X, \Omega^1_{X/K}) \longrightarrow H^1_{\mathrm{dR}}(X/K) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow 0\,. \tag{2.1.4}$$

Proof. This is classical, but we outline a proof (using that K has characteristic zero). Since the edge map map

$$d : H^0(X, \mathscr{O}_X) \to H^0(X, \Omega^1_{X/K})$$

induced by the differential $d : \mathscr{O}_X \to \Omega^1_{X/K}$ is clearly zero (as X is proper), we in any case have an exact sequence

$$0 \longrightarrow H^0(X, \Omega^1_{X/K}) \longrightarrow H^1_{\mathrm{dR}}(X/K) \longrightarrow H^1(X, \mathscr{O}_X) \tag{2.1.5}$$

Let $g = \dim_K H^0(X, \Omega^1_{X/K})$ be the (geometric) genus of X. Thanks to the isomorphism (2.1.2), we know that $H^1_{\mathrm{dR}}(X/K)$ has the same dimension as $H^1(X^{\mathrm{an}}_{\mathbf{C}}, \mathbf{C})$, which is just $2g$ as $X^{\mathrm{an}}_{\mathbf{C}}$ is a smooth and compact complex manifold of genus g. On the other hand, Serre duality gives a canonical isomorphism of K-vector spaces

$$H^0(X, \Omega^1_{X/K}) \simeq H^1(X, \mathscr{O}_X)^\vee,$$

so we know that $H^1(X, \mathscr{O}_X)$ has dimension g (where $(\cdot)^\vee$ denotes K-linear dual). Thus, dimension considerations show that the left exact sequence (2.1.5) must be right exact too, giving the exact sequence (2.1.4). Thus, we see that the edge map

$$H^1(X, \mathscr{O}_X) \to H^1(X, \Omega^1_{X/K})$$

induced by d is 0, which gives the claimed degeneration. ■

Remark 2.1.2. In fact, the Hodge to de Rham spectral sequence (which is a purely algebraic construction, and so exists for K of any characteristic) *always* degenerates at the E_1 stage when X is a smooth and proper curve over a field. Our argument of course does not work for K of characteristic $p > 0$, and one must appeal to the algebraic characteristic p methods of Deligne-Illusie [18]. Moreover, by using standard techniques to reduce problems in characteristic zero to problems in characteristic p, one can apply the results of [18] to give a purely algebraic proof of the degeneration of the Hodge to de Rham spectral sequence of a curve over a field of characteristic zero as well.

By a slight abuse of language, we will call the exact sequence (2.1.4) the *Hodge filtration* of $H^1_{\mathrm{dR}}(X/K)$, or even more liberally, just the Hodge filtration of X.

2.2 Cup product and duality

In this section, we recall the definition of cup product pairing on de Rham cohomology and prove that it induces a canonical autoduality of the Hodge filtration of a smooth curve.

For any ringed space $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ and bounded below complexes of $\mathscr{O}_{\mathscr{X}}$-modules $\mathscr{G}^\bullet$ and $\mathscr{F}^\bullet$, abstract nonsense with canonical flasque resolutions provides a natural $\Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$-linear morphism

$$\mathbf{H}^n(\mathscr{X}, \mathscr{F}^\bullet) \otimes_{\Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})} \mathbf{H}^m(\mathscr{X}, \mathscr{G}^\bullet) \longrightarrow \mathbf{H}^{n+m}(\mathscr{X}, \mathscr{F}^\bullet \otimes_{\mathscr{O}_{\mathscr{X}}} \mathscr{G}^\bullet)\,. \tag{2.2.1}$$

Applying this with $\mathscr{X} = X$ a smooth proper and geometrically connected curve over K and $\mathscr{G}^\bullet = \mathscr{F}^\bullet = \Omega^\bullet_{X/K}$, and composing with the natural map of complexes

$$\Omega^\bullet_{X/K} \otimes_{\mathscr{O}_X} \Omega^\bullet_{X/K} \to \Omega^\bullet_{X/K}$$

given by wedge product of differential forms, one obtains a natural K-linear cup product map

$$H^n_{\mathrm{dR}}(X/K) \otimes_K H^m_{\mathrm{dR}}(X/K) \to H^{n+m}_{\mathrm{dR}}(X/K). \tag{2.2.2}$$

Since the spectral sequence (2.1.3) degenerates already at E_1, the edge map

$$H^2_{\mathrm{dR}}(X/K) \to H^1(X, \Omega^1_{X/K}) \tag{2.2.3}$$

is an isomorphism. Specializing (2.2.2) to $n = m = 1$ and composing with the edge map (2.2.3) followed by the canonical trace map of Serre-Grothendieck [15, Corollary 3.6.6] (see also §1.2, §5.2, and Appendix B of [15])

$$\mathrm{Tr}_X : H^1(X, \Omega^1_{X/K}) \to K$$

we get a K-bilinear skew-symmetric pairing

$$(\cdot, \cdot)_X : H^1_{\mathrm{dR}}(X/K) \times H^1_{\mathrm{dR}}(X/K) \to K. \tag{2.2.4}$$

Since the cup product of two sections of $\Omega^1_{X/K}$ is zero, the pairing $(\cdot, \cdot)_X$ is zero when restricted to the submodule $H^0(X, \Omega^1_{X/K})$ of $H^1_{\mathrm{dR}}(X/K)$, and it follows that we get a pairing between $H^0(X, \Omega^1_{X/K})$ and $H^1(X, \mathscr{O}_X)$ that is readily checked in either order to be the usual Serre duality pairing. Thus, the pairing $(\cdot, \cdot)_X$ provides a canonical morphism of exact sequences of K-vector spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X, \Omega^1_{X/K}) & \longrightarrow & H^1_{\mathrm{dR}}(X/K) & \longrightarrow & H^1(X, \mathscr{O}_X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1(X, \mathscr{O}_X)^\vee & \longrightarrow & H^1_{\mathrm{dR}}(X/K)^\vee & \longrightarrow & H^0(X, \Omega^1_{X/K})^\vee & \longrightarrow & 0 \end{array} \tag{2.2.5}$$

Proposition 2.2.1. *The morphism of short exact sequences* (2.2.5) *is an isomorphism.*

Proof. Serre-Grothendieck duality [15, Theorem 3.4.4] shows that the flanking morphisms in (2.2.5) are isomorphisms. Thus, the middle one is too. ∎

2.3 Functoriality: pullback and trace by finite morphisms

In this section, we show that the association of the exact sequence (2.1.4) to the smooth proper and geometrically connected curve X is both a covariant and contravariant functor from the category of smooth proper and geometrically connected curves over K with finite morphisms between them to the category of short exact sequences of K-vector spaces. The contravariance is provided simply by pullback of differentials, while the covariance results from the existence of a trace map on de Rham complexes attached to a finite flat morphism of curves.

Let $f : X \to Y$ be any morphism of smooth curves over K. Pullback of differential forms gives a morphism of complexes

$$\Omega^\bullet_{Y/K} \to f_*\Omega^\bullet_{X/K} \tag{2.3.1}$$

that is $\mathscr{O}_Y$-linear in each degree. Composing with the natural map

$$f_*\Omega^\bullet_{X/K} \to \mathbf{R}f_*\Omega^\bullet_{X/K}$$

and applying the functor $\mathbf{R}\Gamma(Y,\cdot)$ yields a K-linear pullback map

$$f^* : H^\bullet_{\mathrm{dR}}(Y/K) \to \mathbf{H}^\bullet(Y, \mathbf{R}f_*\Omega^\bullet_{X/K}) \simeq \mathbf{H}^\bullet(X, \Omega^\bullet_{X/K}) = H^\bullet_{\mathrm{dR}}(X/K),$$

where the middle isomorphism comes from the Leray spectral sequence for the composition of the two functors $\mathbf{R}f_*$ and $\mathbf{R}\Gamma(Y,\cdot)$; see Remark A.1.2. Since (2.3.1) clearly preserves the evident filtrations on these complexes, we get a K-linear pullback map f^* on Hodge filtrations.

When f is moreover *finite* (hence finite flat), there is a classical construction associating a K-linear "trace" morphism

$$f_* : H^\bullet_{\mathrm{dR}}(X/K) \to H^\bullet_{\mathrm{dR}}(Y/K)$$

that proceeds as follows. Since f is finite, the coherent sheaves $\Omega^i_{X/K}$ are f_*-acyclic (i.e. $R^j f_* \Omega^i_{X/K} = 0$ for all i and all $j > 0$), so the canonical map $f_* \Omega^\bullet_{X/K} \to \mathbf{R} f_* \Omega^\bullet_{X/K}$ is a quasi-isomorphism. In particular, to make the map f_*, it suffices to construct a K-linear morphism of complexes

$$\mathrm{tr}_f : f_* \Omega^\bullet_{X/K} \to \Omega^\bullet_{Y/K}. \tag{2.3.2}$$

Since f is finite flat, the finite $\mathscr{O}_X$-module $f_* \mathscr{O}_Y$ is locally free, and tr_f in degree zero is simply the ring-theoretic trace map $f_* \mathscr{O}_Y \to \mathscr{O}_X$. In degree one, the $\mathscr{O}_Y$-linear map

$$\mathrm{tr}_f : f_* \Omega^1_{X/K} \to \Omega^1_{Y/K} \tag{2.3.3}$$

is provided by Grothendieck's theory of the trace morphism (see [15, 2.7.36]). In our situation, this map has a very explicit description, which we now recall.

When f is finite étale, the map (2.3.3) is simply the composite

$$f_* \Omega^1_{X/K} \simeq f_* f^* \Omega^1_{Y/K} \xrightarrow{\mathrm{tr}} \Omega^1_{Y/K}$$

induced by the canonical isomorphism

$$f^* \Omega^1_{Y/K} \xrightarrow{\simeq} \Omega^1_{X/K} \tag{2.3.4}$$

and the natural transformation $\mathrm{tr} : f_* f^* \to 1$ on vector bundles over Y coming from the trace map $f_* \mathscr{O}_X \to \mathscr{O}_Y$ in degree zero. In the general case, one knows that f is generically étale (due to the fact that K has characteristic zero), so if $i : U \hookrightarrow Y$ is the étale locus of f in Y with preimage $j : V \hookrightarrow X$ in X, we have a trace morphism

$$f_* \Omega^1_{V/K} \to \Omega^1_{U/K}$$

and we claim that the composite map

$$f_* \Omega^1_{X/K} \longrightarrow f_* j_* \Omega^1_{V/K} = i_* f_* \Omega^1_{V/K} \longrightarrow i_* \Omega^1_{U/K}$$

has image landing in $\Omega^1_{Y/K}$, thus giving the desired trace map. One checks this claim via a local calculation; see chapter VI. We note that this "concrete" trace morphism agrees with Grothendieck's trace map as defined in [15, 2.7.36] by virtue of how the latter is constructed; see [15].

That we really do get a morphism of complexes may be checked over the dense open set U, where it is clear as the map (2.3.3) is induced by the ring-theoretic trace map in degree zero via the canonical isomorphism (2.3.4). Moreover, since (2.3.2) is clearly compatible with filtrations, we get a pushforward map f_* on Hodge filtrations.

Clearly $f_* f^*$ is an endomorphism of the Hodge filtration of Y, and we claim that this map is simply multiplication by $n := \deg f$. This claim may be checked on the level of de Rham complexes, and we may even work over the dense open étale locus of f in Y. Thus our claim reduces to the fact that the composite

$$\mathscr{O}_U \xrightarrow{f^*} f_*\mathscr{O}_V \xrightarrow{\mathrm{tr}_f} \mathscr{O}_U$$

of pullback with the ring-theoretic trace map is multiplication by n.

We conclude this section by relating the cup product pairing $(\cdot,\cdot)_X$ on the Hodge filtration of $H^1_{\mathrm{dR}}(X/K)$ to pullback and trace by a finite morphism.

Theorem 2.3.1. *Let $f : X \to Y$ by a finite morphism of smooth proper and geometrically connected curves over a field K of characteristic zero. Then the morphisms f^* and f_* on Hodge filtrations are adjoint with respect to cup product pairing of* (2.2.4)*. Precisely, the diagram*

(2.3.5)
$$\begin{tikzcd} H^1_{\mathrm{dR}}(Y/K) \times H^1_{\mathrm{dR}}(Y/K) \arrow[r, "(\cdot,\cdot)_Y"] \arrow[d, shift left=8ex, "f^*"] & K \arrow[d, equal] \\ H^1_{\mathrm{dR}}(X/K) \times H^1_{\mathrm{dR}}(X/K) \arrow[r, "(\cdot,\cdot)_X"'] \arrow[u, shift left=8ex, "f_*"] & K \end{tikzcd}$$

commutes.

To prove Theorem 2.3.1, we will need the following result:

Lemma 2.3.2. *Let $f : X \to Y$ be a finite morphism of smooth proper and geometrically connected curves over a field K. Then for any $\eta, \omega \in H^1_{\mathrm{dR}}(Y/K)$ the equality*

$$(f^*\eta, f^*\omega)_X = (\deg f) \cdot (\eta, \omega)_Y$$

holds.

Proof. Since both cup product of differential forms and the Hodge to de Rham degeneration isomorphism (2.2.3) are clearly compatible with pullback by f, we see that the diagram

$$\begin{tikzcd}
H^1_{\mathrm{dR}}(Y/K) \times H^1_{\mathrm{dR}}(Y/K) \arrow[r, "\cup"] \arrow[d, "f^*"'] & H^2_{\mathrm{dR}}(Y/K) \arrow[r, "(2.2.3)"] \arrow[d, "f^*"] & H^1(Y, \Omega^1_{Y/K}) \arrow[d, "f^*"] \\
H^1_{\mathrm{dR}}(X/K) \times H^1_{\mathrm{dR}}(X/K) \arrow[r, "\cup"'] & H^2_{\mathrm{dR}}(X/K) \arrow[r, "(2.2.3)"'] & H^1(X, \Omega^1_{X/K})
\end{tikzcd}$$

commutes. Therefore, by the definition of $(\cdot, \cdot)_X$, it suffices to show that the diagram

(2.3.6)
$$\begin{tikzcd}
H^1(Y, \Omega^1_{Y/K}) \arrow[r, "f^*"] \arrow[d, "\mathrm{Tr}_Y"'] & H^1(X, \Omega^1_{X/K}) \arrow[d, "\mathrm{Tr}_X"] \\
K \arrow[r, "\cdot \deg f"'] & K
\end{tikzcd}$$

commutes. Now since $f_* f^* = \deg f$ on cohomology, the diagram

$$\begin{tikzcd}
H^1(Y, \Omega^1_{Y/K}) \arrow[r, "f^*"] \arrow[d, "\mathrm{Tr}_Y"'] & H^1(X, \Omega^1_{X/K}) \arrow[r, "f_*"] & H^1(Y, \Omega^1_{Y/K}) \arrow[d, "\mathrm{Tr}_Y"] \\
K \arrow[rr, "\cdot \deg f"'] & & K
\end{tikzcd}$$

commutes, so in order to show that (2.3.6) commutes, it is enough to show that the composite map

$$H^1(X, \Omega^1_{X/K}) \xrightarrow{f_*} H^1(Y, \Omega^1_{Y/K}) \xrightarrow{\mathrm{Tr}_Y} K$$

coincides with Tr_X. Such agreement is a special case of the general fact that Grothendieck's trace morphism is compatible with composition of morphisms, which comes out of how the maps f_* and Tr are constructed. See [15, Corollary 3.6.6] ■

Note in particular that Lemma 2.3.2 implies that the pairing $(\cdot,\cdot)_X$ is invariant under the natural action of $\mathrm{Aut}_K(X)$ (e.g. pullback in both variables).

Proof of Theorem 2.3.1. The idea of the proof is to reduce to the case that f is Galois with group G and then use the explicit description of f^*f_* as "sum of G-translates."

We first claim that we may assume that the finite map $f: X \to Y$ is a Galois covering. Indeed, let $f': X' \to X$ be a Galois closure of $X \to Y$, so X' is Galois over both X and X'. Assuming the theorem holds in the Galois case, we form the diagram

$$\begin{tikzcd}
H^1_{\mathrm{dR}}(Y/K) \times H^1_{\mathrm{dR}}(Y/K) \arrow[r, "(\cdot,\cdot)_Y"] & K \\
H^1_{\mathrm{dR}}(X/K) \times H^1_{\mathrm{dR}}(X/K) \arrow[r, "(\cdot,\cdot)_X"] \arrow[u, "f_*", shift left=8em] & K \arrow[u, equal] \\
H^1_{\mathrm{dR}}(X'/K) \times H^1_{\mathrm{dR}}(X'/K) \arrow[r, "(\cdot,\cdot)_{X'}"] \arrow[u, "f'_*", shift left=8em] & K \arrow[u, equal]
\end{tikzcd}$$

where the large total "outside" rectangle and the bottom rectangle both commute; we wish to show that the top rectangle also commutes. Note that since $f'_*f'^* = n$ on $H^1_{\mathrm{dR}}(X/K)$, the map f'_* is surjective (as the characteristic of K is zero). Letting $\eta \in H^1_{\mathrm{dR}}(Y/K)$, and $\omega \in H^1_{\mathrm{dR}}(X/K)$, we can thus find $\omega' \in H^1_{\mathrm{dR}}(X'/K)$ with $f'_*\omega' = \omega$. We then compute

$$\begin{aligned}
(f^*\eta,\omega)_X = (f^*\eta, f'_*\omega')_X = (f'^*f^*\eta,\omega')_{X'} = ((f\circ f')^*\eta,\omega')_{X'} &= (\eta,(f\circ f')_*\omega')_Y \\
&= (\eta, f_*f'_*\omega')_Y \\
&= (\eta, f_*\omega)_Y,
\end{aligned}$$

where the second and fourth equalities hold by our assumption that the theorem is true for the Galois maps f' and $f \circ f'$ respectively. Thus it suffices to treat the Galois case, and we assume that f is Galois with Galois group G, so $\deg f = \#G$.

We claim that for any $\eta \in H^1_{\mathrm{dR}}(X/K)$, we have the identity

$$f^* f_* \eta = \sum_{g \in G} g^* \eta.$$

To prove our claim, it clearly suffices to show that on the level of de Rham complexes, the maps

$$f^* f_*, \ \sum_{g \in G} f_*(g^*) \in \mathrm{End}_{\mathscr{O}_Y}\left(f_* \Omega^\bullet_{X/K}\right)$$

agree. As $f_* \Omega^\bullet_{X/K}$ is a complex of locally free $\mathscr{O}_Y$-modules, to check such agreement we may work at the generic point of Y, where by the description of f_* in the étale case it reduces to the usual interpretation of the trace map on function fields $\mathrm{Tr}_{K(X)/K(Y)} : K(X) \to K(Y)$ as a sum of Galois conjugates.

Now fix any $\eta \in H^1_{\mathrm{dR}}(Y/K)$ and $\omega \in H^1_{\mathrm{dR}}(X/K)$, and choose $\eta' \in H^1_{\mathrm{dR}}(X/K)$ with $\eta = f_* \eta'$ (e.g. $\eta' = (1/\#G) \cdot f^* \eta$ will do). Using Lemma 2.3.2 and the above description of $f^* f_*$, we have

$$(\eta, f_* \omega)_Y = (f_* \eta', f_* \omega)_Y = \frac{1}{\#G}(f^* f_* \eta', f^* f_* \omega)_X = \frac{1}{\#G} \sum_{g, g' \in G} (g'^* \eta', g^* \omega)_X.$$

Reindexing the sum by $g'' = g^{-1} g'$ and using that $(\cdot, \cdot)_X$ is G-invariant (thanks to Lemma 2.3.2, as noted above), we find that the above is equal to

$$\sum_{g'' \in G} (g''^* \eta', \omega)_X,$$

which by our description of $f^* f_*$, is

$$(f^* f_* \eta', \omega)_X = (f^* \eta, \omega)_X$$

as desired. ■

CHAPTER III

Admissible curves and regular differentials

Let R be any discrete valuation ring with fraction field K of characteristic zero and residue field k. Fix a uniformizer $\pi \in R$ and set $S = \operatorname{Spec} R$. The hypothesis that K has characteristic zero implies in particular that R is excellent [19, IV_2, 7.8.3 (iii)], so the normalization of any domain A of finite type over R is finite over A. Additionally, we note that we can resolve singularities of normal excellent surfaces, due to work of Lipman [31], [2]. We will use both of these facts repeatedly and without further comment in what follows.

The goal of this chapter is to equip the Hodge filtration (2.1.4) studied in chapter II with a canonical integral structure; i.e. a short exact sequence of free R-modules recovering (2.1.4) after extending scalars to K that is moreover functorial (both covariantly via trace and contravariantly via pullback) in finite morphisms of curves over K. The virtue of such a canonical integral structure in the case that R has mixed characteristic $(0, p)$ is that it can be reduced modulo π, allowing one to study the "characteristic zero" object (2.1.4) using techniques available only over the characteristic p residue field k (e.g. Frobenius).

One such integral structure uses Jacobians and their Néron models as we explain in chapter VIII. In this chapter, we give a construction (for a certain broad class of curves) using only the geometry of curves over K and their models over R. The advantage of this approach is that it allows one to work with the curve itself, rather than its Jacobian, and one can then hope to directly exploit the geometry of the curve.

3.1 Definition and properties

By a *curve* over S, we mean a flat, finite type, and separated S-scheme $f : X \to S$ of pure relative dimension 1 that is normal with geometrically connected generic fiber. For technical convenience, we do not require relative curves to be proper over S. Note that the generic fiber X_K is regular, and hence X_K is smooth over K (since K is of characteristic zero). Thus, the non-regular locus of X is a finite subset of (necessarily closed) points of the closed fiber. Observe, however, that the S-smooth locus of X need not have dense intersection with the closed fiber X_k of X, as we do not require X_k to be generically reduced. In order to define the class of S-curves that yield a good "integral" de Rham cohomology theory we must first recall the concepts of *cohomologically flat* and of *rational singularities*.

We fix a proper relative curve $f : X \to S$. Recall that f is said to be *cohomologically flat* (in dimension 0) if the formation of $f_*\mathscr{O}_X$ commutes with arbitrary base change.

Lemma 3.1.1. *Let f be as above. Then f is cohomologically flat if and only if the R-module $H^1(X, \mathscr{O}_X)$ is torsion free.*

Proof. Since X is flat by hypothesis, we have an exact sequence

$$0 \longrightarrow H^0(X, \mathscr{O}_X) \otimes_R k \longrightarrow H^0(X_0, \mathscr{O}_{X_0}) \longrightarrow H^1(X, \mathscr{O}_X)[\pi] \longrightarrow 0\ ,$$

where $H^1(X, \mathscr{O}_X)[\pi]$ denotes the π-torsion submodule of $H^1(X, \mathscr{O}_X)$. We conclude that $f_*\mathscr{O}_X$ commutes with the base change $\operatorname{Spec} k \to S$ (and hence arbitrary base change) if and only if $H^1(X, \mathscr{O}_X)$ is torsion free. ∎

Definition 3.1.2. Let x be a point of X of codimension 2. We say that x is a *rational singularity* of X if there exists a resolution of singularities $\rho : X' \to X$ of X such that the stalk of $R^1\rho_*\mathscr{O}_{X'}$ at x is zero.

From this definition, it may seem that whether or not a point x of X of codimension 2 is a rational singularity is a difficult thing to check in practice. The following proposition shows that being a rational singularity is intrinsic to x, and does not depend on the choice of resolution.

Proposition 3.1.3. *A point x of X of codimension 2 is a rational singularity if and only if for every proper birational morphism $\rho : X' \to X$ with X' normal, the stalk of $R^1\rho_*\mathscr{O}_{X'}$ at x is zero.*

Proof. See [31], Proposition 1.2, (2). ∎

We will say that X has *rational singularities* if every closed point in the closed fiber is a rational singularity; this is a nontrivial condition at only finitely many points (since any regular point is a rational singularity). Moreover, the property of having rational singularities is preserved under proper birational morphisms:

Lemma 3.1.4. *Suppose that $\rho : X' \to X$ is any proper birational map of relative S-curves and X has rational singularities. Then X' has rational singularities.*

Proof. See [2, Proposition 3.2 (ii)]. ∎

We can now introduce our primary object of study in this secton:

Definition 3.1.5. A relative curve $f : X \to S$ is *admissible* provided f is proper and cohomologically flat, and X has rational singularities.

Remark 3.1.6. Our definition of admissible curve generalizes that of [36] §2 in that we impose weaker conditions on the possible singularities of X and do not require the closed fiber to be reduced. Any normal semistable curve over R with smooth and geometrically connected generic fiber is admissible, and this is a primary example of interest. We note that for certain later proofs to work (see, for example, Theorem 3.2.2), one can not expect to get by with semistable curves, and it is essential to work with the larger category of admissible curves.

We have formulated the notion of admissibility to ensure that every admissible curve over $S = \operatorname{Spec} R$ possesses certain properties. For example, in order for the R-module $H^1(X, \mathscr{O}_X)$ to yield a "good" integral structure on $H^1(X_K, \mathscr{O}_{X_K})$, it is crucial that it be torsion free. Besides, we will later work with the relative Picard functor $\mathrm{Pic}_{X/S}$, which has much better representability properties when X is cohomologically flat. The hypothesis that X have rational singularities is also crucial for our purposes, as it enables us to blow up X at closed points of the closed fiber without changing the cohomology $H^1(X, \mathscr{O}_X)$. It is via such invariance of cohomology that we will be able to provide a canonical integral structure on the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$. We now summarize some basic properties of admissible curves.

Lemma 3.1.7. *Let $f : X \to S$ be a proper curve with rational singularities, and $\rho : X' \to X$ any proper birational morphism with X' normal. Then the canonical pullback map $H^1(X, \mathscr{O}_X) \to H^1(X', \mathscr{O}_{X'})$ is an isomorphism. In particular, X' is admissible if and only if X is admissible.*

Proof. Denote by $f' : X' \to S$ the structural morphism. The exact sequence of terms of low degree for the spectral sequence $R^p f_* R^q \rho_* \mathscr{O}_{X'} \implies R^{p+q} f'_* \mathscr{O}_{X'}$ of the composite $f' = f \circ \rho$ reads

$$0 \longrightarrow R^1 f_*(\rho_* \mathscr{O}_{X'}) \longrightarrow R^1 f'_* \mathscr{O}_{X'} \longrightarrow f_* R^1 \rho_* \mathscr{O}_{X'} \ .$$

Since X' is normal and X has rational singularities, Proposition 3.1.3 ensures that $R^1 \rho_* \mathscr{O}_{X'} = 0$. Thus, since the canonical map $\mathscr{O}_X \to \rho_* \mathscr{O}_{X'}$ is an isomorphism (as ρ is proper birational and, by our definition of S-curve, X is normal), we get an isomorphism $R^1 f_* \mathscr{O}_X \simeq R^1 f'_* \mathscr{O}_{X'}$ that is the canonical map, giving the first assertion of the lemma.

Observe that X' is a relative S-curve. Thanks to Lemma 3.1.4, we know that X' has rational singularities. The isomorphism $H^1(X, \mathscr{O}_X) \simeq H^1(X', \mathscr{O}_{X'})$ shows that $H^1(X', \mathscr{O}_{X'})$ is torsion free if and only if $H^1(X, \mathscr{O}_X)$ is torsion free, so we are done by Lemma 3.1.1. ■

We can now state a useful criterion for a curve X to be admissible:

Proposition 3.1.8. *Suppose that $X \to S$ is a proper curve with rational singularities. If* $\mathrm{char}(k) = 0$ *then X is admissible. If* $\mathrm{char}(k) = p > 0$ *then X is admissible if the greatest common divisor of the geometric multiplicities of the components of the closed fiber is prime to p, or if X admits an étale quasi-section (i.e. $X(S') \neq \emptyset$ for S' a strict henselization of S).*

Proof. Let $\rho : X' \to X$ be a resolution of singularities of X and denote by f' the structural morphism of X'. By Lemma 3.1.7, it suffices to show that X' is admissible, and so it suffices to show that X' is cohomologically flat. When $\mathrm{char}(k) = 0$ this is automatic, due to the normality of X' (cf. the introduction of [43]).

If $\mathrm{char}(k) = p > 0$, we proceed as follows. Since X' is regular, all the local rings of the 1-dimensional closed fiber X'_0 are S_1, so the closed fiber of X' has no embedded components. Since $\mathscr{O}_X \simeq \rho_*\mathscr{O}_{X'}$ as in the proof of Lemma 3.1.7, we see that $f'_*\mathscr{O}_{X'} \simeq \mathscr{O}_S$. Again by regularity of X', the local rings on X' are normal. It follows that X' satisfies the hypotheses of Raynaud's "critère de platitude cohomologique" [43, Théorème 7.2.1], which shows that X' is cohomologically flat if the greatest common divisor of the geometric multiplicities of the closed fiber of X' is prime to p.

In the case that the greatest common divisor of the geometric multiplicities of the components of the closed fiber of X is prime to p, we observe that the same is true of X', as $\rho : X' \to X$ is an isomorphism over an open $U \subseteq X$ meeting X_0 densely (since the generic points of X_0 have codimension one in X). Thus we conclude from Raynaud's criterion that X' is cohomologically flat.

In the case that X admits an étale quasi-section $\sigma : S' \to X$, we observe that σ automatically lifts to an étale quasi-section $\sigma' : S' \to X'$, which must land in the smooth locus of the regular scheme X'. Thus, the closed fiber of X' has an irreducible component of geometric multiplicity one, and we again conclude from Raynaud's criterion that X' is cohomologically flat. ■

Of course, the reader might object to the usefulness of Lemma 3.1.8 on the grounds that the concept of rational singularity is rather abstractly defined. Trivially, every regular S-curve has rational singularities, but as we very much wish to apply our results to relative curves that may not be regular (for example integral models of high p-level modular curves over a p-adic field), we would like some concrete criterion for deciding when a relative curve has rational singularities. As a step in this direction, we have:

Proposition 3.1.9. *Let $f : X' \to X$ be a finite, surjective map of 2-dimensional* *noetherian excellent schemes that are normal and integral. Assume that* $\deg f$ *is a unit in* $\mathscr{O}_X$*. If* X' *has rational singularities, so does* X*.*

Proof. We follow the proof of [30, Proposition 5.13], using spectral sequences in place of derived categories.

Let $\rho : Y \to X$ be a resolution of singularities (which exists thanks to Lipman [31], [2]), and let $g : Y' \to Y$ be the normalization of Y in the function field of X'. Due to the excellence hypothesis, the map g is finite; we set $n := \deg g = \deg f$ (generic degree). We thus have a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "\rho'"'] & Y \arrow[d, "\rho"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

of normal 2-dimensional noetherian excellent schemes with g and f finite of generic degree n.

Since Y is regular and Y' is normal (hence Cohen-Macaulay, since Y' has dimension 2) and the map g is finite, we conclude by [34, Theorem 23.1] that g is flat. Thus, $g_*\mathscr{O}_{Y'}$ is finite locally free over $\mathscr{O}_Y$, and we have a (ring-theoretic) trace map

$$\mathrm{tr}_g : g_*\mathscr{O}_{Y'} \to \mathscr{O}_Y$$

with the property that the composite map

$$\mathrm{tr}_g \circ g^* : \mathscr{O}_Y \to \mathscr{O}_Y$$

is multiplication by n. Since n is a unit in $\mathscr{O}_X$ (hence in $\mathscr{O}_Y$), we see that the map

$$(1/n) \cdot \mathrm{tr}_g : g_*\mathscr{O}_{Y'} \to \mathscr{O}_Y$$

is a section to g^*, so $\mathscr{O}_Y$ is a direct summand of $g_*\mathscr{O}_{Y'}$.

Letting $\psi : Y'' \to Y'$ be a resolution of singularities of Y', the exact sequence of terms of low degree for the spectral sequence of the composite $\varphi := \rho' \circ \psi$ reads

$$0 \longrightarrow R^1\rho'_*\mathscr{O}_{Y'} \longrightarrow R^1\varphi_*\mathscr{O}_{Y''} \longrightarrow \rho'_*R^1\psi'_*\mathscr{O}_{Y''} \ ,$$

so since X' has rational singularities by hypothesis and $\varphi : Y'' \to X'$ is a resolution, we conclude that

$$R^1\rho'_*\mathscr{O}_{Y'} = 0. \tag{3.1.1}$$

Similarly, the exact sequences of terms of low degree for the spectral sequences of the composite $\alpha := f \circ \rho' = \rho \circ g$ read

$$0 \longrightarrow R^1f_*(\rho'_*\mathscr{O}_{Y'}) \longrightarrow R^1\alpha_*\mathscr{O}_{Y'} \longrightarrow f_*R^1\rho'_*\mathscr{O}_{Y'} \tag{3.1.2}$$

and

$$0 \longrightarrow R^1\rho_*(g_*\mathscr{O}_{Y'}) \longrightarrow R^1\alpha_*\mathscr{O}_{Y'} \longrightarrow \rho_*R^1g_*\mathscr{O}_{Y'} \ . \tag{3.1.3}$$

From (3.1.2) and (3.1.1) we deduce that the edge map

$$R^1f_*(\rho'_*\mathscr{O}_{Y'}) \to R^1\alpha_*\mathscr{O}_{Y'}$$

is an isomorphism. But f is finite, so $R^1f_* = 0$ on coherent $\mathscr{O}_{X'}$-modules, so $R^1\alpha_*\mathscr{O}_{Y'} = 0$. It then follows from (3.1.3) that $R^1\rho_*(g_*\mathscr{O}_{Y'}) = 0$, Since $\mathscr{O}_Y$ is a direct summand of $g_*\mathscr{O}'_Y$, we finally conclude that $R^1\rho_*\mathscr{O}_Y$ must vanish, and hence that X has rational singularities as desired. ■

The following corollary is usually attributed to Elkik [20, II, Lemme 1]:

Corollary 3.1.10. *Let A be a noetherian excellent 2-dimensional regular domain that is essentially finite type over a noetherian domain R, and let G be a finite group acting on A by R-linear automorphisms. If $\#G$ is a unit in A then the ring of G-invariants A^G has rational singularities.*

Proof. It is easy to see that A^G is normal, and our assumptions ensure that $A^G \to A$ is finite. Thus, the corollary follows at once from Proposition 3.1.9 with $X = \mathrm{Spec}(A^G)$ and $X' = \mathrm{Spec}\, A$, using Artin's lemma to know that $K^G \to K$ is finite Galois of degree $\#G$, where $K = \mathrm{Frac}(A)$. ∎

Example 3.1.11. Using Corollary 3.1.10, we can show that many relative S-curves of arithmetic interest have rational singularities. Quite generally, let $\mathscr{X}$ be any regular Deligne-Mumford stack of relative dimension one over S and suppose that the finite order of the automorphism group at each geometric point of $\mathscr{X}$ is a unit in R (automatic if $\mathrm{char}(k) = 0$). If the underlying coarse moduli space X is a scheme, then it is a relative curve over S with all local rings of the form A^G with A regular and G a finite group of R-linear automorphisms of A with order a unit in R. It follows that in this situation, X has rational singularities.

Remark 3.1.12. Consider the modular curve $X_0(N)_{\mathbf{Q}}$ over $\mathbf{Q}$ that is the compactification of the coarse moduli space associated to the stack classifying pairs (E, C) with E an elliptic curve over a $\mathbf{Q}$-scheme and C a finite subgroup scheme of E that is cyclic of order N. It is smooth, proper, and geometrically connected. Thanks to [28, 5.1.1], $X_0(N)_{\mathbf{Q}}$ is the base change to $\mathbf{Q}$ of the coarse moduli space $X_0(N)$ associated to a certain regular Deligne-Mumford stack $\mathscr{M}(\Gamma_0(N))$ over $\mathbf{Z}$. Since the automorphism group of any geometric point must divide 24 (as the automorphism group of an elliptic curve has order dividing 24) we conclude from the example above that $X_0(N)_{\mathbf{Z}_p}$ has rational singularities for $p > 3$. A more careful analysis of the possible automorphism groups at supersingular points in characteristic p and at the cusps shows that $X_0(N)_{\mathbf{Z}_p}$ has rational singularities when $p = 2$ or 3 provided N is divisible by some prime other than p. One further knows that the closed fiber of $X_0(N)_{\mathbf{Z}_p}$ *always* has a component of geometric multiplicity one [28, 13.4.7](although when $p^2|N$ there can be non-reduced components). It follows from Proposition 3.1.8 that $X_0(N)_{\mathbf{Q}}$ has an admissible model over $\mathbf{Z}_p$ for $p > 3$ and for $p = 2, 3$ when N is divisible by some prime distinct from p.

Similar reasoning shows that the modular curve $X_1(N)_{\mathbf{Q}}$ that is the compactification of the coarse moduli space associated to the stack classifying pairs (E, α) with E an elliptic curve over some $\mathbf{Q}$-scheme and α an embedding of group schemes $\alpha : \mu_N \hookrightarrow E[N]$ has an admissible model over $\mathbf{Z}_p[\zeta_{p^n}]$ where $n = \mathrm{ord}_p(N)$ for $p > 3$ arbitrary, and for $p = 2, 3$ when N is divisible by some prime distinct from p

To conclude this section, let us prove that admissibility is stable under a certain class of base changes.

Lemma 3.1.13. *If X is an admissible curve over $S = \mathrm{Spec}\, R$ and $R \to R'$ is any essentially étale extension of discrete valuation rings, then $X' := X \times_S S'$ is an admissible curve over $S' = \mathrm{Spec}\, R'$.*

Proof. Since the base change of any smooth morphism is smooth, it is clear that $X' \to S'$ has smooth generic fiber. Obviously the generic fiber is geometrically connected as this is true for $X \to S$. Since $X' \to X$ is essentially étale and X is normal, we see that X' is normal (by the Corollary to Theorem 23.9 in [34]). That X' is cohomologically flat follows from the fact that cohomology commutes with flat base change and the criterion of Lemma 3.1.1. To see that X' has rational singularities, it suffices to observe that the essentially étale base change by $S' \to S$ of a resolution of singularities $\rho : \widetilde{X} \to X$ of X is a resolution of singularities of X'. ■

3.2 Admissible models

We will say that a smooth proper and geometrically connected curve X_K over K has an *admissible model* over S if there exists an admissible curve $X \to S$ with generic fiber X_K. It is precisely the curves X_K over K that admit an admissible model over S for which we will be able to provide a canonical integral structure on the Hodge filtration of X_K. In this section, we fix a smooth proper and geometrically connected curve X_K and study its admissible models.

Lemma 3.2.1. *Any two admissible models can be dominated by a third.*

Proof. If X_1 and X_2 are admissible models of X_K, we let Γ_K denote the graph in $(X_1)_K \times_K (X_2)_K$ of the generic fiber isomorphism $(X_1)_K \simeq (X_2)_K$. Let Γ be the closure of Γ_K in $X_1 \times_S X_2$. Then Γ has K-fiber Γ_K because each X_i is S-separated. We take X_3 to be the normalization of Γ. The normalization map $X_3 \to \Gamma$ is finite (as R is excellent), so the natural maps $X_3 \rightrightarrows X_i$ induced by the projection maps $\mathrm{pr}_i : X_1 \times_S X_2 \rightrightarrows X_i$ for $i = 1, 2$ are proper and birational, whence by Lemma 3.1.7, X_3 is an admissible model of X_K. ■

In §3.4, we will see how an admissible model of X_K over S yields an integral structure on $H^1_{\mathrm{dR}}(X_K/K)$; using Lemma 3.2.1, we will further be able to show that this integral structure does not depend on the choice of admissible model. However, we also wish to show that the integral structure provided by an admissible model is functorial in finite K-morphisms of curves $f_K : X_K \to Y_K$ over K. One of the main difficulties in doing this is to find admissible models X and Y of X_K and Y_K and a finite S-morphism $f : X \to Y$ recovering f_K on generic fibers. That this can be done is a subtle matter, and depends crucially on the fact that we do not require admissible models to be regular (regularity is too strong a condition to impose, roughly speaking, because a resolution of singularities will almost never be finite).

Theorem 3.2.2. *Suppose that X and Y are admissible curves and $f_K : X_K \to Y_K$ is a finite morphism of their generic fibers. Then there exist admissible models X' of X_K and Y' of Y_K which dominate X and Y, respectively, and a finite morphism $f' : X' \to Y'$ recovering f_K on generic fibers.*

Proof. Let $\Gamma_K \subseteq X_K \times Y_K$ be the graph of f_K. As Y is S-separated, the closure Γ of Γ_K in $X \times Y$ has K-fiber Γ_K. Let $\widetilde{\Gamma}$ be the normalization of Γ. Since the map $\widetilde{\Gamma} \to \Gamma$ is finite, it is proper, and clearly $\widetilde{\Gamma}_K \simeq \Gamma_K \simeq X_K$ via the first projection. Thus, composing the normalization map with the projection maps, we obtain a proper birational S-morphism $\widetilde{\Gamma} \to X$ and an S-morphism $f : \widetilde{\Gamma} \to Y$ that recovers f_K on generic fibers. By Lemma 3.1.7, we know that $\widetilde{\Gamma}$ is admissible.

If f happens to be quasifinite, then it is finite and so we are done. However, this may not be the case. We use results of Raynaud-Gruson [44]. Applying [44, Théorème 5.2.2] with $n = 1$ to the morphism f, which is generically finite, we find that there is a blowup Y'' of Y such that the strict transform $\widetilde{\Gamma}'$ of $\widetilde{\Gamma}$ with respect to $Y'' \to Y$ is quasi-finite over Y''. We let X' be the normalization of $\widetilde{\Gamma}'$ and Y' the normalization of Y''. The quasi-finite dominant map $X' \to Y''$ necessarily factors through Y' (due to the universal property of normalization), so we obtain a quasi-finite S-morphism $f' : X' \to Y'$. Now X' and Y' are evidently proper over S and are admissible by Lemma 3.1.7; moreover, the proper quasi-finite morphism f' must be finite, so we are done (see the diagram below).

$$
\begin{tikzcd}
X' \arrow[d] \arrow[drr, dashed, "f'"] & & \\
\widetilde{\Gamma}' \arrow[d] \arrow[drr] & & Y' \arrow[d] \\
\widetilde{\Gamma} \arrow[d] \arrow[drr, "f"] & & Y'' \arrow[d] \\
\Gamma \arrow[r] & X \times_S Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & Y \arrow[d] \\
 & X \arrow[r] & S
\end{tikzcd}
$$

■

As we have indicated, Theorem 3.2.2 is almost certainly not true if one were to require admissible curves to be regular, or even semistable; cf. Remark 3.1.6. Our definition of admissible was formulated so as to be able to prove Theorem 3.2.2.

3.3 The dualizing sheaf of a relative curve

In order to associate to an admissible curve X an integral structure on the Hodge filtration of its generic fiber X_K, we will construct an "integral version" of the de Rham complex of X_K using the relative dualizing sheaf of X over S. When X is smooth, our construction will recover the de Rham complex of X. In this section, we recall the facts about the relative dualizing sheaf and Grothendieck duality that we will need.

Recall that a locally finite type map of locally noetherian schemes is said to be *Cohen-Macaulay* (which we abbreviate to "CM") if it is flat and all fibers are Cohen-Macaulay schemes. Let $f : X \to Y$ be any CM map between locally noetherian schemes with pure relative dimension n, and suppose that Y admits a dualizing complex. By [15, Theorem 3.5.1], the relative dualizing complex $f^!\mathscr{O}_Y$ has a unique nonzero cohomology sheaf, which is in degree $-n$, and we define

$$\omega_{X/Y} := H^{-n}(f^!\mathscr{O}_Y). \tag{3.3.1}$$

We call the sheaf $\omega_{X/Y}$ the dualizing sheaf (for X over Y). It is flat over Y by [15, Theorem 3.5.1], and its formation is compatible with arbitrary base change thanks to [15, Theorem 3.6.1]. The discussion preceding [15, Corollary 4.4.5] shows moreover that the formation of $\omega_{X/Y}$ is compatible with étale localization on X.

If f is in addition proper, Grothendieck's theory provides a natural transformation of functors (see [15], §3.4)

$$\mathrm{Tr}_f : \mathbf{R}f_*f^! \to 1 \tag{3.3.2}$$

which induces a unique trace map

$$\gamma_f : R^nf_*\omega_{X/Y} \to \mathscr{O}_Y \tag{3.3.3}$$

whose formation commutes with arbitrary base change [15, Theorem 3.6.5] (we will only use this fact for flat extension of scalars). Moreover, γ_f is an isomorphism when f is smooth with geometrically connected fibers [15, Corollary 3.6.6]. In this context, Grothendieck's duality theorem may be stated as:

Theorem 3.3.1. *Let $f : X \to Y$ be a CM morphism of locally noetherian schemes of pure relative dimension n. Suppose that $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of finite rank such that the higher direct images $R^jf_*\mathscr{F}$ are locally free of finite rank for all $j \geq 0$. Then for all $j \geq 0$, the natural map*

$$R^{n-j}f_*(\mathscr{F}^\vee \otimes \omega_{X/Y}) \to R^jf_*(\mathscr{F})^\vee$$

induced by the composite

$$R^j f_*(\mathscr{F}) \otimes R^{n-j} f_*(\mathscr{F}^\vee \otimes \omega_{X/Y}) \xrightarrow{\cup} R^n f_*\omega_{X/Y} \xrightarrow{\gamma_f} \mathscr{O}_Y$$

is an isomorphism.

Proof. This is a special case of [15], Theorem 5.1.2. ∎

Now suppose that $p_X : X \to S$ is any relative curve. Since $S = \operatorname{Spec} R$ with R a discrete valuation ring, S is regular and of finite Krull dimension; it follows from [26, V, §10] that $\mathscr{O}_S$ is a dualizing complex for S. Moreover, since X is normal and of pure relative dimension one over S, it is automatically Cohen-Macaulay by Serre's criterion for normality. In particular, the dualizing complex is a coherent sheaf $\omega_{X/S}$ concentrated in some degree. By our discussion above, the sheaf $\omega_{X/S}$ is flat over S and its formation is compatible with arbitrary base change on S and étale localization on X.

If X is in addition proper and cohomologically flat in dimension 0, then the sheaves $R^i p_{X*}\mathscr{O}_X$ are locally free $\mathscr{O}_S$-modules for every i. This holds for $i = 0$ since X is flat over S and for $i \geq 2$ because these sheaves vanish (by the theorem on formal functions). For $i = 1$, it is equivalent to cohomological flatness. Thus, we may apply Theorem 3.3.1 with $\mathscr{F} = \mathscr{O}_X$: we find that the natural morphism of R-modules

$$H^{1-j}(X, \omega_{X/S}) \to H^j(X, \mathscr{O}_X)^\vee \tag{3.3.4}$$

induced by cup-product and the trace map is an isomorphism for $j = 0, 1$. We have proved:

Proposition 3.3.2. *Let $X \to S$ be a proper and cohomologically flat curve. Then the R-modules $H^j(X, \mathscr{O}_X)$ are free for all j. Moreover, $H^j(X, \omega_{X/S})$ is canonically R-dual to $H^{1-j}(X, \mathscr{O}_X)$, hence free as an R-module for $j = 0, 1$.*

Remark 3.3.3. The need to have Proposition 3.3.2 is why we impose the condition of cohomologically flat in the definition of admissibility. Without this hypothesis, Grothendieck duality does not take on a particularly useful form. Indeed, many of our later proofs concerning integral structures on the Hodge filtration of a smooth curve over K will critically use the duality statement of Proposition 3.3.2.

Remark 3.3.4. We sometimes call $\omega_{X/S}$ the sheaf of "regular differentials" as in [16], I, §2.1.1. This terminology is partially justified since $\omega_{X/S}$ is a subsheaf of the sheaf of differential 1-forms on the generic fiber of X. Indeed, let $i : X_K \hookrightarrow X$ be the inclusion of the smooth and geometrically connected generic fiber into X. Since $\omega_{X/S}$ is S-flat, the natural map $\omega_{X/S} \to i_*i^*\omega_{X/S}$ is injective. Moreover, the general theory of the dualizing sheaf provides a canonical isomorphism $i^*\omega_{X/S} \simeq \Omega^1_{X_K/K}$, so we see that we have an injective map of $\mathscr{O}_X$-modules

$$\omega_{X/S} \hookrightarrow i_*\Omega^1_{X_K/K}. \tag{3.3.5}$$

This observation is crucial for our purposes, and we think of $\omega_{X/S}$ as an "integral version" of $\Omega^1_{X_K/K}$ which is a suitable replacement for $\Omega^1_{X/S}$ when X is not S-smooth (and coincides with $\Omega^1_{X/S}$ when X is S-smooth).

Considering $\omega_{X/S}$ as a subsheaf of $i_*\Omega^1_{X_K/K}$ as in Remark 3.3.4, we will show in the next section (Theorem 3.4.1) that the usual differential $d : \mathscr{O}_{X_K} \to \Omega^1_{X_K/K}$ carries $\mathscr{O}_X$ into $\omega_{X/S}$, and thus that we have a subcomplex $d : \mathscr{O}_X \to \omega_{X/S}$ of the de Rham complex of X_K. It is this complex whose hypercohomology will give us the desired integral structure on $H^1_{\mathrm{dR}}(X_K/K)$. Since one of our goals is to produce an integral structure that is functorial via pullback and trace of finite morphisms, we must first associate to any finite morphism of admissible curves pullback and trace maps of the complex $d : \mathscr{O}_X \to \omega_{X/S}$ that recover pullback and trace on the de Rham complex as defined in §2.3 after base change to K. To do this, we begin by defining the desired pullback and trace maps on $\mathscr{O}_X$ and $\omega_{X/S}$.

Suppose that $f : X \to Y$ is a finite S-morphism of relative curves. Then we have an induced "pullback" map

$$f^* : \mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X \; . \tag{3.3.6}$$

We claim that we also have an $\mathscr{O}_Y$-linear trace map

$$f_* : f_*\mathscr{O}_X \longrightarrow \mathscr{O}_Y \; . \tag{3.3.7}$$

If f is finite *flat*, then this is clear, as we take f_* to be the usual trace map attached to the finite locally free extension of algebras $\mathscr{O}_Y \to f_*\mathscr{O}_X$. To define f_* for general f (i.e. finite but not necessarily flat) we need:

Lemma 3.3.5. *For any finite morphism $f : X \to Y$ of relative S-curves, there exists a Zariski open subset $V \subseteq Y$ with scheme-theoretic preimage $U := f^{-1}V$ verifying the following properties:*

1. *The complements of V in Y and of U in X consist of finitely many points of codimension 2 (necessarily in the closed fibers).*
2. *The morphism f is flat over V.*

Proof. This follows from the fact that f *must* carry generic points of X_k to generic points of Y_k, and that any local extension of discrete valuation rings is flat. ■

For any finite morphism $f : X \to Y$ of relative curves over S, we let V and $U := f^{-1}(V)$ be as in Lemma 3.3.5, and we denote by f_U the restriction of f to U; it is finite and flat. Since X and Y are normal, the natural maps $\mathscr{O}_X \to i_*\mathscr{O}_U$ and $\mathscr{O}_Y \to j_*\mathscr{O}_V$ are isomorphisms, and we define (3.3.7) to be the composite

$$f_*\mathscr{O}_X \xrightarrow{\simeq} f_*i_*\mathscr{O}_U = j_*f_{U*}\mathscr{O}_U \xrightarrow{j_*(f_{U*})} j_*\mathscr{O}_V \xleftarrow{\simeq} \mathscr{O}_Y \; .$$

Observe that on generic fibers, pullback (3.3.6) and trace (3.3.7) recover the usual pullback and trace morphisms associated to the finite flat morphism f_K. In particular, the preceding construction is independent of the choice of V.

We generalize the existence of pullback and trace maps on differentials as in §2.3 to the "integral level" by showing that any finite S-morphism $f : X \to Y$ of curves induces pullback and trace maps on dualizing sheaves:

Theorem 3.3.6. *Let $f : X \to Y$ be a finite morphism of* (*not necessarily proper*) *relative curves over S. Then there exist unique $\mathscr{O}_Y$-linear morphisms*

$$f^* : \omega_{Y/S} \to f_*\omega_{X/S}$$

$$f_* : f_*\omega_{X/S} \to \omega_{Y/S}$$

that recover the usual pullback and trace morphisms of differentials associated to the finite flat f_K in §2.3 *on passing to generic fibers and using the identifications of Remark* 3.3.4.

Since $\omega_{Y/S}$ is flat, the uniqueness aspect of the theorem is clear (cf. Remark 3.3.4). The existence of f_* will follow from general facts about Grothendieck's trace morphism (3.3.2) as we explain below. To show the existence of the pullback map f^*, we will first reduce to the case of finite *flat* f, and then use the theory of duality for a finite flat morphism.

To reduce to the finite flat case, we will proceed as in the construction of $f_* : f_*\mathscr{O}_X \to \mathscr{O}_Y$ and work over an open V in Y, complementary to finitely many points of codimension 2, over which f is flat. To ensure that such a strategy can be carried out, we must first understand how the dualizing sheaf behaves under restriction to such an open. So suppose that $i : U \hookrightarrow X$ is any open subset of a curve which is complementary to finitely many points of codimension 2 (necessarily in the closed fiber of X). The general theory of the relative dualizing sheaf ensures that there is a canonical isomorphism

$$i^*\omega_{X/S} \simeq \omega_{U/S},$$

and hence a natural map

$$\omega_{X/S} \to i_* i^* \omega_{X/S} \simeq i_* \omega_{U/S}. \tag{3.3.8}$$

Lemma 3.3.7. *The morphism* (3.3.8) *is an isomorphism.*

Proof. This is proved as in the argument below 5.2.7 in [15], which we explain in more detail. Thanks to the flatness of $\omega_{X/S}$, the map (3.3.8) is injective and it is an isomorphism over X_K (since U contains the entire generic fiber of X).

To show that it is surjective, we have to analyze stalks at closed points of the closed fiber X_0, and we proceed as follows. Let $j : U \hookrightarrow X$ denote the natural inclusion and let Z denote the complement of U in X; it is a closed subset. If $\mathscr{F}$ is any abelian sheaf on X and $\mathscr{I}^\bullet$ is an injective resolution of $\mathscr{F}$, we have a short exact sequence of complexes of abelian sheaves on X

$$0 \longrightarrow \mathscr{H}^0_Z(\mathscr{I}^\bullet) \longrightarrow \mathscr{I}^\bullet \longrightarrow j_*\left(\mathscr{I}^\bullet\big|_U\right) \longrightarrow 0,$$

where (as usual) $\mathscr{H}^0_Z(\mathscr{F})$ denotes the subsheaf with supports in Z and the right exactness is due to the fact that injective sheaves are flasque. Passing to cohomology, we thus get an exact sequence

$$0 \longrightarrow \mathscr{H}^0_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*\left(\mathscr{F}\big|_U\right) \longrightarrow \mathscr{H}^1_Z(\mathscr{F}) \longrightarrow 0\,.$$

Applying this to $\mathscr{F} = \omega_{X/S}$, we already know that $\mathscr{H}^0_Z(\omega_{X/S}) = 0$ (thanks to the S-flatness of $\omega_{X/S}$, as noted above), and we wish to show that $\mathscr{H}^1_Z(\omega_{X/S}) = 0$ as well.

Thus, we must show that for each $x \in X - U$, the local cohomology group $H^1_x(X, \omega_{X/S})$ vanishes. By [25, Exp. III, Example 3.4], such vanishing is equivalent to

$$\text{depth}_{\mathscr{O}_{X,x}}(\omega_{X/S,x}) \geq 2 \tag{3.3.9}$$

for each $x \in X - U$. If x is a regular point, then this inequality is trivially verified as $\omega_{X/S,x}$ is a free $\mathscr{O}_{X,x}$ module of rank 1 for such x, and $\mathscr{O}_{X,x}$ is 2-dimensional and normal (whence it has depth 2, by Serre's criterion for normality).

In general, by [25, III, Corollary 2.5] (and since $\omega_{X/S}$ is flat and of formation compatible with arbitrary base change), it is enough to show that for each non-regular point x on the closed fiber X_0 of X we have

$$\text{depth}_{\mathscr{O}_{X_0,x}}(\omega_{X_0/k,x}) \geq 1. \tag{3.3.10}$$

If this is not the case, then the ideal $\mathfrak{m}_x$ of $\mathscr{O}_{X_0,x}$ consists entirely of zero divisors for the finite $\mathscr{O}_{X_0,x}$-module $\omega_{X_0/k,x}$, so it must be an associated prime of $\omega_{X_0/k,x}$. We would then have $\mathfrak{m}_x = \text{Ann}(s)$ for some nonzero $s \in \omega_{X_0/k,x}$, whence $\text{Hom}_{X_0}(k(x), \omega_{X_0/k}) \neq 0$. However, we have

$$\text{Hom}_{X_0}(k(x), \omega_{X_0/k}) = H^1(X_0, k(x))^\vee \tag{3.3.11}$$

by Grothendieck duality for the k-scheme X_0 (see Corollary 5.1.3 and the bottom half of page 224 in [15]), and we know that the right side of (3.3.11) is zero (since $k(x)$ is a skyscraper sheaf supported at the point x), which is a contradiction. Thus, $\mathfrak{m}_x$ contains an $\omega_{X_0/k,x}$-regular element, so (3.3.10) holds, as desired. ∎

We may now prove Theorem 3.3.6.

Proof of Theorem 3.3.6. We begin by defining the trace morphism f_*. To do this, we will proceed as in [15, pg. 97] and use Grothendieck's trace morphism (3.3.2). For any complex $(C^\bullet, d^\bullet)$ of sheaves and any integer n, we denote by $C^\bullet[n]$ the complex whose i th term is C^{i+n} and whose differential in degree i is $(-1)^n d^{i+n}$. Further, we view any sheaf as a complex concentrated in degree zero. Denote by $p_X : X \to S$ and $p_Y : Y \to S$ the structural morphisms. From the definition (3.3.1) of the relative dualizing sheaf, we have quasi-isomorphisms

$$\omega_{X/S}[1] \simeq p_X^! \mathscr{O}_S$$

and

$$\omega_{Y/S}[1] \simeq p_Y^! \mathscr{O}_S.$$

Since f is an S-morphism, we have $p_X = p_Y \circ f$, and there is a natural isomorphism of functors $p_X^! \simeq f^! p_Y^!$ given as in [15, 3.3.14]. Thus, we have a quasi-isomorphism

$$\omega_{X/S}[1] \simeq p_X^! \mathscr{O}_S \simeq f^! p_Y^! \mathscr{O}_S \simeq f^! \omega_{Y/S}[1].$$

Applying the functor $\mathbf{R}f_*$ and composing on the left with the natural map

$$f_*\omega_{X/S}[1] \to \mathbf{R}f_*\omega_{X/S}[1]$$

yields a canonical map

$$f_*\omega_{X/S}[1] \longrightarrow \mathbf{R}f_*\omega_{X/S}[1] \simeq \mathbf{R}f_* f^! \omega_{Y/S}[1] \,.$$

Since f is proper, Grothendieck's theory provides a trace map (3.3.2), which yields a trace morphism

$$f_*\omega_{X/S}[1] \longrightarrow \mathbf{R}f_*\omega_{X/S}[1] \simeq \mathbf{R}f_* f^! \omega_{Y/S}[1] \xrightarrow{\mathrm{Tr}_f} \omega_{Y/S}[1] \,. \tag{3.3.12}$$

We define $f_* : f_*\omega_{X/S} \to \omega_{Y/S}$ to be the map on cohomology groups in degree -1 induced by (3.3.12); observe that this construction commutes with Zariski localization on S and Y. We claim that f_* recovers the usual trace map of §2.3 on differentials on generic fibers after using the canonical identifications of Remark 3.3.4.

We have two morphisms of locally free sheaves on Y_K that we wish to compare, and to do this it suffices to work at the generic point of Y_K. Since K has characteristic zero, the map f_K is generically étale, so our claim is reduced to the assertion that for a finite étale map map $f : X_K \to Y_K$ of smooth (perhaps non-proper) curves over a field K, Grothendieck's trace map (whose formation commutes with base change)

$$f_{K*}\Omega^1_{X_K/K} \simeq f_{K*}f_K^*\Omega^1_{Y_K/K} \to \Omega^1_{Y_K/K}$$

is the canonical trace map (2.3.2) induced by $f_{K*}\mathscr{O}_{X_K} \to \mathscr{O}_{Y_K}$ via the canonical isomorphism $\Omega^1_{X_K/K} \simeq f_K^*\Omega^1_{Y_K/K}$. As we noted in §2.3, such agreement is a consequence of how the theory of the trace map is constructed, or more precisely how Grothendieck's trace map is built in the finite étale case; see for instance the description of (2.7.37) in [15], or [26, III, §6].

We now construct the desired pullback map f^* on dualizing sheaves. First suppose that we have f^* for finite flat f. Let $j : V \hookrightarrow Y$ and $i : U \hookrightarrow X$ be as in Lemma 3.3.5. Since the restriction f_U of f to U is finite flat, and U and V are curves over S (note that it was crucial for this step *not* to require relative curves to be proper), by our temporary hypothesis that the finite flat case is settled, there exists a pullback map $f_U^* : \omega_{V/S} \to f_{U*}\omega_{U/S}$ recovering the usual pullback map on differentials over the generic fiber of V (which *is* the generic fiber of Y). We push f_U^* forward by j and use the identification of Lemma 3.3.7 to obtain the desired map:

$$\omega_{Y/S} \xrightarrow[(3.3.8)]{\simeq} j_*\omega_{V/S} \xrightarrow{j_*(f_U^*)} j_*f_{U*}\omega_{U/S} = f_*i_*\omega_{U/S} \xleftarrow[(3.3.8)]{\simeq} f_*\omega_{X/S}$$

To construct f^* for general finite f, we are thus reduced to the case of finite flat f. The theory of the trace map for finite flat morphisms provides a canonical isomorphism [15, (2.7.38)] (cf. [26], III, Theorem 6.7)

$$f_*\omega_{X/S} \xrightarrow{\simeq} \mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \omega_{Y/S})\,. \tag{3.3.13}$$

We define

$$f^* : \omega_{Y/S} \longrightarrow \mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \omega_{Y/S}) \xleftarrow[(3.3.13)]{\simeq} f_*\omega_{X/S} \tag{3.3.14}$$

by sending a section η of $\omega_{Y/S}$ to the map $f_*\mathscr{O}_X \to \omega_{Y/S}$ which takes any section t of $f_*\mathscr{O}_X$ to the section $\mathrm{Tr}(t)\cdot\eta$ of $\omega_{Y/S}$. Here, $\mathrm{Tr} : f_*\mathscr{O}_X \to \mathscr{O}_Y$ is the usual trace map attached to the finite locally free (f is flat) extension of algebras $\mathscr{O}_Y \to f_*\mathscr{O}_X$.

The map (3.3.14) is obviously $\mathscr{O}_Y$-linear, so it remains to check that it induces pullback of differentials by f_K on generic fibers. As before, we now have two maps of sheaves on Y_K that we wish to compare and by working at the generic point of Y_K we may and do assume that f_K is étale, and hence that the canonical morphism

$$f_{K*}\mathscr{O}_{X_K} \otimes \Omega^1_{Y_K/K} \to f_{K*}\Omega^1_{X_K/K}$$

is an isomorphism. We then wish to show that the map (3.3.14) on generic fibers

$$\Omega^1_{Y_K/K} \longrightarrow \mathscr{H}om_{\mathscr{O}_{Y_K}}(f_{K*}\mathscr{O}_{X_K}, \Omega^1_{Y_K/K}) \xleftarrow{\simeq} f_{K*}\mathscr{O}_{X_K} \otimes \Omega^1_{Y_K/K}$$

agrees with the usual pullback map

$$\Omega^1_{Y_K/K} \to f_{K*}\mathscr{O}_{X_K} \otimes \Omega^1_{Y_K/K}$$

which takes a section η of $\Omega^1_{Y_K/K}$ to $1 \otimes \eta$. In other words, we want to show that the identification

$$f_{K*}\mathscr{O}_{X_K} \otimes \Omega^1_{Y_K/K} \xrightarrow{\simeq} \mathscr{H}om_{\mathscr{O}_{Y_K}}(f_{K*}\mathscr{O}_{X_K}, \Omega^1_{Y_K/K})$$

of (3.3.13) on generic fibers (in the finite étale case) is given by

$$\alpha \otimes \eta \mapsto (t \mapsto \mathrm{Tr}(\alpha t)\cdot\eta)\,.$$

This is a direct consequence of how the theory of the trace is constructed, and in particular how the map (3.3.13) is defined; see (2.7.10) and (2.7.9) in [15], and compare to [26, III, pg. 187]. ■

Remark 3.3.8. Observe that the construction of $f_* : f_*\omega_{X/S} \to \omega_{Y/S}$ used only that f was proper, so we have a trace morphism on dualizing sheaves associated to any proper (which forces finiteness on K-fibers) morphism of relative curves over S.

We conclude this discussion of the relative dualizing sheaf with the following description of its behavior under proper birational morphisms, which will be essential in the next section for proving that the integral structure we provide for the Hodge filtration of X_K is independent of the choice of admissible model X of X_K.

Proposition 3.3.9. *For any proper birational morphism of admissible curves $\rho : X' \to X$ over S, the canonical morphism*

$$\rho_*\omega_{X'/S} \to \omega_{X/S}$$

arising from the trace map on relative dualizing complexes (*cf. Remark* 3.3.8) *is an isomorphism.*

Proof. See [2], Corollary 3.4. ■

3.4 Cohomology of regular differentials

In this section, for any proper relative curve X, we will define a subcomplex $d : \mathscr{O}_X \to \omega_{X/S}$ of the de Rham complex of X_K and study its hypercohomology. When X is admissible, we will use this complex to (finally) produce the promised canonical integral structure on the Hodge filtration of X_K.

Fix a proper relative curve X. We wish to form a complex of sheaves on X that generalizes the usual de Rham complex (and indeed recovers the de Rham complex over the generic fiber of X). Let $i : X_K \hookrightarrow X$ be the inclusion of the generic fiber. Since $\mathscr{O}_X$ and $\omega_{X/S}$ are $\mathscr{O}_S$-flat, the canonical maps $\mathscr{O}_X \to i_*\mathscr{O}_{X_K}$ and $\omega_{X/S} \to i_*\Omega^1_{X_K/K}$ (cf. Remark 3.3.4) are injective. Thus, we can ask whether or not the K-linear derivation

$$d : \mathscr{O}_{X_K} \to \Omega^1_{X_K/K}$$

restricts to an $\mathscr{O}_S$-linear derivation $\mathscr{O}_X \to \omega_{X/S}$. We claim that it does:

Theorem 3.4.1. *There is a unique complex of sheaves*

$$\omega^\bullet_{X/S} := \mathscr{O}_X \xrightarrow{d} \omega_{X/S}$$

whose restriction to X_K is the usual de Rham complex of X_K/K.

Proof. First observe that the problem of showing that $d : \mathscr{O}_{X_K} \to \Omega^1_{X_K/K}$ carries $\mathscr{O}_X$ into $\omega_{X/S}$ is étale-local around the closed fiber of X, since the statement is a tautology on the generic fiber and the formation of $\omega_{X/S}$ is compatible with étale localization on X. Moreover, thanks to the isomorphisms $\mathscr{O}_X \xrightarrow{\simeq} i_*\mathscr{O}_U$ and $\omega_{X/S} \xrightarrow{\simeq} i_*\omega_{U/S}$ for any open subscheme $i : U \hookrightarrow X$ with zero-dimensional complement, it is enough to work étale locally at the generic points of the closed fiber.

We claim that, étale locally at generic points of the closed fiber, there exists a smooth curve C over S and a finite flat morphism $f : X \to C$. Indeed, fix a closed point x in the closed fiber of X, and let π be a uniformizer of R. Since X is normal, it is S_2 by Serre's criterion, and since π is a non-zerodivisor in the local ring $\mathscr{O}_{X,x}$ (due to the flatness of X over S) we may complete it to a regular sequence $\{\pi, t\}$ of the 2-dimensional local ring $\mathscr{O}_{X,x}$. There exists an open neighborhood U of x to which t extends, giving an S- morphism $\varphi : U \to \mathbf{A}^1_S$. Observe that since $\{\pi, t\}$ is a regular sequence in $\mathscr{O}_{X,x}$, the quotient ring $\mathscr{O}_{X_0,x}/(t) = \mathscr{O}_{X,x}/(\pi, t)$ is zero-dimensional, from which we conclude that x is an isolated point in the fiber of φ over $\varphi(x)$. As fiber dimension is upper semi-continuous on the source, we may shrink U around x so that $\varphi : U \to \mathbf{A}^1_S$ is quasi-finite. Observe that φ is flat since any quasi-finite morphism from a Cohen-Macaulay surface to a regular surface is automatically flat [34, Theorem 23.1]. As we have observed, we are free to pass to an étale neighborhood of x and of its image $\varphi(x)$ in $\mathbf{A}^1_S$, and by the general structure of quasi-finite maps [19, IV_4 18.5.11 (c)] we may choose such an étale localization to obtain a *finite* flat S-morphism $f : U \to C$, with C a smooth curve over S. We may further shrink C to ensure that the locally free sheaf $\Omega^1_{C/S}$ is generated as an $\mathscr{O}_C$-module by a single global section, dT for some global function T on C.

The general theory of duality for a finite flat morphism provides a canonical isomorphism

$$f_*\omega_{U/S} \simeq \mathscr{H}om_{\mathscr{O}_C}(f_*\mathscr{O}_U, \omega_{C/S}),$$

(see [15, 2.7.38]) and since C is S-smooth, this is an isomorphism

$$f_*\omega_{U/S} \simeq \mathscr{H}om_{\mathscr{O}_C}(f_*\mathscr{O}_U, \Omega^1_{C/S}).$$

As in the construction of the pullback map on dualizing sheaves (3.3.14), on generic stalks at K-fibers this isomorphism is induced by the canonical morphism

$$f_*\mathscr{O}_U \otimes \Omega^1_{C/S} \longrightarrow \mathscr{H}om_{\mathscr{O}_C}(f_*\mathscr{O}_U, \Omega^1_{C/S})$$

given by

$$\alpha \otimes \eta \mapsto (t \mapsto \mathrm{Tr}(\alpha t) \cdot \eta)\,;$$

see [15, 2.7.10].

Our problem of showing that $d : \mathscr{O}_{X_K} \to \Omega^1_{X_K/K}$ carries $\mathscr{O}_X$ into $\omega_{X/S}$ is thus reduced to the following concrete commutative algebra problem. Let $A \to B$ be a finite flat map of R-algebras, with A, B normal domains such that the extension of fields $Q(B)/Q(A)$ is separable and $\Omega^1_{A/R} = A \cdot dT$ for some $T \in A$. Then for any $b \in B$, we claim that the restriction to B of the natural map $Q(B) \to Q(A)$ given by

$$s \mapsto \mathrm{Tr}(db/dT \cdot s)$$

has image contained in A. Since A is normal, this is a problem at height 1 primes, so we can assume that A is a discrete valuation ring and that B is a semi-local Dedekind domain. Thus, we have to show that the element db/dT lies in the inverse different $D^{-1}_{B/A}$ of the extension $A \to B$ for all $b \in B$.

We have an exact sequence of B-modules

$$B \otimes_A \Omega^1_{A/R} \longrightarrow \Omega^1_{B/R} \longrightarrow \Omega^1_{B/A} \longrightarrow 0$$

which is also left exact, since this can be checked on the level of fraction fields (where it is clear as $Q(B)/Q(A)$ is separable). We wish to show that the B-submodule $D_{B/A} \cdot db$ of $\Omega^1_{B/R}$ is in the image of $B \otimes_A \Omega^1_{A/R}$, or in other words that $D_{B/A}$ annhilates $\Omega^1_{B/A}$. But A is a discrete valuation ring and B is a semi-local Dedekind domain, so our assertion is precisely the "differential characterization of the different" as in [45, III, §7, Proposition 14], and we are done. ∎

The functoriality of the de Rham complex via pullback and trace with respect to finite morphisms as discussed in §2.3 extends to the integral level as follows. If $g : X \to Y$ is any finite S-morphism of relative curves, Theorem 3.3.6 shows that there are natural maps of sheaves $\omega_{Y/S} \to g_*\omega_{X/S}$ and $g_*\omega_{X/S} \to \omega_{Y/S}$ that recover pullback and trace of differentials over K. Using the canonical pullback map $\mathscr{O}_Y \to g_*\mathscr{O}_X$ on functions (3.3.6) and the trace map $g_*\mathscr{O}_X \to \mathscr{O}_Y$ as in (3.3.7) we obtain a pullback diagram

$$\begin{tikzcd} \mathscr{O}_Y \arrow[r, "d"] \arrow[d] & \omega_{Y/S} \arrow[d] \\ g_*\mathscr{O}_X \arrow[r, "g_*(d)"'] & g_*\omega_{X/S} \end{tikzcd}$$

and a trace diagram

$$\begin{tikzcd} g_*\mathscr{O}_X \arrow[r, "g_*(d)"] \arrow[d] & g_*\omega_{X/S} \arrow[d] \\ \mathscr{O}_Y \arrow[r, "d"'] & \omega_{Y/S} \end{tikzcd}$$

that commute because this may be checked over K (as the sheaves in question are all flat), where it is clear by the construction of pullback and trace in §2.3. Thus, $\omega^\bullet_{X/S}$ is contravariant via pullback and covariant via trace in finite morphisms of curves.

Definition 3.4.2. Let $f : X \to S$ be a proper relative curve. Define

$$H^i(X) := \mathbf{H}^i(X, \omega^\bullet_{X/S}).$$

We call the R-modules H^i the *cohomology of regular differentials.*

Let $f : X \to Y$ be a finite morphism of relative curves over S. As in our construction of the pushforward map on cohomology in §2.3, we note that the natural map

$$f_*\omega^\bullet_{X/S} \to \mathbf{R}f_*\omega^\bullet_{X/S}$$

is a quasi-isomorphism due to the fact that f is finite (so any coherent sheaf is f_*-acyclic). Thus we have a canonical isomorphism of δ-functors on complexes of quasicoherent sheaves on X

$$\mathbf{H}^\bullet(Y, f_*(\cdot)) \simeq \mathbf{H}^\bullet(X, (\cdot))$$

(cf. Remark A.1.2). It follows from the preceding discussion that H^i is both a contravariant functor (using pullback) and a covariant functor (using trace) from the category of proper S-curves with finite morphisms to the category of R-modules. Moreover, as the restriction of $\omega^\bullet_{X/S}$ to the generic fiber of X is the de Rham complex of X_K/K, we see that naturally

$$H^i(X) \otimes_R K \simeq H^i_{\mathrm{dR}}(X_K/K).$$

When X is proper and cohomologically flat (for example, admissible), the R-module $H^1(X)$ fits into a short exact sequence that provides an integral structure on the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$:

Proposition 3.4.3. *Suppose that $X \to S$ is a proper and cohomologically flat relative curve. Then there is an exact sequence of free R-modules of finite rank*

$$0 \longrightarrow H^0(X, \omega_{X/S}) \longrightarrow H^1(X) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow 0 \tag{3.4.1}$$

which is functorial (both contravariant and covariant) in finite morphisms of proper and cohomologically flat curves, and recovers the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$ after tensoring with K.

Proof. Associated to the evident filtration of $\omega^\bullet_{X/S}$ ("la filtration bête" [17]) is the usual hypercohomology spectral sequence

$$E_1^{p,q} = H^p(X, \omega^q_{X/S}) \implies H^{p+q}(X)$$

which we claim degenerates at the E_1-stage. To see this, we need only show that the differentials $d : H^i(X, \mathscr{O}_X) \to H^i(X, \omega_{X/S})$ are zero for all i since $E_1^{p,q} = 0$ except possibly for $0 \leq p, q \leq 1$. Since the Hodge to de Rham spectral sequence for the smooth proper geometrically connected curve X_K degenerates at the E_1-stage, the maps

$$d \otimes 1 : H^i(X, \mathscr{O}_X) \otimes K \to H^i(X, \omega_{X/S}) \otimes K$$

are zero; it follows that the image of d is torsion. But $H^i(X, \omega_{X/S})$ is free over R by Proposition 3.3.2, so the image of d must be zero, as desired. Since X is proper, the flanking R-modules in the exact sequence (3.4.1) are finite, and we know they are free by Proposition 3.3.2. We conclude that $H^i(X)$ is also finite and free. By construction, the exact sequence (3.4.1) is functorial (both contravariant and covariant) with respect to finite S-maps in X and recovers the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$ after tensoring with K. ■

We can consider the exact sequence (3.4.1) as an *integral structure* on the Hodge filtration of the first de Rham cohomology group of the smooth proper and geometrically connected curve X_K over K. We wish to show that this integral structure is preserved by the cup product pairing map (2.2.5). More precisely:

Proposition 3.4.4. *Let X be a proper and cohomologically flat relative curve over S. The pairing $(\cdot,\cdot)_{X_K}$ of* (2.2.4) *uniquely extends to a canonical R-bilinear pairing*

$$(\cdot,\cdot)_X : H^1(X) \times H^1(X) \to R. \tag{3.4.2}$$

This pairing induces a morphism of exact sequences of free R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X,\omega_{X/S}) & \longrightarrow & H^1(X) & \longrightarrow & H^1(X,\mathscr{O}_X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1(X,\mathscr{O}_X)^\vee & \longrightarrow & H^1(X)^\vee & \longrightarrow & H^0(X,\omega_{X/S})^\vee & \longrightarrow & 0 \end{array} \tag{3.4.3}$$

recovering (2.2.5) *after extending scalars to K. This morphism is moreover an isomorphism.*

Proof. Uniqueness is obvious. Specializing the "abstract nonsense" map (2.2.1) to the case $\mathscr{X} = X$ and $\mathscr{G}^\bullet = \mathscr{F}^\bullet = \omega^\bullet_{X/S}$ and composing with the natural "wedge product of forms" map of complexes

$$\omega^\bullet_{X/S} \otimes_{\mathscr{O}_X} \omega^\bullet_{X/S} \to \omega^\bullet_{X/S},$$

one obtains a natural R-linear cup product map

$$H^n(X) \otimes_R H^m(X) \to H^{n+m}(X)$$

that clearly recovers the usual cup product on de Rham cohomology after extending scalars to K. Since the spectral sequence

$$E_1^{p,q} = H^p(X, \omega^q_{X/S}) \implies H^{p+q}(X)$$

degenerates already at E_1, the edge map

$$H^2(X) \to H^1(X, \omega_{X/S}) \tag{3.4.4}$$

is an isomorphism and obviously recovers (2.2.3) after tensoring with K. Thus, composing the cup product map

$$H^1(X) \otimes_R H^1(X) \to H^2(X)$$

with the edge map (3.4.4) followed by Grothendieck's trace map (3.3.3) we get an R-bilinear pairing (3.4.2) which by construction recovers the pairing $(\cdot,\cdot)_{X_K}$ on $H^1_{\mathrm{dR}}(X/K)$ after tensoring with K.

Since the cup product of two sections of $\omega_{X/S}$ is zero (by definition), the pairing $(\cdot,\cdot)_X$ restricts to zero on the submodule $H^0(X, \omega_{X/S})$ of $H^1(X)$, and so induces a pairing between $H^0(X, \omega_{X/S})$ and $H^1(X, \mathscr{O}_X)$ which is *a fortiori* the cup product pairing inducing the duality (3.3.4) of Grothendieck duality. Thus, $(\cdot,\cdot)_X$ provides the desired morphism (3.4.3). By Proposition 3.3.2, the flanking vertical maps in (3.4.3) are isomorphisms. Thus, the middle one is too. ■

In general, the integral structure (3.4.1) on the first de Rham cohomology of X_K can depend on the choice of proper and cohomologically flat model X of X_K.

Remarkably, when X_K has an *admissible* model, the integral structure (3.4.1) is intrinsic to X_K, and does not depend on the choice of admissible model of X_K. This fact is made precise by the following theorem, which provides the sought-after canonical integral structure on the Hodge filtration of X_K and is our main result in this section:

Theorem 3.4.5. *Suppose that X_K is a proper smooth and geometrically connected curve over K with an admissible model.*

1. *For any two admissible models X and X' of X_K, there is a unique isomorphism of exact sequences of finite free R-modules*

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\omega_{X/S}) \arrow[r] \arrow[d, "\simeq"] & H^1(X) \arrow[r] \arrow[d, "\simeq"] & H^1(X,\mathscr{O}_X) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & H^0(X',\omega_{X'/S}) \arrow[r] & H^1(X') \arrow[r] & H^1(X',\mathscr{O}_{X'}) \arrow[r] & 0 \end{tikzcd}$$

   respecting the K-fiber identifications.

2. *For any finite morphism $f : X_K \to Y_K$ of proper smooth geometrically connected curves over K that admit admissible models X and Y, there are canonical R-linear morphisms of integral structures*

$$\begin{tikzcd} 0 \arrow[r] & H^0(Y,\omega_{Y/S}) \arrow[r] \arrow[d, "f^*"'] & H^1(Y) \arrow[r] \arrow[d, "f^*"] & H^1(Y,\mathscr{O}_Y) \arrow[r] \arrow[d, "f^*"] & 0 \\ 0 \arrow[r] & H^0(X,\omega_{X/S}) \arrow[r] & H^1(X) \arrow[r] & H^1(X,\mathscr{O}_X) \arrow[r] & 0 \end{tikzcd} \tag{3.4.5}$$

   and

$$\begin{tikzcd} 0 \arrow[r] & H^0(Y,\omega_{Y/S}) \arrow[r] & H^1(Y) \arrow[r] & H^1(Y,\mathscr{O}_Y) \arrow[r] & 0 \\ 0 \arrow[r] & H^0(X,\omega_{X/S}) \arrow[r] \arrow[u, "f_*"] & H^1(X) \arrow[r] \arrow[u, "f_*"'] & H^1(X,\mathscr{O}_X) \arrow[r] \arrow[u, "f_*"'] & 0 \end{tikzcd} \tag{3.4.6}$$

which recover the natural K-linear pullback and trace maps, respectively, on the Hodge filtrations of $H^1_{\mathrm{dR}}(Y_K/K)$ and $H^1_{\mathrm{dR}}(X_K/K)$ induced by f_K (as defined in §2.3) *after extending scalars to K.*

3. *The morphisms* (3.4.5) *and* (3.4.6) *are adjoint with respect to the cup product pairing* (3.4.2)*; i.e., the diagram*

(3.4.7)
$$\begin{array}{ccc} H^1(Y)\times H^1(Y) & \xrightarrow{(\cdot,\cdot)_Y} & R \\ {\scriptstyle f_*}\uparrow \quad \downarrow{\scriptstyle f^*} & & \| \\ H^1(X)\times H^1(X) & \xrightarrow[(\cdot,\cdot)_X]{} & R \end{array}$$

commutes.

Proof. (1). Uniqueness is obvious, by R-freeness. Suppose X and X' are admissible models of X_K. Since any two such models can be dominated by a third by Lemma 3.2.1, we may suppose there is a (necessarily unique) proper birational map $\rho : X' \to X$ inducing the identity on K-fibers. Since ρ is birational and X is normal, the natural map $\mathscr{O}_X \to \rho_*\mathscr{O}_{X'}$ is an isomorphism. Moreover, Proposition 3.3.9 shows that the trace map on relative dualizing complexes $\rho_*\omega_{X'/S} \to \omega_{X/S}$ is an isomorphism, so we obtain a diagram

$$\begin{array}{ccc} \mathscr{O}_X & \longrightarrow & \omega_{X/S} \\ \simeq\downarrow & & \uparrow\simeq \\ \rho_*\mathscr{O}_{X'} & \longrightarrow & \rho_*\omega_{X'/S} \end{array}$$

which commutes because this may be checked over K, where it is the canonical isomorphism induced by the isomorphism $X_K \simeq X'_K$. This yields a map of complexes

$$\omega^\bullet_{X/S} \xrightarrow{\simeq} \rho_*\omega^\bullet_{X'/S} \longrightarrow \mathbf{R}\rho_*\omega^\bullet_{X'/S}$$

that respects the canonical filtrations on both sides, and hence (upon applying

$\mathbf{R}\Gamma(X, \cdot)$) gives a commutative diagram

(3.4.8)
$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \omega_{X/S}) \arrow[r] \arrow[d] & H^1(X) \arrow[r] \arrow[d] & H^1(X, \mathscr{O}_X) \arrow[r] \arrow[d] & 0 . \\ 0 \arrow[r] & H^0(X', \omega_{X'/S}) \arrow[r] & H^1(X') \arrow[r] & H^1(X', \mathscr{O}_{X'}) \arrow[r] & 0 \end{tikzcd}$$

recovering the identification of the Hodge filtrations of $H^1_{\mathrm{dR}}(X_K/K)$ and $H^1_{\mathrm{dR}}(X'_K/K)$ after extending scalars to K.

By Lemma 3.1.7, the right vertical map of (3.4.8) is an isomorphism. By definition, the left vertical map is the inverse of the map on global sections over X induced from the isomorphism $\rho_*\omega_{X'/S} \to \omega_{X/S}$ of Proposition 3.3.9, and is hence an isomorphism too. Over K, this map *is* the canonical pullback map on differentials. It follows that the map $H^1(X) \to H^1(X')$ is also an isomorphism.

(2). Let $f : X_K \to Y_K$ be a finite morphism of smooth proper geometrically connected curves over K that have admissible models X and Y over S. By Theorem 3.2.2, there exist admissible models X' of X_K and Y' of Y_K that dominate X and Y, respectively, and a *finite* morphism $f' : X' \to Y'$ recovering f on generic fibers. By Proposition 3.4.3, there exists an R-linear morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & H^0(Y', \omega_{Y'/S}) \arrow[r] \arrow[d, "f'^*"'] & H^1(Y') \arrow[r] \arrow[d, "f'^*"] & H^1(Y', \mathscr{O}_{Y'}) \arrow[r] \arrow[d, "f'^*"] & 0 \\ 0 \arrow[r] & H^0(X', \omega_{X'/S}) \arrow[r] & H^1(X') \arrow[r] & H^1(X', \mathscr{O}_{X'}) \arrow[r] & 0 \end{tikzcd}$$

recovering the canonical pullback map after extending scalars to K, and an R-linear morphism

$$\begin{tikzcd} 0 \arrow[r] & H^0(Y', \omega_{Y'/S}) \arrow[r] & H^1(Y') \arrow[r] & H^1(Y', \mathscr{O}_{Y'}) \arrow[r] & 0 \\ 0 \arrow[r] & H^0(X', \omega_{X'/S}) \arrow[r] \arrow[u, "f'_*"] & H^1(X') \arrow[r] \arrow[u, "f'_*"'] & H^1(X', \mathscr{O}_{X'}) \arrow[r] \arrow[u, "f'_*"'] & 0 \end{tikzcd}$$

recovering the canonical trace map after tensoring over R with K. Since Y' dominates Y, part (1) ensures that the top row of each diagram is naturally isomorphic to

$$0 \longrightarrow H^0(Y, \omega_{Y/S}) \longrightarrow H^1(Y) \longrightarrow H^1(Y, \mathscr{O}_Y) \longrightarrow 0$$

in a manner that recovers the canonical isomorphism induced by pullback after extending scalars to K. The same reasoning applies to the bottom rows, whence we obtain the desired maps of integral structures.

(3). The claimed adjointness of the maps (3.4.5) and (3.4.6) with respect to the pairing (3.4.2) (i.e., the commutativity of (3.4.7)) may be checked over K, where it reduces to Theorem 2.3.1. ■

CHAPTER IV

p-adic cohomology

In chapter III we saw how to equip the Hodge filtration of a smooth curve over the fraction field K of a discrete valuation ring R with a canonical integral structure (at least when the curve has an admissible model over R). We will ultimately be interested in the case that R is complete of mixed characteristic $(0, p)$, and we will want to employ p-adic cohomology to study the Hodge filtration of a curve over K. We will specifically consider the de Rham cohomology of (topologically finite type) formal schemes over R, schemes and rigid spaces over K, and the rigid and crystalline cohomology of schemes over the residue field k of R. Since one of our main goals in this thesis is to prove certain compatibility results about comparison isomorphisms between these different cohomology theories, it will be essential for future arguments to know how these theories are defined. In this chapter, we therefore recall the definitions of the cohomology theories we will use, and record some of the basic properties of these theories for later use. We remark that this chapter and the next one are primarily intended for reference, and the reader who is content to skip ahead can safely do so.

For the remainder of the chapter, we fix a complete discrete valuation ring R of mixed characteristic $(0, p)$ with field of fractions K and residue field k, and we set

$S = \operatorname{Spec} R$ and $\widehat{S} = \operatorname{Spf} R$.

4.1 de Rham cohomology

Let us begin by recalling the definition of de Rham cohomology in the context of schemes, formal schemes, and rigid spaces.

If $X \to S$ is an S-scheme we denote by $\Omega^1_{X/S}$ the sheaf of relative *algebraic* differentials [19, 0_{III}, §16]. If X is a rigid space over K, then $\Omega^1_{X/K}$ denotes the sheaf of *continuous* relative differentials (cf. [10, §1]). Finally, if $X \to \widehat{S}$ is a topologically finite type formal $\widehat{S}$-scheme, we write $\widehat{\Omega}^1_{X/\widehat{S}}$ for the sheaf of *continuous* relative differentials. In each case, the de Rham complex is the complex whose nth term is the $\mathscr{O}_X$-module $\wedge^n\Omega^1$, with the usual boundary maps d. By *de Rham cohomology*, H^n_{dR} we *always* mean the hypercohomology of the de Rham complex. By pullback of differentials and (A.1.3), we see that H^n_{dR} is a contravariant functor from the category of S-schemes (respectively rigid spaces over K, topologically finite type formal $\widehat{S}$-schemes) to the category of R-modules (respectively K-vector spaces, R-modules).

The formation of the de Rham complex is compatible with formal completion and analytification. More precisely, let X be a locally finite type S-scheme, and denote by $\widehat{X}$ its formal completion along the closed fiber. The natural map $\widehat{X} \to X$ yields a morphism of coherent $\mathscr{O}_{\widehat{X}}$-modules

$$\widehat{\Omega^1_{X/S}} \to \widehat{\Omega}^1_{\widehat{X}/\widehat{S}} \tag{4.1.1}$$

which we claim is an isomorphism. Similarly, for any K-scheme X that is locally of finite type, we denote by X^{an} the corresponding rigid space, endowed with the usual Tate topology. The natural map $X^{\mathrm{an}} \to X$ yields an $\mathscr{O}_{X^{\mathrm{an}}}$-linear map of coherent sheaves

$$(\Omega^1_{X/K})^{\mathrm{an}} \to \Omega^1_{X^{\mathrm{an}}/K} \tag{4.1.2}$$

which we claim is an isomorphism. To see our claims that (4.1.1) and (4.1.2) are isomorphisms, one passes to completed stalks, where it is clear. These isomorphisms uniquely extend to de Rham complexes in an evident manner.

4.2 Rigid cohomology

Let X be a separated k-scheme of finite type. In this section, we will define the *rigid cohomology* groups of X relative to K (under a mild hypothesis on X that is satisfied, for example, when X is quasiprojective). These groups are finite-dimensional K-vector spaces that are functorial in (k-morphisms of) X and provide a reasonably "universal" p-adic cohomology theory in the sense that there are comparison isomorphisms with crystalline cohomology when X is smooth and proper on the one hand, and Monsky-Washnitzer cohomology when X is smooth and affine on the other hand (the comparison with crystalline cohomology will be reviewed in the next section). The main references for this section are Berthelot's papers [3] and [5].

Let us begin by reviewing some basic facts and constructions in formal and rigid geometry that will be needed for the construction of rigid cohomology.

For a topologically finite type formal R-scheme $\mathscr{P}$, we denote by $\mathscr{P}^{\mathrm{rig}}$ the Raynaud generic fiber of $\mathscr{P}$; it is a rigid analytic space over K. When $\mathscr{P} = \operatorname{Spf} A$ is affine, the K-algebra $A \otimes K$ is K-affinoid (as A is topologically of finite type) and we have $\mathscr{P}^{\mathrm{rig}} = \mathrm{Sp}(A \otimes K)$. The general construction follows from this one by gluing.

Let $\mathrm{sp} : \mathscr{P}^{\mathrm{rig}} \to \mathscr{P}$ be the *specialization morphism*; this morphism of ringed sites has as target the Zariski site of $\mathscr{P}$, which is the Zariski site of $\mathscr{P}_0$, where $\mathscr{P}_0$ denotes the closed fiber of $\mathscr{P}$. First we construct it as a map of sets. For $\mathscr{P} = \operatorname{Spf} A$, a map of sets

$$\mathrm{sp} : \mathrm{Sp}(A \otimes K) \to \operatorname{Spf} A$$

is defined as follows (cf. [5, §1.1] and [8, §7.1.5]). For each point x of $\mathrm{Sp}(A \otimes K)$ corresponding to a maximal ideal $\mathfrak{m}_x$ and each $a \in A$, we denote by $a(x)$ the image of a in the residue field $K(x) := (A \otimes K)/\mathfrak{m}_x$ under the canonical map

$$A \to A \otimes K \to K(x). \tag{4.2.1}$$

The point $\mathrm{sp}(x)$ of $\operatorname{Spf} A$ corresponds to the maximal ideal

$$\{a \in A \mid |a(x)| < 1\},$$

where $|\cdot|$ is the unique norm on $K(x)$ extending the one on K. In other words, $\mathrm{sp}(x)$ is the maximal ideal of A that is the pullback under (4.2.1) of the maximal ideal of topological nilpotents in the valuation ring of $K(x)$.

For any Zariski open U of $\operatorname{Spf} A$, one has

$$\mathrm{sp}^{-1}(U) = U^{\mathrm{rig}},$$

so the case of general $\mathscr{P}$ follows from this one by gluing. In general, we have $\mathrm{sp}^{-1}(U) = U^{\mathrm{rig}}$ for any Zariski-open $U \subseteq \mathscr{P}$, and this defines sp_* as well as an evident map of sheaves of R-algebras

$$\mathscr{O}_{\mathscr{P}} \to \mathrm{sp}_* \mathscr{O}_{\mathscr{P}^{\mathrm{rig}}}.$$

The adjoint sp^* is easily constructed and is seen to be left exact, so sp_* defined a morphism of ringed sites.

Example 4.2.1. Let $R = \mathbf{Z}_p$ and let $\mathscr{P} = \operatorname{Spf} \mathbf{Z}_p\{T_1, T_2\}$ be the formal spectrum of the p-adic completion of $\mathbf{Z}[T_1, T_2]$, so $\mathscr{P}^{\mathrm{rig}} = \mathrm{Sp}\, \mathbf{Z}_p\langle T_1, T_2\rangle$ is the closed unit ball in rigid analytic affine 2-space. Let X be the locally closed subset of $\mathscr{P}_0$ defined by $T_1 = 0$, $T_2 \neq 0$. Then

$$\mathrm{sp}^{-1}(X) = \{x \in \mathrm{Sp}\,\mathbf{Z}_p\langle T_1, T_2\rangle \ : \ |T_1(x)| < 1,\ |T_2(x)| = 1\}.$$

For a locally closed subset W of $\mathscr{P}_0$, the *tube* of W in $\mathscr{P}^{\mathrm{rig}}$, denoted $]W[_{\mathscr{P}}$, is the admissible open set $\mathrm{sp}^{-1}(W)$. If $W = V(f_1, \ldots, f_n) \cap D(g_1, \ldots, g_m)$ for sections $f_i, g_j \in \Gamma(\mathscr{P}, \mathscr{O}_{\mathscr{P}})$, then as in Example 4.2.1 one calculates

$$]W[_{\mathscr{P}} = \{x \in \mathscr{P}^{\mathrm{rig}} \mid |f_i(x)| < 1,\ |g_j(x)| = 1 \text{ for all } i, j\}.$$

There is another type of tube that we will need later, which we define now. First suppose that W is a closed subset of $\mathscr{P}_0$ defined by the vanishing of the reductions of sections $f_1, \ldots, f_n$ of $\Gamma(\mathscr{P}, \mathscr{O}_{\mathscr{P}})$. For $\eta \in |\overline{K}^{\times}|$ with $\eta < 1$ we define the *closed tube of radius* η, denoted $[W]_{\mathscr{P},\eta}$, to be the locus of points

$$\{x \in \mathscr{P}^{\mathrm{rig}} \mid |f_i(x)| \leq \eta \text{ for all } i\}. \tag{4.2.2}$$

This is evidently an open affinoid subdomain of $\mathscr{P}^{\mathrm{rig}}$. For general η, this definition depends on the choice of the f_i. However, for η sufficiently close to 1, the locus (4.2.2) is intrinsic to W. Indeed, for a different choice of sections $\{g_j\}_{j=1}^m$ whose reductions cut out W, we can write

$$g_j = \sum_{i=1}^{n} a_{ij} f_i + b_j$$

with $a_{ij} \in \Gamma(\mathscr{P}, \mathscr{O}_{\mathscr{P}})$ and $b_j \in \pi\Gamma(\mathscr{P}, \mathscr{O}_{\mathscr{P}})$. Since $a_{ij} \leq 1$ and $|b_j| \leq |\pi|$, we see that for x in the set (4.2.2) we have

$$|g_j(x)| \leq \max_i\{|f_i(x)|, |b_j(x)|\} \leq \max\{\eta, |\pi|\}.$$

Thus, for $\eta \geq |\pi|$ the set (4.2.2) is intrinsic to W (and it makes sense to denote it $[W]_{\mathscr{P},\eta})$. We will only use this type of tube when $\eta \geq |\pi|$. In general, if $|\eta| \geq |\pi|$, then for a closed subset W defined by a coherent ideal sheaf $\mathscr{I}$ of $\mathscr{O}_{\mathscr{P}}$, we can glue the above constructions and so define an admissible open $[W]_{\mathscr{P},\eta}$ of $\mathscr{P}^{\mathrm{rig}}$ with the above local description.

We now recall the construction of rigid cohomology. Let X be a separated k-scheme of finite type, and let $j : X \hookrightarrow \overline{X}$ be a compactification of X (*i.e.* an open immersion into a proper k-scheme $\overline{X}$). Such a j always exists, as proved by Nagata [33]. Suppose that there exists a formal R-scheme $\mathscr{P}$, topologically of finite type, and a closed immersion $i : \overline{X} \hookrightarrow \mathscr{P}$ such that $\mathscr{P}$ is smooth at the points of X (*i.e.* $\mathscr{P} \to \operatorname{Spf} R$ is formally smooth at all points of X, viewed as points of $\mathscr{P}$ via $i \circ j$). We will *always* assume that all schemes over k that we work with admit a compatible pair consisting of a compactification j and an immersion i as above; this is certainly the case if we restrict our attention to *quasi-projective* schemes over k.

Let Z be the complement of X in $\overline{X}$. We say that an open neighborhood V of $]X[_{\mathscr{P}}$ in $]\overline{X}[_{\mathscr{P}}$ is *strict* if the open covering of $]\overline{X}[_{\mathscr{P}}$ furnished by $]Z[_{\mathscr{P}}$ and V is admissible.

Define a functor $j^\dagger$ from the category of abelian sheaves on $]\overline{X}[_{\mathscr{P}}$ to itself by

$$j^\dagger \mathscr{E} := \varinjlim_{V} j_{V*} j_V^{-1} \mathscr{E}, \tag{4.2.3}$$

where the direct limit is taken over all strict open neighborhoods V of $]X[_{\mathscr{P}}$ in $]\overline{X}[_{\mathscr{P}}$ and $j_V : V \hookrightarrow]\overline{X}[_{\mathscr{P}}$ is the inclusion map. The functor $j^\dagger$ is exact [3, (2.1.3)].

For an abelian sheaf $\mathscr{F}$ on $]\overline{X}[_{\mathscr{P}}$, we will often call $j^\dagger \mathscr{F}$ the sheaf of "germs of sections of $\mathscr{F}$ that are overconvergent along Z", or for short, the sheaf of "overconvergent sections." This is because for any quasi-compact open W of $]\overline{X}[_{\mathscr{P}}$, one has

$$\Gamma(W, j^\dagger \mathscr{F}) = \varinjlim_{V} \Gamma(V \cap W, \mathscr{F}),$$

where the direct limit is taken over all strict opens V of $]X[_{\mathscr{P}}$ in $]\overline{X}[_{\mathscr{P}}$ (see Remark B.1.3).

As explained in §4.1, we denote by $\Omega^{\bullet}_{]\overline{X}[_{\mathscr{P}}/K}$ the de Rham complex of the rigid space $]\overline{X}[_{\mathscr{P}}$ over K.

Proposition 4.2.2. *Up to canonical quasi-isomorphism, the complex*

$$\mathbf{R}\,\mathrm{sp}_*\, j^{\dagger}\Omega^{\bullet}_{]\overline{X}[_{\mathscr{P}}/K} \tag{4.2.4}$$

is independent of the choices of compactification $j : X \hookrightarrow \overline{X}$ and closed immersion $i : \overline{X} \hookrightarrow \mathscr{P}$. Consequently, the hypercohomology groups

$$\mathbf{H}^{\bullet}(\overline{X}, \mathbf{R}\,\mathrm{sp}_*\, j^{\dagger}\Omega^{\bullet}_{]\overline{X}[_{\mathscr{P}}/K}) \tag{4.2.5}$$

are canonically independent of these choices.

Proof. This follows immediately from Théorèmes 1.4 and 1.6 of [5]. ∎

We claim that the complex (4.2.4) is functorial in X. First, suppose we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "w"] & Y' \arrow[r, "i'"] \arrow[d, "v"] & \mathscr{P}' \arrow[d, "u"] \\ X \arrow[r, "j"] & Y \arrow[r, "i"] & \mathscr{P} \end{tikzcd} \tag{4.2.6}$$

where j and j' are open immersions, i and i' are closed immersions and $\mathscr{P}$ and $\mathscr{P}'$ are topologically finite type formal R-schemes that are smooth at the points of X and X', respectively. For any admissible open U of $]Y[_{\mathscr{P}}$ and any strict neighborhood V of $]X[_{\mathscr{P}}$ in $]Y[_{\mathscr{P}}$, the admissible open set $(u^{\mathrm{rig}})^{-1}(V \cap U)$ of $]Y'[_{\mathscr{P}'}$ is a strict open neighborhood of $]X'[_{\mathscr{P}'}$ in $]Y'[_{\mathscr{P}'}$ (see [3, Proposition 1.2.7]). This enables us to define, for any abelian sheaf $\mathscr{F}$ on $]Y[_{\mathscr{P}}$, a canonical morphism of sheaves

$$j^{\dagger}\mathscr{F} \to u^{\mathrm{rig}}_* j'^{\dagger}(u^{\mathrm{rig}})^{-1}\mathscr{F}. \tag{4.2.7}$$

Applying this to the de Rham complex of $]Y[_{\mathscr{P}}$ over K and composing with the natural pullback map of differentials, we obtain a K-linear pullback map of overconvergent de Rham complexes

$$(4.2.8) \qquad j^\dagger\Omega^\bullet_{]Y[_{\mathscr{P}}/K} \longrightarrow \mathbf{R}u^{\mathrm{rig}}_*(j'^\dagger\Omega^\bullet_{]Y'[_{\mathscr{P}'}/K})\ .$$

As we have a commutative diagram of sites

$$\begin{tikzcd} \mathscr{P}'^{\,\mathrm{rig}} \arrow[r, "\mathrm{sp}'"] \arrow[d, "u^{\mathrm{rig}}"'] & \mathscr{P}' \arrow[d, "u"] \\ \mathscr{P}^{\mathrm{rig}} \arrow[r, "\mathrm{sp}"'] & \mathscr{P} \end{tikzcd}$$

we thus obtain a morphism of complexes

$$(4.2.9) \qquad \mathbf{R}\,\mathrm{sp}_*\, j^\dagger\Omega^\bullet_{]Y[_{\mathscr{P}}/K} \longrightarrow \mathbf{R}u_*\mathbf{R}\,\mathrm{sp}'_*(j'^\dagger\Omega^\bullet_{]Y'[_{\mathscr{P}'}/K})$$

so we conclude that (4.2.4) is functorial in the triple $(X, Y, \mathscr{P})$.

In fact, since it is canonically independent of the choice of $\mathscr{P}$, one can see by an easy argument with fiber products that the complex (4.2.4) is even functorial in the pair (X, Y); see the discussion below Corollaire 1.5 of [5]. Repeating this fiber product argument and using that (4.2.4) is canonically independent of the choice of compactification Y, we conclude that the complex (4.2.4), and hence its hypercohomology (4.2.5), is functorial in X alone. Thus, these hypercohomology groups give a reasonable p-adic cohomology:

Definition 4.2.3. The *rigid cohomology* groups of X relative to K are

$$H^\bullet_{\mathrm{rig}}(X/K) := \mathbf{H}^\bullet(\overline{X}, \mathbf{R}\,\mathrm{sp}_*\, j^\dagger\Omega^\bullet_{]\overline{X}[_{\mathscr{P}}/K}).$$

Remark 4.2.4. Applying Remark A.1.2 to the morphism of topoi associated to the morphism of sites $\mathrm{sp} :]\overline{X}[_{\mathscr{P}} \to \mathscr{P}$ and the complex $j^\dagger\Omega^\bullet_{]\overline{X}[_{\mathscr{P}}/K}$ of sheaves on $]\overline{X}[_{\mathscr{P}}$, we find that there is a natural isomorphism

$$\mathbf{H}^\bullet(]\overline{X}[_{\mathscr{P}}, j^\dagger\Omega^\bullet_{]\overline{X}[_{\mathscr{P}}/K}) \to H^\bullet_{\mathrm{rig}}(X/K).$$

Remark 4.2.5. In this work, we will only be concerned with the rigid cohomology groups of certain open subschemes U of the smooth locus of the special fiber of a proper formal R-scheme $\mathscr{P}$ that is smooth at the points of U. Such U come equipped with a canonical compactification $j : U \hookrightarrow \overline{U}$ (take $\overline{U}$ to be the closure of U in the special fiber $\mathscr{P}_0$ of $\mathscr{P}$) and a canonical closed immersion of $\overline{U}$ into a proper formal R-scheme (the composite of the two closed immersions $i : \overline{U} \hookrightarrow \mathscr{P}_0 \hookrightarrow \mathscr{P}$). Therefore, in our situations of interest, the rigid cohomology groups will always be defined.

Suppose that the absolute Frobenius endomorphism of k admits a lift σ to K. We want to equip the K-vector spaces $H^i_{\mathrm{rig}}(X/K)$ with a canonical σ-semilinear endomorphism F coming from the relative Frobenius on X. In order to do this, we need first to prove an appropriate base change theorem for rigid cohomology.

Suppose that $R \to R'$ is a finite flat map of discrete valuation rings and denote by K' and k' the fraction field and residue field of R'. Let X be any finite type k-scheme and let X' be the base change $X_{k'}$. Choosing a compactification $j : X \hookrightarrow Y$ and a closed immersion $Y \hookrightarrow \mathscr{P}$ into a formal $\operatorname{Spf} R$-scheme $\mathscr{P}$ that is smooth at the points of X, we obtain via base change a diagram of the form (4.2.6) with cartesian squares. Since K' is finite over K, we may consider $\mathscr{P}'^{\mathrm{rig}}$ as a rigid space over K that is finite over $\mathscr{P}^{\mathrm{rig}}$. The morphism (4.2.7) induces a canonical K'-linear pullback morphism of complexes on $]Y[_{\mathscr{P}}$

$$K' \otimes_K j^\dagger \Omega^\bullet_{]Y[_{\mathscr{P}}/K} \longrightarrow \mathbf{R}u^{\mathrm{rig}}_* j'^\dagger \Omega^\bullet_{]Y'[_{\mathscr{P}'}/K'}. \tag{4.2.10}$$

Theorem 4.2.6. *In the situation above, the map* (4.2.10) *is an isomorphism. Thus, for each i there is a canonical isomorphism of K'-vector spaces*

$$K' \otimes_K H^i_{\mathrm{rig}}(X/K) \xrightarrow{\sim} H^i_{\mathrm{rig}}(X'/K').$$

Proof of Theorem 4.2.6. That (4.2.10) is an isomorphism follows immediately from the proof of [5, Proposition 1.8]. The second assertion of the theorem follows from the first after noting that $(u^{\mathrm{rig}})^{-1}(]Y[_{\mathscr{P}}) =]Y'[_{\mathscr{P}'}$ and that $]Y[_{\mathscr{P}}$ is quasi-compact and quasi-separated, so hypercohomology commutes with flat extension of scalars (as one sees via Čech theory). ∎

Remark 4.2.7. The proof of [5, Proposition 1.8] crucially uses that the map $K \to K'$ is finite (note that we needed finiteness to even define the map (4.2.10)). It is undoubtedly true that base change for rigid cohomology holds with K' of infinite degree over K, but we do not have a proof of this fact.

In the case that the Frobenius endomorphism of k admits a lift to K, Theorem 4.2.6 enables us to equip the rigid cohomology of any finite type separated k-scheme with a semilinear Frobenius endomorphism:

Corollary 4.2.8. *Suppose that the Frobenius endomorphism of k admits a lift σ to K and let X be any finite type and separated k-scheme. For each i, Frobenius induces a canonical σ-linear endomorphism F of the K-vector space $H^i_{\mathrm{rig}}(X/K)$.*

Proof. Denote by $X^{(p)}$ the Frobenius twist of X; i.e. the base change of $X \to \operatorname{Spec} k$ by the absolute Frobenius of k. By the functoriality of rigid cohomology, the relative Frobenius morphism $F_{X/k} : X \to X^{(p)}$ induces a K-linear pullback map

$$H^i_{\mathrm{rig}}(X^{(p)}/K) \to H^i_{\mathrm{rig}}(X/K). \tag{4.2.11}$$

Using Theorem 4.2.6, we see that the projection map $X^{(p)} \to X$ yields a canonical isomorphism

$$H^i_{\mathrm{rig}}(X/K) \otimes_K K \xrightarrow{\sim} H^i_{\mathrm{rig}}(X^{(p)}/K), \tag{4.2.12}$$

where the tensor product is over $\sigma : K \to K$. The composition of (4.2.11) and (4.2.12) gives the desired σ-linear endomorphism F of $H^i_{\mathrm{rig}}(X/K)$. ■

4.3 Crystalline cohomology

We now define the crystalline cohomology of a smooth and proper scheme over a perfect field of characteristic p. Keeping our notation from before, we impose the additional hypothesis that k is perfect. Denote by e the absolute ramification index of R and by π a uniformizer.

We suppose that $e \leq p-1$, so the ideal (π) of R is equipped with a canonical divided power structure γ [7, 3.28]. Set $S = \operatorname{Spec} R$ and $\widehat{S} = \operatorname{Spf} R$. Let X be any quasi-compact k-scheme. Following Berthelot [7, 7.17], define the crystalline site $\mathrm{Cris}(X/\widehat{S})$ as follows. The objects of (the underlying category of) $\mathrm{Cris}(X/\widehat{S})$ are divided power thickenings $(U \hookrightarrow T, \delta)$, where U is a Zariski open subset of X and $U \hookrightarrow T$ is a closed S-immersion defined by an ideal sheaf $\mathscr{J}$ with divided power structure δ that is compatible with γ, and such that p is locally nilpotent on T. Morphisms are commutative squares

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & T \arrow[d] \\ U' \arrow[r, hook] & T' \end{tikzcd}$$

with $U \to U'$ an inclusion in the Zariski topology of X and $T \to T'$ a divided power morphism. Finally, the covering families are sets of morphisms

$$\{(U_i \hookrightarrow T_i, \delta_i) \to (U \hookrightarrow T, \delta)\}$$

with each $T_i \to T$ an open immersion and $\cup T_i = T$. This gives a pretopology in the sense of [49, Exp. II, 1.3], and hence defines a site. We write $(X/\widehat{S})_{\mathrm{cris}}$ for the topos associated to $\mathrm{Cris}(X/\widehat{S})$. There is a sheaf of rings in $(X/\widehat{S})_{\mathrm{cris}}$, the "structure sheaf" denoted $\mathscr{O}_{X/\widehat{S}}$, which assigns to any object $(U \hookrightarrow T, \delta)$ the ring $\mathscr{O}_T(T)$.

Definition 4.3.1. The *crystalline cohomology* of X with respect to R is the cohomology

$$H^\bullet_{\mathrm{cris}}(X/R) := H^\bullet((X/\widehat{S})_{\mathrm{cris}}, \mathscr{O}_{X/\widehat{S}})$$

in the sense of Definition A.1.1.

Proposition 4.3.2. *The assignment $X \mapsto H^\bullet_{\mathrm{cris}}(X/R)$ defines a contravariant functor from the category of smooth and proper k-schemes to the category of finitely generated R-modules.*

Proof. See [7], 7.26. ■

In the next section, we will relate relate crystalline cohomology to other p-adic cohomology theories (e.g. rigid cohomology). In order to do this, we will need a description of the crystalline cohomology of X in terms of the hypercohomology of a complex of sheaves on the Zariski site of X. Such a description is essentially given by the crystalline Poincaré lemma, as we now explain.

Let X_{zar} denote the topos of sheaves on the Zariski site of X, and for any Zariski open immersion $j : U \hookrightarrow X$ let

$$j_{\mathrm{cris}} : (U/\widehat{S})_{\mathrm{cris}} \to (X/\widehat{S})_{\mathrm{cris}}$$

be the morphism of ringed topoi induced by j (see [7, 5.6] or [4, III, §2]). There is a canonical morphism of ringed topoi

$$u_{X/\widehat{S}} : (X/\widehat{S})_{\mathrm{cris}} \to X_{\mathrm{zar}}$$

defined as in [7, 5.18]:

1. If $\mathscr{F} \in (X/\widehat{S})_{\mathrm{cris}}$ then $u_{X/\widehat{S}_*}\mathscr{F}$ is the Zariski sheaf on X that assigns to an open set $j : U \hookrightarrow X$ the set
$$\Gamma((U/\widehat{S})_{\mathrm{cris}}, j^*_{\mathrm{cris}}\mathscr{F}).$$

2. If $\mathscr{G}$ is an $\mathscr{O}_X$-module, then $u^*_{X/\widehat{S}}(\mathscr{G})$ is the sheaf on $\mathrm{Cris}(X/\widehat{S})$ that assigns to any object $(U \hookrightarrow T, \delta)$ of $\mathrm{Cris}(X/\widehat{S})$ the $\mathscr{O}_T(T)$-module (via $\mathscr{O}_T(T) \to \mathscr{O}_U(U)$) $\Gamma(U, \mathscr{G})$.

By Remark A.1.2, we then have a natural R-linear isomorphism

$$H^\bullet_{\mathrm{cris}}(X/R) \simeq \mathbf{H}^\bullet(X, \mathbf{R}u_{X/\widehat{S}_*}\mathscr{O}_{X/\widehat{S}}). \tag{4.3.1}$$

The “abstract nonsense” isomorphism (4.3.1) at least allows us to compute crystalline cohomology as the hypercohomology of a complex on the Zariski site of X. However, the complex $\mathbf{R}u_{X/\widehat{S}_*}\mathscr{O}_{X/\widehat{S}}$ is difficult to work with. Roughly speaking, the crystalline Poincaré lemma says that this abstract complex is canonically quasi-isomorphic to a certain de Rham complex. When X admits a smooth lift $\mathscr{X}$ to R, one knows that the crystalline cohomology of X is canonically isomorphic to the de Rham cohomology of $\mathscr{X}$ (this is the simplest form of the crystalline Poincaré lemma; cf. Proposition 5.3.2). For general X, it is too much to hope for a smooth lifting $\mathscr{X}$ of X over R. In the cases of interest to us, we will however be able to embed X as a closed subscheme of a topologically finite type formal $\widehat{S}$-scheme $\mathscr{P}$ (this is the case, for example, when X is projective). One might then hope that the crystalline cohomology of X can be understood in terms of the de Rham complex of $\mathscr{P}$. In fact, this is indeed the case; we make this precise below.

Suppose that there is a closed immersion $i : X \hookrightarrow \mathscr{P}$ of X into a topologically finite type formal $\widehat{S}$-scheme $\mathscr{P}$ that is smooth at the points of X and let $\mathscr{I}$ be the corresponding coherent ideal sheaf. For each positive integer n, we denote by $\mathscr{P}_n$

the reduction of $\mathscr{P}$ mod π^n; it is a finite type $R/\pi^n R$-scheme. The closed immersion i : gives a compatible system of closed immersions $i_n : X \hookrightarrow \mathscr{P}_n$.

Let $\mathscr{D}_X(\mathscr{P})$ be the divided power envelope of $\mathscr{I}$ and define

$$\widehat{\mathscr{D}}_X(\mathscr{P}) := \varprojlim_n \mathscr{D}_X(\mathscr{P}_n);$$

this is a quasi-coherent sheaf of $\mathscr{O}_{\mathscr{P}}$-algebras (see [7, §3]).

The canonical connection (see [7, 6.4]) $\nabla : \mathscr{D}_X(\mathscr{P}) \to \mathscr{D}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^1_{\mathscr{P}/\widehat{S}}$ given for any section s of $\mathscr{I}$ by

$$\nabla(s^{[n]}) = s^{[n-1]} \otimes ds \tag{4.3.2}$$

induces an integrable connection on $\widehat{\mathscr{D}}_X(\mathscr{P})$, making $\widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^{\bullet}_{\mathscr{P}/\widehat{S}}$ into a complex (via $d(f \otimes \omega) = \nabla(f) \wedge \omega + f \otimes d\omega$). The desired description of crystalline cohomology is furnished by:

Proposition 4.3.3. *In the above situation, there is a natural isomorphism in the derived category of bounded below complexes of sheaves of R-modules on the Zariski site of $\mathscr{P}$*

$$\mathbf{R}i_*\mathbf{R}u_{X/\widehat{S}_*}\mathscr{O}_{X/\widehat{S}} \xrightarrow{\simeq} \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^{\bullet}_{\mathscr{P}/\widehat{S}}, \tag{4.3.3}$$

which is functorial in the pair $(X, \mathscr{P})$.

Proof. Since $\widehat{\mathscr{D}}_X(\mathscr{P})$ depends only on a neighborhood of X in $\mathscr{P}$ [7, 3.34], and since $\mathscr{P}$ is smooth at the points of X (hence in a neighborhood of X) we may without loss of generality suppose that $\mathscr{P}$ is smooth. The proposition then follows at once from Theorem 7.23 of [7]. ■

Remark 4.3.4. There is a slight abuse of notation in [7, Theorem 7.23]. Namely, $\mathbf{R}i_*$ is omitted from the left hand side of (4.3.3).

Using Proposition 4.3.3, let us show that crystalline cohomology commutes with finite flat base change. Let R' be a discrete valuation ring with field of fractions K' and residue field k', and fix a uniformizer π' of R'. We further suppose that the ideal (π') is equipped with a divided power structure γ' (i.e. that the absolute ramification index e' of R' satisfies $e' \leq p-1$). In addition, suppose that R' is a finite (flat) R-algebra. The canonical map $R \to R'$ is automatically a divided power morphism (as one sees by working over K'). We set $S' = \operatorname{Spec} R'$ and $\widehat{S}' = \operatorname{Spf} R'$.

Let X be any smooth and proper k-scheme, and suppose that X admits a closed immersion into a topologically finite type formal $\widehat{S}$-scheme $\mathscr{P}$ that is smooth at the points of X. Via the base change $S' \to S$ we obtain a closed immersion $X' \hookrightarrow \mathscr{P}'$ (with obvious notation) and a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & \mathscr{P}' \arrow[d, "u"] \\ X \arrow[r, hook] & \mathscr{P} \end{tikzcd} \tag{4.3.4}$$

Due to the functoriality of divided power envelopes (see [4, I, Proposition 4.1.5]), there is a canonical morphism of sheaves on $\mathscr{P}$

$$\widehat{\mathscr{D}}_X(\mathscr{P}) \longrightarrow u_*\widehat{\mathscr{D}}_{X'}(\mathscr{P}') \,. \tag{4.3.5}$$

that is functorial in $(X, \mathscr{P})$ and compatible with the respective connections ∇ and ∇' via the canonical map

$$\widehat{\Omega}^1_{\mathscr{P}/\widehat{S}} \to u_*\widehat{\Omega}^1_{\mathscr{P}'/\widehat{S}'}.$$

Together with the usual pullback map on differentials, this yields a natural map in the derived category of bounded below complexes of sheaves on $\mathscr{P}$

$$\widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^\bullet_{\mathscr{P}/\widehat{S}} \longrightarrow \mathbf{R}u_* \left(\widehat{\mathscr{D}}_{X'}(\mathscr{P}') \otimes_{\mathscr{O}_{\mathscr{P}'}} \widehat{\Omega}^\bullet_{\mathscr{P}'/\widehat{S}'}\right) \tag{4.3.6}$$

that is linear over $R \to R'$ and so by extension of scalars an R'-linear morphism

$$R' \otimes_R \left(\widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^\bullet_{\mathscr{P}/\widehat{S}}\right) \longrightarrow \mathbf{R}u_* \left(\widehat{\mathscr{D}}_{X'}(\mathscr{P}') \otimes_{\mathscr{O}_{\mathscr{P}'}} \widehat{\Omega}^\bullet_{\mathscr{P}'/\widehat{S}'}\right) \tag{4.3.7}$$

that is functorial in $(X, \mathscr{P})$.

Proposition 4.3.5. *The map* (4.3.7) *is an isomorphism. Hence, there is a canonical isomorphism of R'-modules*

$$R' \otimes_R H^i_{\mathrm{cris}}(X/R) \longrightarrow H^i_{\mathrm{cris}}(X'/R')\ . \tag{4.3.8}$$

Proof. By hypothesis, the map $R \to R'$ is flat. It follows from [4, I, Proposition 4.1.6, (iv)] (or [7, Proposition 3.21]) that the natural map

$$R' \otimes_R \widehat{\mathscr{D}}_X(\mathscr{P}) \to \mathbf{R}u_* \widehat{\mathscr{D}}_{X'}(\mathscr{P}')$$

deduced from (4.3.5) is an isomorphism. As the canonical pullback map

$$R' \otimes_R \widehat{\Omega}^\bullet_{\mathscr{P}/\widehat{S}} \to \mathbf{R}u_* \widehat{\Omega}^\bullet_{\mathscr{P}'/\widehat{S}'}$$

is also an isomorphism, we conclude that (4.3.7) is an isomorphism. Passing to hypercohomology and using the fact (proved via Čech theory) that hypercohomology of bounded below complexes of coherent sheaves commutes with the finite flat extension of scalars $R \to R'$, we deduce the second statement of the proposition from Proposition 4.3.3, (4.3.1), and Remark A.1.2. ■

Remark 4.3.6. Since the base change map (4.3.7) and the map (4.3.3) are functorial in $(X, \mathscr{P})$, one sees by a standard argument with fiber products that the resulting map (4.3.8) is canonically independent of the choice of $\mathscr{P}$. In fact, one can define a base change map for crystalline cohomology without the auxiliary device of a the closed immersion $X \hookrightarrow \mathscr{P}$ by appealing to the crystalline adjunction formula; see

[7, Theorem 7.8] and [4, V, Théorème 3.5.1]. This more general base change map is compatible with the map (4.3.7) that we have defined, as follows from the diagram on the bottom of page 349 in [4]. However, let us note that the proof that the more general base change map defined by adjunction is an isomorphism in the case of flat base change proceeds by working locally (using cohomological descent) to reduce to the case that there is a closed embedding as above and by arguing as we have done. See [7, 7.8] and the proof of [4, V, Théorème 3.5.1], especially [4, V,§3.5.4] for details. We have restricted ourselves to the situation considered above because we only need the base change map for schemes that admit a closed immersion as above, and because we later require the more explicit description (4.3.7) of the base change map.

Corollary 4.3.7. *Let X be a smooth and proper k-scheme and suppose that the absolute Frobenius endomorphism of k admits a lift σ to R. Then for each i, Frobenius induces a canonical σ-semilinear endomorphism of the R-module $H^i_{\mathrm{cris}}(X/R)$.*

Proof. Observe that $\sigma : R \to R$ is a divided power morphism (by working in K), so we may apply Proposition 4.3.5. The proof is then the same as that of Corollary 4.2.8. ∎

CHAPTER V

Comparisons

In this chapter we record several comparison results between the cohomology theories defined in chapter IV. As we wish to ultimately show that comparison isomorphisms between the cohomology theories considered in chapter IV are compatible with certain structures (e.g. Frobenius and correspondences on curves) it will be essential later to have descriptions of these comparison maps at our disposal. However, like chapter IV, this chapter is primarily for reference and contains no new results. Therefore, it can be safely skipped by the reader content to refer back as needed.

For the remainder of this chapter, we fix a perfect field k of characteristic $p > 0$ and a complete discrete valuation ring R of mixed characteristic $(0, p)$ having residue field k and fraction field K. We denote by π a uniformizer of R and by e the absolute ramification index of R. As usual, $W(k)$ denotes the ring of Witt vectors of k. We set $S = \operatorname{Spec} R$ and $\widehat{S} = \operatorname{Spf} R$ (for the p-adic topology, as always).

The basic issue that we address in this chapter is the following. Let X be a finite type R-scheme. Then we can associate to X a number of "spaces" of geometric interest: the generic fiber X_K over K, the closed fiber X_0 over k, the formal completion $\widehat{X}$ over $\widehat{S}$ of X along the closed fiber, and the K-rigid analytic spaces $\widehat{X}^{\text{rig}}$ and X_K^{an}. There are of course natural maps of sites between these spaces (for example, $X_K^{\mathrm{an}} \to X_K$, $\widehat{X} \to X$, $\widehat{X}^{\mathrm{rig}} \to \widehat{X}$, and so on) that allow us to compare the different cohomology theories native to these sites as discussed in §IV. (Note that rigid analytic spaces are not topological spaces in the usual sense due to the Tate topology.) We now recall how these comparisons are defined and under what conditions they are isomorphisms.

5.1 Rigid and formal GAGA

Recall that rigid analytification defines a functor from the category of schemes of locally finite type over K to the category of rigid analytic spaces over K. There is a canonical morphism of ringed sites

$$\iota : X^{\mathrm{an}} \to X$$

and we may "pull back" differentials along ι; that is, there is a morphism of complexes

$$\Omega^\bullet_{X/K} \to \iota_* \Omega^\bullet_{X^{\mathrm{an}}/K},$$

over $\mathscr{O}_X \to \iota_* \mathscr{O}_{X^{\mathrm{an}}}$, and by adjunction we obtain a morphism of complexes of sheaves on X^{an}

$$\iota^{-1} \Omega^\bullet_{X/K} \to \Omega^\bullet_{X^{\mathrm{an}}/K}.$$

We note in passing that the induced $\mathscr{O}_{X^{\mathrm{an}}}$-linear map $\iota^* \Omega^\bullet_{X/K} \to \Omega^\bullet_{X^{\mathrm{an}}/K}$ in each degree is an isomorphism (see §4.1), as we can check this by a local calculation on completed stalks.

Specializing the map (A.1.3) to this situation, we thus obtain a K-linear map

$$H^\bullet_{\mathrm{dR}}(X/K) \to H^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}/K) \tag{5.1.1}$$

that is evidently functorial in X.

Proposition 5.1.1. *When X is proper, the map* (5.1.1) *is an isomorphism.*

Proof. The map (5.1.1) is compatible with the obvious K-linear map between the spectral sequences

$$E_1^{p,q} = H^p(X, \Omega^q_{X/K}) \implies H^{p+q}_{\mathrm{dR}}(X)$$
$$E_1^{p,q} = H^p(X^{\mathrm{an}}, \Omega^q_{X^{\mathrm{an}}/K}) \implies H^{p+q}_{\mathrm{dR}}(X^{\mathrm{an}}),$$

and by rigid analytic GAGA (whose proof goes exactly as in the classical case over $\mathbf{C}$) this latter map is already an isomorphism at the E_1-stage via the canonical isomorphisms $(\Omega^q_{X/K})^{\mathrm{an}} \simeq \Omega^q_{X^{\mathrm{an}}/K}$ for all q. ∎

Similarly, there is a functor which associates to any R-scheme X, locally of finite type, the topologically locally finite type formal scheme $\widehat{X}$ obtained by completing X along its special fiber X_0, and there is a natural map of ringed spaces

$$\iota : \widehat{X} \to X$$

which is functorial in X. Exactly as above, we obtain from the canonical R-linear map of complexes

$$\iota^{-1}\Omega^\bullet_{X/S} \to \widehat{\Omega}^\bullet_{\widehat{X}/\widehat{S}}$$

(which induces isomorphisms $\iota^*(\Omega^q_{X/S}) \to \widehat{\Omega}^q_{\widehat{X}/\widehat{S}}$ for all $q \geq 0$) an R-linear map of de Rham cohomology groups

$$H^i_{\mathrm{dR}}(X/S) \to H^i_{\mathrm{dR}}(\widehat{X}/\widehat{S}), \tag{5.1.2}$$

that is functorial in X.

Proposition 5.1.2. *If X is proper, the map* (5.1.2) *is an isomorphism.*

Proof. One proceeds exactly as in the proof of Proposition 5.1.1, using formal GAGA [19, III, Proposition 5.1.2] in place of rigid GAGA. ∎

5.2 de Rham cohomology comparisons

Let X be any locally finite type R-scheme with formal completion $\widehat{X}$ along X_0. For any R-algebra A of finite type, we let $\widehat{A}$ denote the π-adic completion of A. The natural map of K-algebras $A_K \to \widehat{A} \otimes_R K$ corresponds to a unique morphism of ringed spaces of K-algebras

$$\mathrm{Sp}(\widehat{A} \otimes_R K) \to \mathrm{Spec}(A_K)$$

via [14, Lemma 5.1.1] (see also [19, II, Errata, I.1.8.1]). By the universal property of $\mathrm{Spec}(A_K)^{\mathrm{an}}$, the above map factors through $\mathrm{Spec}(A_K)^{\mathrm{an}} \to \mathrm{Spec}(A_K)$. The resulting morphism of analytic spaces

$$i_A : \mathrm{Sp}(\widehat{A} \otimes_R K) \to \mathrm{Spec}(A_K)^{\mathrm{an}}$$

is evidently functorial in A, and this enables us to uniquely globalize the construction to locally finite type S-schemes X as a map of rigid spaces

$$i_X : \widehat{X}^{\mathrm{rig}} \to X_K^{\mathrm{an}}. \tag{5.2.1}$$

Proposition 5.2.1. *If X is R-separated, i_X is a quasi-compact open immersion. If X is moreover proper, i_X is an isomorphism.*

Proof. The former statement is a consequence of the valuative criterion for separatedness; the latter, of the valuative criterion for properness. See [14, Theorem 5.3.1] for a complete proof. ∎

In light of Propositions 5.2.1 and 5.1.1, for any proper S-scheme X with formal completion $\widehat{X}$ along its special fiber, pullback of differentials along the natural map of ringed sites

$$\widehat{X}^{\mathrm{rig}} \to X_K^{\mathrm{an}} \to X_K$$

yields a natural isomorphism

$$H^\bullet_{\mathrm{dR}}(X_K/K) \xrightarrow{\simeq} H^\bullet_{\mathrm{dR}}(\widehat{X}^{\mathrm{rig}}/K)\ . \tag{5.2.2}$$

If $\mathscr{X}$ is any separated formal $\widehat{S}$-scheme topologically of finite type, the natural morphism of ringed topoi

$$\mathrm{sp} : \mathscr{X}^{\mathrm{rig}} \to \mathscr{X}$$

gives rise to a natural R-linear map of de Rham complexes

$$\widehat{\Omega}^\bullet_{\mathscr{X}/\widehat{S}} \to \mathrm{sp}_* \,\Omega^\bullet_{\mathscr{X}^{\mathrm{rig}}/K}$$

and hence an R-linear map

$$H^\bullet_{\mathrm{dR}}(\mathscr{X}/\widehat{S}) \to H^\bullet_{\mathrm{dR}}(\mathscr{X}^{\mathrm{rig}}/K). \tag{5.2.3}$$

Extending scalars to K yields a natural map of K-vector spaces

$$H^\bullet_{\mathrm{dR}}(\mathscr{X}/\widehat{S}) \otimes_R K \to H^\bullet_{\mathrm{dR}}(\mathscr{X}^{\mathrm{rig}}/K). \tag{5.2.4}$$

Proposition 5.2.2. *The natural map* (5.2.4) *is an isomorphism.*

Proof. Since $\mathscr{X}$ is separated of topologically finite type, hypercohomology commutes with tensoring over R with K (as one sees via Čech theory). Thus, it suffices to show that the K-linear composite map of complexes on $\mathscr{X}$

$$\widehat{\Omega}^\bullet_{\mathscr{X}/\widehat{S}} \otimes_R K \longrightarrow \mathrm{sp}_* \,\Omega^\bullet_{\mathscr{X}^{\mathrm{rig}}/K} \longrightarrow \mathbf{R}\,\mathrm{sp}_* \,\Omega^\bullet_{\mathscr{X}^{\mathrm{rig}}/K}$$

is a quasi-isomorphism. We first claim that the natural map $\mathrm{sp}_* \to \mathbf{R}\,\mathrm{sp}_*$ is a quasi-isomorphism on bounded below complexes of coherent sheaves. Indeed, for any formal open affine $U = \mathrm{Spf}\, A$ of $\mathscr{X}$ and any coherent sheaf $\mathscr{F}$ on $\mathscr{X}^{\mathrm{rig}}$, for $i > 0$ we have

$$H^i(U, \mathrm{sp}_* \mathscr{F}) = H^i(\mathrm{Sp}(A \otimes K), \mathscr{F}) = 0$$

thanks to Kiehl's generalization [29, 2.4.2] of Tate's acyclicity theorem [8, §8.2]. Thus, $R^i \operatorname{sp}_* \mathscr{F} = 0$ as desired.

It remains to show that the natural map

$$\widehat{\Omega}^{\bullet}_{\mathscr{X}/\widehat{S}} \otimes_R K \to \operatorname{sp}_* \Omega^{\bullet}_{\mathscr{X}^{\mathrm{rig}}/K}$$

is an isomorphism. We can work locally on $\mathscr{X}$ in any fixed degree, so for $U = \operatorname{Spf} A$ we wish to show that the natural map

$$\widehat{\Omega}^{j}_{A/R} \otimes K \to \Omega^{j}_{A \otimes K/K}$$

is an isomorphism for all j. This is clear, and we are done. ■

5.3 Rigid and crystalline cohomology

In certain favorable situations, both rigid and crystalline cohomology can be related to de Rham cohomology, and it is precisely this type of relationship that will later enable us to use rigid and crystalline cohomology to study the Hodge filtration of a curve over K. We begin this section by recalling the construction of the relevant comparison maps in these situations.

If $\mathscr{X}$ is any proper formal $\operatorname{Spf} R$-scheme and U is any open subscheme of the smooth locus of $\mathscr{X}_0$, we can define a K-linear map

$$H^{\bullet}_{\mathrm{dR}}(\mathscr{X}^{\mathrm{rig}}/K) \to H^{\bullet}_{\mathrm{rig}}(U/K) \tag{5.3.1}$$

which is functorial in the pair $(U, \mathscr{X})$ as follows. Let $\overline{U}$ be the schematic closure of U in $\mathscr{X}_0$; it is a compactification of U as $\mathscr{X}$ is proper. For any abelian sheaf $\mathscr{F}$ on $]\overline{U}[_{\mathscr{X}}$ and any strict open neighborhood $j_V : V \hookrightarrow]\overline{U}[_{\mathscr{X}}$ of $]U[_{\mathscr{X}}$ in $]\overline{U}[_{\mathscr{X}}$, there is a natural map $\mathscr{F} \to j_{V*}j_V^{-1}\mathscr{F}$, and hence a natural map $\mathscr{F} \to j^{\dagger}\mathscr{F}$. We thus obtain a K-linear morphism of complexes on $]\overline{U}[_{\mathscr{X}}$

$$\Omega^{\bullet}_{]\overline{U}[_{\mathscr{X}}/K} \to j^{\dagger}\Omega^{\bullet}_{]\overline{U}[_{\mathscr{X}}/K}, \tag{5.3.2}$$

and hence of hypercohomology groups

$$H^\bullet_{\mathrm{dR}}(]\overline{U}[_{\mathscr{X}}/K) \to \mathbf{H}^\bullet(]\overline{U}[_{\mathscr{X}}, j^\dagger\Omega^\bullet_{]\overline{U}[_{\mathscr{X}}/K}) = H^\bullet_{\mathrm{rig}}(U/K), \tag{5.3.3}$$

where the last equality is the content of Remark 4.2.4. Composing (5.3.3) with the natural "restriction" map

$$H^\bullet_{\mathrm{dR}}(\mathscr{X}^{\mathrm{rig}}/K) \to H^\bullet_{\mathrm{dR}}(]\overline{U}[_{\mathscr{X}}/K)$$

obtained from the inclusion of rigid spaces $\iota :]\overline{U}[_{\mathscr{X}} \hookrightarrow \mathscr{X}^{\mathrm{rig}}$ yields the desired K-linear mapping.

Proposition 5.3.1. *If $\mathscr{X}_0$ is smooth and $U = \mathscr{X}_0$, then* (5.3.1) *is an isomorphism.*

Proof. From the *construction* of (5.3.1), it is enough to show that the morphism of complexes (5.3.2) is an isomorphism. But since $U = \mathscr{X}_0$, we have $U = \overline{U}$ so $j^\dagger$ is the identity. ∎

Similarly, as we mentioned before, the crystalline cohomology of a smooth and proper k-scheme X_0 "captures" the de Rham cohomology of a smooth lift of X_0 to R (when one exists):

Proposition 5.3.2. *Let X/R be a smooth and proper scheme with closed fiber X_0. If $e \leq p-1$ then there is a canonical R-linear isomorphism*

$$H^\bullet_{\mathrm{cris}}(X_0/R) \to H^\bullet_{\mathrm{dR}}(X/R) \tag{5.3.4}$$

which is functorial in X.

Proof. Let $\widehat{X}$ be the formal completion of X along X_0. The hypothesis on e ensures that the ideal (π) of R has a unique (but *not* nilpotent for $e = p-1$) divided power structure [7, 3.28], which extends to the ideal $\mathscr{I} = \pi\mathscr{O}_X$ of X_0 in X as X is smooth, hence R-flat. Let $\mathscr{D}_{X_0}(X)$ be the divided power envelope of $\mathscr{I}$. The structure map $\mathscr{O}_X \to \mathscr{D}_{X_0}(X)$ is injective, as can be checked over K since $\mathscr{O}_X$ is R-flat, and it is surjective by [7, Remark 3.20 (3)], so it is an isomorphism. By Proposition 4.3.3 there is then a natural isomorphism in the derived category of bounded below complexes of sheaves of R-modules on the Zariski site of X_0

$$\mathbf{R}i_*\mathbf{R}u_{X_0/\widehat{S}_*}\mathscr{O}_{X_0/\widehat{S}} \to \widehat{\Omega}^\bullet_{\widehat{X}/\widehat{S}} \tag{5.3.5}$$

where $i : X_0 \hookrightarrow \widehat{X}$ is the canonical inclusion. Passing to hypercohomology and using the fact that the Zariski site of X_0 is the Zariski site of $\widehat{X}$, we obtain a natural isomorphism

$$\mathbf{H}^i(X_0, \mathbf{R}u_{X_0/\widehat{S}_*}\mathscr{O}_{X_0/\widehat{S}}) \to \mathbf{H}^i(\widehat{X}, \widehat{\Omega}^\bullet_{\widehat{X}/\widehat{S}}) = H^i_{\mathrm{dR}}(\widehat{X}/\widehat{S}). \tag{5.3.6}$$

Composing with (4.3.1) yields a natural (in the pair $(X_0, \widehat{X})$) isomorphism

$$H^i_{\mathrm{cris}}(X_0/R) \to H^i_{\mathrm{dR}}(\widehat{X}/\widehat{S}) \tag{5.3.7}$$

as R-modules. The composite of (5.3.7) and the inverse of the isomorphism (5.1.2) gives the required isomorphism. ■

As we mentioned in the introduction, one of the key issues in resolving the "unchecked compatibilities" in Gross' paper [22] is the Frobenius compatibility of a certain map between crystalline cohomology and Monsky-Washnitzer cohomology. Since the former only yields a good theory for smooth proper k-schemes, and the latter only for smooth affine k-schemes, there is no means within either of these theories to make any comparison. It is here that rigid cohomology is essential, for it yields a good cohomology theory both in the proper and the affine case, and there are natural comparison isomorphisms in each case with crystalline cohomology and Monsky-Washnitzer cohomology, respectively. These comparison isomorphisms therefore play a critical role in our subsequent work. As we previously remarked, we will not need to use Monsky-Washnitzer cohomology in what we do (as Gross' proof of the companion form theorem can be reworked using only rigid cohomology), so we say nothing about it here. As far as the comparison isomorphism between rigid and crystalline cohomology in the proper case goes, we have:

Proposition 5.3.3. *Let X be any smooth and proper k-scheme, and assume X admits a closed embedding $i : X \hookrightarrow \mathscr{P}$ into a p-adic formal $\widehat{S}$-scheme that is smooth at the points of X and that $e \leq p-1$. Then there is a canonical isomorphism*

$$H^{\bullet}_{\mathrm{rig}}(X/K) \to H^{\bullet}_{\mathrm{cris}}(X/R) \otimes_R K. \tag{5.3.8}$$

This is natural in (k-morphisms of) X and compatible with any local finite flat base change $R \to R'$ with R' a discrete valuation ring having absolute ramification index $e' \leq p-1$.

Proof. The first part of the Proposition is is essentially [5, Proposition 1.9]. Since we will later need a detailed description of the map involved, we give a more detailed version of Berthelot's argument, and explain the statement about compatibility with base change.

Since X is proper, we may take $\overline{X} = X$ and use the given closed immersion i to calculate the rigid cohomology of X. In this situation, $j : X \hookrightarrow \overline{X}$ is the identity map, so $j^{\dagger}$ is the identity functor, and by Definition 4.2.3 we have

$$H^{\bullet}_{\mathrm{rig}}(X/K) = \mathbf{H}^{\bullet}(X, \mathbf{R}\,\mathrm{sp}_* \,\Omega^{\bullet}_{]X[_{\mathscr{P}}/K}).$$

Further, we claim that in this context, the natural map in the derived category of complexes of sheaves of K-modules on the Zariski site of $\mathscr{P}$

$$\mathrm{sp}_* \,\Omega^n_{]X[_{\mathscr{P}}/K} \to \mathbf{R}\,\mathrm{sp}_* \,\Omega^n_{]X[_{\mathscr{P}}/K}$$

is an isomorphism for all n. To verify our claim, it suffices to show that

$$R^m \operatorname{sp}_* \Omega^n_{]X[_{\mathscr{P}}/K} = 0$$

for all n and all $m > 0$. This can be checked locally, so it is enough to show that for any formal open affine $U = \operatorname{Spf} A$ of $\mathscr{P}$, the cohomology groups

$$H^m(\operatorname{sp}^{-1}(U) \cap]X[_{\mathscr{P}}, \Omega^n_{(\operatorname{sp}^{-1}(U)\cap]X[_{\mathscr{P}})/K})$$

vanish for all n and any $m > 0$. We assert that the rigid space $\operatorname{sp}^{-1}(U) \cap]X[_{\mathscr{P}}$ is *quasi-stein* (in the sense of [29, Definition 2.3]), so (since Ω^n is coherent) such vanishing follows from Kiehl's "Theorem B" [29, 2.4.2]. To conclude, we need to show that $W := \operatorname{sp}^{-1}(U) \cap]X[_{\mathscr{P}}$ has an admissible cover by a rising union of affinoids $U_1 \subseteq U_2 \subseteq \cdots$ with the property the natural restriction maps $\mathscr{O}_W(U_{r+1}) \to \mathscr{O}_W(U_r)$ have dense image for all $r > 0$.

Let $\mathscr{I}$ be the sheaf of ideals in $\mathscr{O}_{\mathscr{P}}$ corresponding to the closed immersion i and choose generators $(f_1, \ldots, f_s)$ of $I := \Gamma(U, \mathscr{I})$. We then have the description

$$W = \{x \in \operatorname{Sp}(A \otimes K) || f_m(x)| < 1 \text{ for all } m\}.$$

Of course, the intrinsic definition $W := \operatorname{sp}^{-1}(U) \cap]X[_{\mathscr{P}}$ shows that W is independent of the choice of generators of I. For each natural number n, define the A-algebra

$$A_n := A\{T_1, \ldots, T_s\}/(T_1 p - f_1^{p+n}, \ldots, T_s p - f_s^{p+n});$$

clearly $A_n \otimes_R K$ is a K-affinoid algebra, and we let U_n be the affinoid

$$U_n := \operatorname{Sp}(A_n \otimes_R K) = \{x \in \operatorname{Sp}(A \otimes K) || f_m(x)| \leq |p|^{1/(p+n)} \text{ for all } m\}.$$

While A_n may depend on the choice of generators f_α of I, the K-affinoid algebra $A_n \otimes K$ does not. Indeed, since $A_n \otimes K$ is the K-algebra of global functions on U_n, it suffices to see that U_n is independent of the f_α's, which is clear from the description

$$U_n = [X]_{U,\eta_n}$$

with $\eta_n = |p|^{1/(p+n)}$, which *is* intrinsic to X, U, and η as

$$\eta = |p|^{1/(p+n)} > |p|^{1/(p-1)} \geq |\pi|$$

by our hypothesis that $e \leq p-1$.

For each pair of natural numbers n, n' with $n \leq n'$, the natural map of A-algebras $A_{n'} \to A_n$ defined by $T_\alpha \mapsto f_\alpha^{n'-n} T_\alpha$ induces the inclusion of U_n as an open affinoid subset of U_{n+1} within W, and it is obvious that the image of $K \otimes A_{n'}$ in $K \otimes A_n$ is dense. Moreover, we clearly have $W = \cup_{n \geq 0} U_n$ with $\{U_n\}_{n \geq 0}$ an admissible cover of W due to the maximum modulus principle, so W is indeed quasi-stein, as claimed. Thus, we have

$$H^\bullet_{\mathrm{rig}}(X/K) = \mathbf{H}^\bullet(X, \mathrm{sp}_* \Omega^\bullet_{]X[_{\mathscr{P}/K}}). \tag{5.3.9}$$

On the other hand, let $\mathscr{D}_X(\mathscr{P})$ be the divided power envelope of $\mathscr{I}$ and as before set $\widehat{\mathscr{D}}_X(\mathscr{P}) = \varprojlim_n \mathscr{D}_X(\mathscr{P}_n)$ where $\mathscr{P}_n = \mathscr{P} \otimes_R (R/\pi^n R)$. By Proposition 4.3.3, there is a canonical isomorphism

$$\mathbf{R}i_* \mathbf{R}u_{X/\widehat{S}_*} \mathscr{O}_{X/\widehat{S}} \xrightarrow{\simeq} \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^\bullet_{\mathscr{P}/\widehat{S}}, \tag{5.3.10}$$

which is functorial in the pair $(X, \mathscr{P})$. Since X is separated and quasi-compact, hypercohomology commutes with tensoring by K (as one sees by Čech theory), so we conclude that the natural map

$$H^\bullet_{\mathrm{cris}}(X/R) \otimes_R K \to \mathbf{H}^\bullet(X, \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^\bullet_{\mathscr{P}/\widehat{S}} \otimes K) \tag{5.3.11}$$

induced by (5.3.10) and the identification (4.3.1) is an isomorphism. Thus, to define an isomorphism

$$H^\bullet_{\mathrm{rig}}(X/K) \to H^\bullet_{\mathrm{cris}}(X/R) \otimes_R K$$

that is functorial in the pair $(X, \mathscr{P})$, it suffices to define a K-linear quasi-isomorphism of complexes

$$\operatorname{sp}_* \Omega^\bullet_{]X[_{\mathscr{P}}/K} \to \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^\bullet_{\mathscr{P}/\widehat{S}} \otimes K. \tag{5.3.12}$$

We construct (5.3.12) as follows. First, we show that there is a canonical integrable connection on the $\mathscr{O}_{\mathscr{P}}$-module $\operatorname{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}}$ and hence a complex $\operatorname{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}} \otimes \Omega^\bullet_{\mathscr{P}/\widehat{S}}$. We then show that there is a natural isomorphism of complexes

$$\operatorname{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}} \otimes \widehat{\Omega}^\bullet_{\mathscr{P}/\widehat{S}} \simeq \operatorname{sp}_* \Omega^\bullet_{]X[_{\mathscr{P}}/K},$$

$\mathscr{O}_{\mathscr{P}}$-linear in each degree, so that we may define (5.3.12) by defining an $\mathscr{O}_{\mathscr{P}}$-algebra morphism

$$\operatorname{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}} \to \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_R K \tag{5.3.13}$$

that is compatible with the connections on each side (the connection on the right side is (4.3.2)). We will finally define this morphism, and show that (5.3.12) is a quasi-isomorphism.

To keep notation unencumbered, for any R-algebra C, we denote by C_K the K-algebra $C \otimes K$ obtained by extending scalars. With the same notation as above, for each formal open affine $U = \operatorname{Spf} A$ of $\mathscr{P}$, the K-affinoid algebras $(A_n)_K = \Gamma(U_n, \mathscr{O}_{U_n})$ come equipped with canonical K-linear connections

$$\nabla'_n : (A_n)_K \to (A_n)_K \otimes_{A_K} \Omega^1_{A_K/K}$$

defined as follows. An easy calculation with the presentation of A_n shows that the module of continuous relative differentials $\widehat{\Omega}^1_{A_n/A}$ is p-torsion, so the natural map

$$(A_n)_K \otimes_{A_K} \Omega^1_{A_K/K} \to \Omega^1_{(A_n)_K/K}$$

is surjective for all n. It is moreover injective because U_n is open affinoid in U^{rig}, so the map on completed stalks at points of U_n is an isomorphism. We define ∇'_n as the composite

$$(5.3.14) \qquad (A_n)_K \xrightarrow{d} \Omega^1_{(A_n)_K/K} \xleftarrow{\simeq} (A_n)_K \otimes_{A_K} \Omega^1_{(A_n)_K/K} .$$

Observe that this map is intrinsic to $U_n \hookrightarrow U^{\mathrm{rig}}$. Since the ∇'_n are evidently compatible with the transition maps $U_n \hookrightarrow U_{n'}$ for $n' \geq n$, we obtain a connection

$$\nabla'_{\{U_n\}}(U) : \mathrm{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}}(U) \to \left(\mathrm{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}} \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^1_{\mathscr{P}/\widehat{S}}\right)(U)$$

that is intrinsic to the directed system $\{U_n\}_{n\geq 0}$. However, for any other choice of directed system $\{V_m\}_{m\geq 0}$ of affinoid opens forming an admissible cover of U^{rig}, for each m we can find $n(m) \geq 0$ such that $V_m \subseteq U_{n(m)}$ (by the admissibility of the cover $\{U_n\}_{n\geq 0}$). Due to how we saw that the connections $\nabla'_{n(m)}$ and ∇'_m respectively defined using $U_{n(m)} \hookrightarrow U^{\mathrm{rig}}$ and $V_m \hookrightarrow U^{\mathrm{rig}}$ are intrinsic to $U_{n(m)}$ and V_m, we see that $\nabla'_{n(m)}$ restricts to ∇'_m on V_m. As we can clearly reverse the roles of $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ in the above, we see that the resulting connections $\nabla'_{\{U_n\}}(U)$ and $\nabla'_{\{V_m\}}(U)$ agree. Thus, we have defined a connection $\nabla'(U)$ that is intrinsic to U. It follows that for variable U these connections glue to give (after sheafifying) a connection on the $\mathscr{O}_{\mathscr{P}}$-module $\mathrm{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}}$ which we denote by ∇'. As the ∇'_n are clearly integrable, the same is true of ∇', so we obtain a complex $\mathrm{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}} \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^\bullet_{\mathscr{P}/\widehat{S}}$, as desired.

We claim that there is a natural isomorphism of complexes

$$(5.3.15) \qquad \mathrm{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}} \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^\bullet_{\mathscr{P}/\widehat{S}} \to \mathrm{sp}_* \Omega^\bullet_{]X[_{\mathscr{P}}/K}$$

that is functorial in the pair $(X, \mathscr{P})$ and $\mathscr{O}_{\mathscr{P}}$-linear in each degree. Over $U = \mathrm{Spf}\, A$, this isomorphism is induced by the compatible system of canonical isomorphisms

$$(A_n)_K \otimes_{A_K} \Omega^1_{A_K/K} \xrightarrow{\simeq} \Omega^1_{(A_n)_K/K} ,$$

and so is clearly intrinsic to the directed system $\{U_n\}_{n\geq 0}$. Comparison with any other such directed system shows, as before, that the resulting map (5.3.15) over U is intrinsic to U, and so by gluing we really do get the claimed global isomorphism (5.3.15).

Thus, to define the desired morphism (5.3.12), it suffices to define a morphism of $\mathscr{O}_{\mathscr{P}}$-algebras

$$\mathrm{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}} \to \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_R K, \tag{5.3.16}$$

functorially in $(X, \mathscr{P})$, that is compatible with the connections ∇ of (4.3.2) and ∇'. We then have to check that (5.3.12) is a quasi-isomorphism.

Set $\eta = |p|^{-1/p}$ and note that our hypothesis on the absolute ramification index e of R ensures that $|\pi| < \eta$ so in particular the closed tube $[X]_{\mathscr{P},\eta}$ defined in (4.2.2) is intrinsic to X, $\mathscr{P}$, and η. We will construct a natural morphism of $\mathscr{O}_{\mathscr{P}}$-algebras

$$\mathrm{sp}_* \mathscr{O}_{[X]_{\mathscr{P},\eta}} \to \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_R K \tag{5.3.17}$$

and then define (5.3.16) as the composite of the natural restriction map $\mathrm{sp}_* \mathscr{O}_{]X[_{\mathscr{P}}} \to \mathrm{sp}_* \mathscr{O}_{[X]_{\mathscr{P},\eta}}$ (coming from the inclusion of rigid spaces $[X]_{\mathscr{P},\eta} \hookrightarrow]X[_{\mathscr{P}}$) with (5.3.17).

We proceed locally on $\mathscr{P}$. Letting $U = \mathrm{Spf}\, A$ be any formal open affine as before, we set $I = \Gamma(U, \mathscr{I})$, $B = \Gamma(U, \widehat{\mathscr{D}}_X(\mathscr{P}))$ and pick generators $f_1, \ldots, f_s$ of I, so

$$\Gamma(U, \mathrm{sp}_* \mathscr{O}_{[X]_{\mathscr{P},\eta}}) = A\{T_1, \ldots, T_s\}/(T_1 p - f_1^p, \ldots, T_s p - f_s^p) \otimes_R K = A_0 \otimes_R K.$$

There is a unique continuous (for the π-adic topologies) homomorphism of A-algebras

$$\varphi : A\{T_1, \ldots, T_r\}/(T_1 p - f_1^p, \ldots, T_s p - f_s^p) \to B$$

defined by sending T_α to $(p-1)! f_\alpha^{[p]}$; this map may depend on the choices of the f_α. Tensoring over R with K yields a morphism

$$\Gamma(U, \mathrm{sp}_* \mathscr{O}_{[X]_{\mathscr{P},\eta}}) \to \Gamma(U, \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes K). \tag{5.3.18}$$

In order to see that this map is functorial in the pair $(U, \mathscr{P})$ and that for variable U these maps can be glued, it suffices to see that (5.3.18) does *not* depend on the choice of generators $\{f_\alpha\}$ for I. To do this, we will give an intrinsic characterization of (5.3.18).

Observe that $\Gamma(U, \mathrm{sp}_* \mathscr{O}_{[X]_{\mathscr{P},\eta}})$ is K-affinoid, so it comes equipped with a unique K-Banach algebra topology. Moreover, the R sub-algebra A_K is dense in this topology. Now since $\mathscr{P}$ is smooth at the points of X and $\mathscr{D}_X(\mathscr{P})$ only depends on a neighborhood of X in $\mathscr{P}$, we may shrink $\mathscr{P}$ around X to suppose that $\mathscr{P}$ is formally smooth. Since X is also smooth, it then follows from [7, Corollary 3.35] that $\widehat{\mathscr{D}}_X(\mathscr{P})$ is R-torsion free. Thus, since the A-algebra $B = \Gamma(U, \widehat{\mathscr{D}}_X(\mathscr{P}))$ is R-torsion free, the A_K-algebra B_K has a unique K-Banach A_K-algebra norm for which B is the unit ball. It follows easily from definitions that the map (5.3.18) is continuous for these K-Banach topologies. Since it is moreover a map of A_K-algebras and A_K is dense in the source, we can characterize (5.3.18) as the *unique* continuous map of K-Banach A_K-algebras. It follows that (5.3.18) is intrinsic to U and compatible with restriction. Hence, (5.3.18) is functorial in $(U, \mathscr{P})$ and for variable U, these morphisms uniquely glue to a morphism of $\mathscr{O}_{\mathscr{P}}$-algebras

$$\mathrm{sp}_* \mathscr{O}_{[X]_{\mathscr{P},\eta}} \longrightarrow \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes K\ .$$

To check that the resulting map (5.3.16) is compatible with the connections ∇ and ∇', it clearly suffices to show that for each n, the composite map

$$A_n \otimes K \to A_0 \otimes K \to B \otimes K$$

intertwines ∇'_n and ∇. Since the canonical maps $A_n \otimes K \to A_0 \otimes K$ are obviously compatible with the connections on both sides, we are reduced to checking that the map $A_0 \otimes K \to B \otimes K$ intertwines ∇'_0 and ∇. From the definition (5.3.14) of

∇'_0, such compatibility amounts to the commutativity of the following diagram of $A \otimes K$-algebras:

$$\begin{tikzcd}
A\{T_1,\dots,T_r\}/(T_1p - f_1^p,\dots,T_sp - f_s^p)\otimes K \arrow[r, "\varphi"] \arrow[d, "d"] & B\otimes K \arrow[d, "\nabla"] \\
A\{T_1,\dots,T_r\}/(T_1p - f_1^p,\dots,T_sp - f_s^p)\otimes_A \widehat{\Omega}^1_{A/R}\otimes K \arrow[r, "\varphi\otimes 1"'] & B\otimes \widehat{\Omega}^1_{A/R}\otimes K
\end{tikzcd}$$

which is clear from the calculations

$$\nabla(\varphi(T_i)) = \nabla((p-1)!f_i^{[p]}) = (p-1)!f_i^{[p-1]}\otimes df_i = f_i^{p-1}\otimes df_i$$
$$(\varphi\otimes 1)(d(T_i)) = f_i^{p-1}\otimes df_i.$$

By construction and the universal mapping property of divided power envelopes (see [7, Theorem 3.19] and [4, I, Proposition 4.1.5]) the map (5.3.12) is easily seen to be functorial in the pair $(X,\mathscr{P})$. Thus, given two closed embeddings $i : X \hookrightarrow \mathscr{P}$ and $i' : X \hookrightarrow \mathscr{P}'$ of X into topologically finite type formal $\widehat{S}$-schemes and a morphism $u : \mathscr{P}' \to \mathscr{P}$ making

$$\begin{tikzcd}
 & \mathscr{P}' \arrow[d, "u"] \\
X \arrow[ur, "i'"] \arrow[r, "i"'] & \mathscr{P}
\end{tikzcd}$$

commute, we obtain a commutative diagram

$$(5.3.19)\qquad \begin{tikzcd}
\mathbf{R}\,\mathrm{sp}_*\,\Omega^\bullet_{]X[_{\mathscr{P}}/K} \arrow[r, "(5.3.12)"] \arrow[d] & \widehat{\mathscr{D}}_X(\mathscr{P})\otimes_{\mathscr{O}_{\mathscr{P}}}\widehat{\Omega}^1_{\mathscr{P}/\widehat{S}}\otimes K \arrow[dd] \\
\mathbf{R}\,\mathrm{sp}_*\,\mathbf{R}u_*^{\mathrm{rig}}\Omega^\bullet_{]X[_{\mathscr{P}'}/K} \arrow[d, equal] & \\
\mathbf{R}u_*\mathbf{R}\,\mathrm{sp}_*\,\Omega^\bullet_{]X[_{\mathscr{P}'}/K} \arrow[r, "\mathbf{R}u_*(5.3.12)"'] & \mathbf{R}u_*\left(\widehat{\mathscr{D}}_X(\mathscr{P}')\otimes_{\mathscr{O}_{\mathscr{P}'}}\widehat{\Omega}^1_{\mathscr{P}'/\widehat{S}}\otimes K\right)
\end{tikzcd}$$

The left vertical arrow is an isomorphism by [5, Théorème 1.4] (i.e. rigid cohomology is independent of the choice of embedding $i : X \hookrightarrow \mathscr{P}$) and we claim that the right vertical arrow is a quasi-isomorphism. Indeed, since the isomorphism (4.3.3) of Proposition 4.3.3 is *natural*, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{R}i_*\mathbf{R}u_{X/\widehat{S}_*}\mathscr{O}_{X/\widehat{S}} \arrow[r, "(4.3.3)"] \arrow[d, equal] & \widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^{\bullet}_{\mathscr{P}/\widehat{S}} \arrow[d] \\
\mathbf{R}u_*\mathbf{R}i'_*\mathbf{R}u_{X/\widehat{S}_*}\mathscr{O}_{X/\widehat{S}} \arrow[r, "\mathbf{R}u_*(4.3.3)"'] & \mathbf{R}u_*\left(\widehat{\mathscr{D}}_X(\mathscr{P}') \otimes_{\mathscr{O}_{\mathscr{P}'}} \widehat{\Omega}^{\bullet}_{\mathscr{P}'/\widehat{S}}\right)
\end{tikzcd} \tag{5.3.20}$$

where the right vertical arrow is the vertical map in (5.3.19). Since the two horizontal maps are quasi-isomorphisms by Proposition 4.3.3, we conclude that the right vertical map is also a quasi-isomorphism. It follows by the standard fiber product argument that (5.3.12) is independent of the choice of embedding $X \hookrightarrow \mathscr{P}$ up to canonical quasi-isomorphism. A similar argument with fiber products then shows that (5.3.12) is functorial in X alone (with morphisms in the subcategory of finite type k-schemes that admit some closed embedding into a topologically finite type formal $\widehat{S}$-scheme).

Let us now show that (5.3.12) is a quasi-isomorphism. We may check this locally, so we may suppose that $\mathscr{P}$ is a formal affine. Moreover, since (5.3.12) is independent of the choice of $\mathscr{P}$ up to canonical quasi-isomorphism, we may pass to the case that $\mathscr{P}$ is a smooth formal lifting of the smooth affine scheme X. In this case, $\mathscr{I} = \pi\mathscr{O}_{\mathscr{P}}$ so $\widehat{\mathscr{D}}_X(\mathscr{P}) = \mathscr{O}_{\mathscr{P}}$ and $]X[_{\mathscr{P}} = \mathscr{P}^{\mathrm{rig}}$ so $\mathrm{sp}_*\,\mathscr{O}_{]X[_{\mathscr{P}}} = \mathscr{O}_{\mathscr{P}} \otimes_R K$. Moreover, I is generated by one element $f_1 := \pi$, and $|\pi| < \eta$ as noted above; it follows that $\mathrm{sp}_*\,\mathscr{O}_{[X]_{\mathscr{P},\eta}} = \mathrm{sp}_*\,\mathscr{O}_{]X[_{\mathscr{P}}}$ and the map (5.3.16), and hence (5.3.12), is the identity.

We have thus constructed a K-linear quasi-isomorphism

$$\mathbf{R}\,\mathrm{sp}_*\,j^{\dagger}\Omega^{\bullet}_{]X[_{\mathscr{P}}/K} \to (\widehat{\mathscr{D}}_X(\mathscr{P}) \otimes_{\mathscr{O}_{\mathscr{P}}} \widehat{\Omega}^{\bullet}_{\mathscr{P}/\widehat{S}}) \otimes_R K \tag{5.3.21}$$

that is functorial in X and canonically independent of the choice of $\mathscr{P}$ for fixed X. The fact that this map is moreover compatible with finite flat base change $R \to R'$ with R' a discrete valuation ring having absolute ramification index $e' \leq p-1$ follows easily from the definitions of the base change maps (4.2.10), (4.3.7) and the fact that (5.3.21) is natural in $(X, \mathscr{P})$.

■

Corollary 5.3.4. *With the same notation as above, suppose that the Frobenius endomorphism of k admits a lift σ to R, so the K-vector spaces*

$$H^i_{\mathrm{rig}}(X/K), \qquad\qquad H^i_{\mathrm{cris}}(X/R) \otimes K$$

are equipped with canonical σ-linear Frobenius endomorphisms by Corollaries 4.2.8 *and* 4.3.7*. The isomorphism* (5.3.8) *is compatible with the Frobenius endomorphisms of both sides.*

Proof. This follows easily from the functoriality of the map (5.3.8) in k-morphisms (in particular relative Frobenius) and its compatibility with base change. ■

Remark 5.3.5. In this work, we will only need Proposition 5.3.3 in the case that X is the special fiber of a smooth and proper S-scheme $\mathscr{X}$ and $e \leq p-1$. In this case, the *inverse* of (5.3.8) admits a simple description which we will use later, and now describe.

To calculate the rigid cohomology of $X = \mathscr{X}_0$, we may take $\overline{X} = X$ and $i : X \hookrightarrow \widehat{\mathscr{X}}$ to be the natural closed immersion of X into the formal completion $\widehat{\mathscr{X}}$ of $\mathscr{X}$ along X. Since $\overline{X} = X$, the map $j : X \hookrightarrow \overline{X}$ is the identity, so the functor $j^\dagger$ is the identity functor. Moreover, $]\overline{X}[_{\widehat{\mathscr{X}}} = \mathrm{sp}^{-1}(X) = \widehat{\mathscr{X}}^{\mathrm{rig}}$, so by Remark 4.2.4 the natural map

$$\mathbf{H}^\bullet(\widehat{\mathscr{X}}^{\mathrm{rig}}, \Omega^\bullet_{\widehat{\mathscr{X}}^{\mathrm{rig}}/K}) \to H^\bullet_{\mathrm{rig}}(X/K)$$

(which is of (5.3.1)) is an isomorphism. On the other hand, the proof of Proposition 5.3.2 shows that there is a natural isomorphism

$$H^\bullet_{\mathrm{cris}}(X/R) \otimes_R K \to \mathbf{H}^\bullet(\widehat{\mathscr{X}}, \widehat{\Omega}^\bullet_{\widehat{\mathscr{X}}/\widehat{S}}) \otimes K.$$

Composing this map with the isomorphism 5.2.4, we obtain an isomorphism

$$H^{\bullet}_{\mathrm{cris}}(X/R) \otimes_R K \to H^{\bullet}_{\mathrm{rig}}(X/K) \tag{5.3.22}$$

It is easy to see that (5.3.22) is the inverse of Berthelot's isomorphism (5.3.8) of Proposition 5.3.3 as follows. In this particular case the natural map $\mathscr{O}_{\widehat{X}} \to \widehat{\mathscr{D}}_X(\widehat{\mathscr{X}})$ is an isomorphism (since $X = \mathscr{X}_0$), so (5.3.13) is just the obvious the natural map

$$\mathrm{sp}_* \mathscr{O}_{\widehat{\mathscr{X}}^{\mathrm{rig}}/K} \to \mathscr{O}_{\widehat{\mathscr{X}}} \otimes_R K.$$

The composition of (5.3.8) with (5.3.22) is thus induced by

$$\mathrm{sp}_* \mathscr{O}_{\widehat{\mathscr{X}}^{\mathrm{rig}}/K} \longrightarrow \mathscr{O}_{\widehat{\mathscr{X}}} \otimes_R K \longrightarrow \mathrm{sp}_* \mathscr{O}_{\widehat{\mathscr{X}}^{\mathrm{rig}}/K}, \tag{5.3.23}$$

where the latter map is the canonical map coming from the map of ringed sites $\mathrm{sp} : \widehat{\mathscr{X}}^{\mathrm{rig}} \to \widehat{\mathscr{X}}$. By working locally over $U = \mathrm{Spf}\, A$ in $\widehat{\mathscr{X}}$, we easily see that (5.3.23) is the identity.

Remark 5.3.6. The reader may wonder why we did not use (5.3.22) as the definition of our comparison isomorphism between crystalline and rigid cohomology, since we only need this comparison in the "liftable" case. The answer is simple: it is not clear that (5.3.22) is functorial in (perhaps non-liftable) k-morphisms. To prove such functoriality, one really needs the full generality of the construction of (5.3.8), as is clear from the proof of Proposition 5.3.3.

5.4 Compatibility of comparison isomorphisms

Now suppose that X/R is a smooth and proper scheme of finite type with closed fiber X_0, and let $\widehat{X}$ be the formal completion of X along X_0. We can then form several cohomology groups: the rigid and crystalline cohomologies of X_0, and the de Rham cohomologies of $\widehat{X}^{\mathrm{rig}}$ and X_K. Moreover, we have defined comparison isomorphisms between these groups, and for later purposes it will be crucial to know how these comparison maps are related. More precisely, we have a diagram of isomorphisms K-vector spaces

(5.4.1)
$$\begin{tikzcd} H^\bullet_{\mathrm{rig}}(X_0/K) \arrow[r, "(5.3.8)"] & H^\bullet_{\mathrm{cris}}(X_0/R)\otimes_R K \arrow[d, "(5.3.4)"] \\ H^\bullet_{\mathrm{dR}}(\widehat{X}^{\mathrm{rig}}) \arrow[u, "(5.3.1)"] & H^\bullet_{\mathrm{dR}}(X_K/K) \arrow[l, "(5.2.2)"] \end{tikzcd}$$

with the indicated maps, and we will need to know:

Lemma 5.4.1. *The diagram* (5.4.1) *commutes.*

Proof. First, form the diagram

$$\begin{tikzcd} H^\bullet_{\mathrm{rig}}(X_0/K) \arrow[rr, "(5.3.8)"] & & H^\bullet_{\mathrm{cris}}(X_0/R)\otimes_R K \arrow[dl, "(5.3.7)"'] \arrow[dd, "(5.3.4)"] \\ & H^\bullet_{\mathrm{dR}}(\widehat{X}/\widehat{S})\otimes_R K \arrow[dl, "(5.2.4)"'] & \\ H^\bullet_{\mathrm{dR}}(\widehat{X}^{\mathrm{rig}}) \arrow[uu, "(5.3.1)"] & & H^\bullet_{\mathrm{dR}}(X_K/K) \arrow[ll, "(5.2.2)"] \arrow[ul, "(5.1.2)"'] \end{tikzcd}$$

with the given maps. We claim that each of the three triangular sub-diagrams commute. Granting this, it follows at once that (5.4.1) commutes.

To see our claim, observe that the right triangle commutes since *by definition* the map (5.3.4) is the composite of (5.3.7) and the inverse of (5.1.2). That the top triangle commutes is the content of Remark 5.3.5, which realizes the composite of the long diagonal with (5.3.1) as the inverse of (5.3.8).

We are therefore reduced to proving that the bottom triangle commutes. By definition, the map (5.1.2) arises from the natural pullback map on differentials associated to the morphism of sites $\iota : \widehat{X} \to X$. Similarly, (5.2.4) comes from the morphism of sites $\mathrm{sp} : \widehat{X}^{\mathrm{rig}} \to \widehat{X}$ and (5.2.2) comes from the map of sites

$\eta : \widehat{X}^{\mathrm{rig}} \to X_K$ constructed in §5.2. Thus, the commutativity of the bottom triangle in the above diagram amounts to the commutativity of the diagram of ringed sites

$$\begin{tikzcd} \widehat{X} \arrow[r, "\iota"] & X \\ \widehat{X}^{\mathrm{rig}} \arrow[u, "\mathrm{sp}"] \arrow[r, "\eta"'] & X_K \arrow[u] \end{tikzcd}$$

where $X_K \to X$ is the natural inclusion of the generic fiber. Such commutativity is obvious. $\blacksquare$

CHAPTER VI

Trace morphisms and correspondences

In chapters II and III we studied the first de Rham cohomology of a smooth proper and geometrically connected curve X over a field K of characteristic zero, and we saw that for a finite morphism $f : X \to Y$ of such curves over K, one has a K-linear map

$$f_* : H^1_{\mathrm{dR}}(X/K) \to H^1_{\mathrm{dR}}(Y/K)$$

induced via the trace map tr_f on differentials. We wish to have a similar construction for rigid analytic curves over a nonarchimedean field of characteristic zero. However, the generalization of trace morphisms on differentials to the analytic setting is not immediate, as one does not have Grothendieck's trace morphism in this context. Instead, one might hope to mimic the classical construction of the trace map tr_f alluded to in §II: first one constructs tr_f over the dense étale locus of f in Y, where it is induced from the usual trace map attached to the finite (locally) free extension of algebras $\mathscr{O}_Y \to f_*\mathscr{O}_X$, and then one shows by a local calculation that this map uniquely extends to the desired trace map over all of Y. This strategy does indeed work in the rigid-analytic setting, but more care is required than in the algebraic case.

In this chapter we will construct a trace map on de Rham cohomology associated to a finite flat surjection $f : X \to Y$ of smooth rigid spaces over a nonarchimedean field K of characteristic zero; our method of construction will also recover the map tr_f above in the algebraic case for smooth schemes over a field K of characteristic zero (in particular, here we carry out the local calculation alluded to in §2.3). The algebraic and analytic trace maps will be compatible (in a precise sense) via the rigid analytification functor. Our construction is more or less a straightforward globalization of that of [39, §8] (cf. [21, §7.6], especially Proposition 7.6.2).

We will then show how this construction provides (in certain cases) trace maps in rigid cohomology that are compatible with the natural comparison map (5.3.1) from de Rham cohomology to rigid cohomology.

Recall that a *correspondence* on a smooth proper and geometrically connected curve X over K is a pair of finite K-morphisms $f : Y \rightrightarrows X$ with Y over K smooth proper and geometrically connected. By pullback via one morphism and trace via the other, any correspondence yields an endomorphism of $H^1_{\mathrm{dR}}(X/K)$. Our construction of trace maps in the cohomology of rigid analytic curves and on rigid cohomology will allow us to generalize this algebraic construction to these situations. We will explain this in §6.2, and will then show that the resulting endomorphisms of cohomology are compatible with comparison maps between cohomology theories. This will settle the Hecke compatibility issue of [22, 16.6].

6.1 Trace morphisms for smooth rigid spaces and varieties

Fix a complete nonarchimedean field K of characteristic zero and let $f : X \to Y$ be a finite morphism of smooth rigid spaces over K of the same pure dimension, so f is automatically flat and carries each connected component of X onto one of Y.

Since $f_*\mathscr{O}_X$ is a finite and flat $\mathscr{O}_Y$-algebra, one has the usual ring-theoretic trace map

$$\mathrm{tr}_f : f_*\mathscr{O}_X \to \mathscr{O}_Y.$$

We will extend this to a morphism of complexes

$$\mathrm{tr}_f : f_*\Omega^\bullet_{X/K} \to \Omega^\bullet_{Y/K}$$

which is $\mathscr{O}_Y$-linear in each degree.

We begin by constructing our trace morphism in the case that f is étale. In this situation, for any i the natural map

$$f^*\Omega^i_{Y/K} \to \Omega^i_{X/K}$$

is an isomorphism [10, Proposition 2.6], and we define

$$\mathrm{tr}_f : f_*\Omega^i_{X/K} \to \Omega^i_{Y/K}$$

as the $\mathscr{O}_Y$-linear composite

$$\mathrm{tr}_f : f_*\Omega^i_{X/K} \simeq f_*f^*\Omega^i_{Y/K} \simeq f_*\mathscr{O}_{X/K} \otimes_{\mathscr{O}_{Y/K}} \Omega^i_{Y/K} \xrightarrow{\mathrm{tr}_f \otimes 1} \Omega^i_{Y/K}. \tag{6.1.1}$$

By local considerations, tr_f is compatible with the d-maps in each degree and is compatible with any étale base change $Y' \to Y$ ($X' := X \times_Y Y'$ is K-smooth since X is).

Theorem 6.1.1. *Let $f : X \to Y$ be a finite flat map of smooth rigid spaces over K. Then there is a unique $\mathscr{O}_Y$-linear morphism of complexes*

$$\mathrm{tr}_f : f_*\Omega^\bullet_{X/K} \to \Omega^\bullet_{Y/K}$$

which recovers (6.1.1) *over the étale locus of f in Y.*

Observe that the uniqueness aspect implies that tr_f is compatible with étale base change on Y. In order to prove Theorem 6.1.1, we first establish a lemma from which the uniqueness aspect of the theorem follows immediately.

Lemma 6.1.2. *Let U be the locus in Y over which f is étale. Suppose*

$$\varphi : \mathscr{F} \to \mathscr{G}$$

is a morphism of abelian sheaves with $\mathscr{G}$ locally free of finite rank over $\mathscr{O}_Y$. If the restriction of φ to U is zero, then φ is zero.

Proof. Since f is finite flat, $X \to Y$ is étale at $x \in X$ if and only if the fiber $X_{f(x)}$ over $f(x)$ is étale at x. Thus, $Z := Y - U$ is exactly the support of the coherent sheaf $f_*\Omega^1_{X/Y}$. We claim that Z is a nowhere-dense analytic set. By working locally, we may assume that $Y = \mathrm{Sp}\, A$ and $X = \mathrm{Sp}\, B$ with A a domain and B a finite A-algebra, and we wish to show that the support of the A-module $\Omega^1_{B/A}$ associated to $\Omega^1_{X/Y}$ is a proper analytic set in the irreducible rigid space $\mathrm{Sp}\, A$. Since B is finite over A, the module $\Omega^1_{B/A}$ may be identified with the module of algebraic differentials. Letting $Q(A)$ denote the fraction field of A, we therefore have

$$\Omega^1_{B/A} \otimes_A Q(A) = \Omega^1_{B \otimes Q(A)/Q(A)} = 0$$

since $Q(A)$ is of characteristic zero (as K is), so $B \otimes Q(A)$ is a product of *separable* field extensions of $Q(A)$. We conclude that $\mathrm{Ann}_A(\Omega^1_{B/A})$ is a nonzero ideal of A and hence that the support of the A-module $\Omega^1_{B/A}$ is a proper analytic set in $\mathrm{Sp}\, A$, as desired.

Now if s is any section of $\mathscr{O}_Y$ whose restriction to U is zero, then the locus of points where s is zero is an analytic set containing the everywhere-dense Zariski open set U. According to [14, Lemma 2.1.4], any analytic subset of a connected normal rigid space containing a nonempty admissible open must be the whole space, so by treating the connected components of Y separately, we conclude that s vanishes identically, so $\mathscr{O}_Y \to \iota_* \mathscr{O}_U$ is injective where $\iota : U \to Y$ is the inclusion. Since $\mathscr{G}$ is locally free, it follows that $\mathscr{G} \to \iota_* \left(\mathscr{G}\big|_U\right)$ is injective. Therefore, φ is zero if and only if the composite

$$\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \longrightarrow \iota_* \left(\mathscr{G}\big|_U\right) \tag{6.1.2}$$

is zero. Obviously, (6.1.2) is zero if and only if $\varphi\big|_U$ is zero, and we are done. ■

We now proceed to construct the map $\mathrm{tr}_f : f_* \Omega^\bullet_{X/K} \to \Omega^\bullet_{Y/K}$ for general finite (flat) $f : X \to Y$.

Proof of Theorem 6.1.1. We claim that we may work locally on Y. Indeed, first observe that the properties of Theorem 6.1.1 are local on Y, so the restriction of any morphism with these properties to an open subset again has the same properties. Fix an admissible open cover $\{U_j\}$ of Y, put $V_j = f^{-1}(U_j)$ and suppose given trace morphisms $\mathrm{tr}_{f,j} : f_* \Omega^i_{V_j/K} \to \Omega^i_{U_j/K}$ with the claimed properties. Then the restrictions of $\mathrm{tr}_{f,j}$ and $\mathrm{tr}_{f,j'}$ to the open $U_j \cap U_{j'}$ agree by Lemma 6.1.2 and hence glue. Moreover, their gluing is unique by Lemma 6.1.2.

We are thus reduced to the situation $Y = \mathrm{Sp}\, A$ and $X = \mathrm{Sp}\, B$ where A and B are K-affinoid algebras with B finite flat over A. Clearly we may assume that Y is connected, and so may suppose that A is a domain. By treating the connected components of X one at a time, we may further assume that B is a domain. For the remainder of the proof, we closely follow [39], §8.

Let $Q(A)$ and $Q(B)$ be the fraction fields of A and B, respectively. Consider the natural $Q(B)$-linear map

$$Q(B) \otimes_A \Omega^i_{A/K} \to Q(B) \otimes_B \Omega^i_{B/K}, \tag{6.1.3}$$

in which we use (as usual) the modules of *continuous* Kähler differentials for the K-Banach topologies of A and B. We assert that (6.1.3) is an isomorphism. To show this, since exterior product commutes with extension of scalars, it is enough to treat the case $i = 1$, and we will construct an inverse mapping

$$Q(B) \otimes_B \Omega^1_{B/K} \to Q(B) \otimes_A \Omega^1_{A/K}. \tag{6.1.4}$$

By the universal property of $\Omega^1_{B/K}$ [10, §1], to make a $Q(B)$-linear map (6.1.4) it suffices to construct an K-linear derivation

$$Q(B) \to Q(B) \otimes_A \Omega^1_{A/K}$$

which carries B into a finite B-submodule of the target and, when restricted to A, recovers the universal K-derivation $D : A \to \Omega^1_{A/K}$. Now the composite

$$Q(A) \xrightarrow{D} Q(A) \otimes \Omega^1_{A/K} \to Q(B) \otimes_A \Omega^1_{A/K}$$

is certainly an K-derivation, and since the map $Q(A) \to Q(B)$ is finite étale (as K has characteristic zero), this derivation uniquely lifts to a K-derivation

$$\widetilde{D} : Q(B) \to Q(B) \otimes_A \Omega^1_{A/K}.$$

For an A-module spanning set $\{b_1, \ldots, b_n\}$ of B, we have for any $a_1, \ldots, a_n \in A$

$$\widetilde{D}(\sum_{i=1}^n a_i b_i) = \sum_{i=1}^n b_i D(a_i) + \sum_{i=1}^n a_i \widetilde{D}(b_i).$$

We conclude that the image of $\widetilde{D}$ lies in the B-submodule of $Q(B) \otimes_A \Omega^1_{A/K}$ generated by the B-finite module $B \otimes_A \Omega^1_{A/K}$ and the finite set $\{\widetilde{D}(b_i)\}_{i=1}^n$, which is indeed a finite B-submodule of the target.

To show that we have constructed an inverse to (6.1.3), first note that A and B have the same pure dimension, say d, so $Q(B) \otimes_B \Omega^1_{B/K}$ and $Q(B) \otimes_A \Omega^1_{A/K}$ are

$Q(B)$-vector spaces with the same dimension d. Thus, to check the inverse property, it suffices to compute the composite in one direction to be the identity (as $Q(B)$ is a finite extension of the field $Q(A)$). Obviously, the $Q(B)$-linear map

$$Q(B) \otimes_A \Omega^1_{A/K} \to Q(B) \otimes_B \Omega^1_{B/K} \to Q(B) \otimes_A \Omega^1_{A/K}$$

is the identity.

Now consider the composite

$$Q(B) \otimes_B \Omega^i_{B/K} \xleftarrow{\sim} Q(B) \otimes_A \Omega^i_{A/K} \xrightarrow{\mathrm{Tr}\otimes 1} Q(A) \otimes_A \Omega^1_{A/K}, \tag{6.1.5}$$

where $\mathrm{Tr} = \mathrm{Tr}_{Q(B)/Q(A)} : Q(B) \to Q(A)$ is the natural trace map associated to the finite (separable) extension of fields $Q(B)/Q(A)$. Since $\Omega^i_{B/K}$ and $\Omega^i_{A/K}$ are locally free, they are submodules of $Q(B) \otimes \Omega^i_{B/K}$ and $Q(A) \otimes \Omega^i_{A/K}$, respectively. Thus, we may ask whether or not (6.1.5) restricts to a map $\Omega^i_{B/K} \to \Omega^i_{A/K}$; we claim that it does, and the resulting map will be the desired trace morphism.

Note that since $X = \mathrm{Sp}\, B$ and $Y = \mathrm{Sp}\, A$ are smooth, A and B are normal rings. Since A and B are moreover domains, for any height 1 prime $\mathfrak{p}$ of A, the local ring $A_{\mathfrak{p}}$ is a discrete valuation ring, and $B_{\mathfrak{p}} := B \otimes_A A_{\mathfrak{p}}$ is a semi-local Dedekind domain, hence a principal ideal domain. Since A is normal, we know that it is the intersection inside $Q(A)$ of the rings $A_{\mathfrak{p}}$ as $\mathfrak{p}$ runs over the height-1 primes of A. Therefore, since $\Omega^i_{A/K}$ is a locally free A-module of finite rank, to prove that the image of $\Omega^i_{B/K}$ under (6.1.5) is contained in $\Omega^i_{A/K}$, it suffices to prove that the image of $\Omega^i_{B/K} \otimes_B B_{\mathfrak{p}}$ under (6.1.5) is contained in $\Omega^i_{A/K} \otimes_A A_{\mathfrak{p}}$ for each height-1 prime $\mathfrak{p}$ of A.

As in the algebraic case we have an exact sequence

$$\Omega^1_{A/K} \otimes_A B \longrightarrow \Omega^1_{B/K} \longrightarrow \Omega^1_{B/A} \longrightarrow 0\,,$$

(see [10, Proposition 1.1]), where $\Omega^1_{B/A}$ may be identified with the *algebraic* module of Kähler differentials since B is finite over A. The first map is injective. Indeed, since (6.1.3) is an isomorphism, the kernel of $\Omega^1_{A/K} \otimes_A B \to \Omega^1_{B/K}$ is torsion as a B-module. But since X is smooth, the A-module $\Omega^1_{A/K}$ is finite flat, so we conclude that $\Omega^1_{A/K} \otimes_A B$ is torsion free over B, whence the kernel is zero. Localizing the resulting short exact sequence at $\mathfrak{p}$ we obtain an exact sequence of $B_{\mathfrak{p}}$-modules

$$
(6.1.6) \qquad 0 \longrightarrow \Omega^1_{A/K} \otimes_A B_{\mathfrak{p}} \longrightarrow \Omega^1_{B/K} \otimes_B B_{\mathfrak{p}} \longrightarrow \Omega^1_{B_{\mathfrak{p}}/A_{\mathfrak{p}}} \longrightarrow 0 \,,
$$

where

$$
\Omega^1_{B/A} \otimes_B B_{\mathfrak{p}} \simeq \Omega^1_{B/A} \otimes_A A_{\mathfrak{p}} \simeq \Omega^1_{B_{\mathfrak{p}}/A_{\mathfrak{p}}}
$$

is the algebraic module of Kähler differentials.

We know that $\Omega^1_{B_{\mathfrak{p}}/A_{\mathfrak{p}}}$ is a finite torsion $B_{\mathfrak{p}}$-module since $Q(B)$ is separable over $Q(A)$, and since $B_{\mathfrak{p}}$ is a principal ideal domain there exists a nonzero $\gamma \in B$ with $(\gamma)B_{\mathfrak{p}} = \mathrm{Ann}_{B_{\mathfrak{p}}}(\Omega^1_{B_{\mathfrak{p}}/A_{\mathfrak{p}}})$. Moreover, the method of proof of [39, §8 Lemma 2] shows that γ annihilates the cokernel of

$$
\Omega^i_{A/K} \otimes_A B_{\mathfrak{p}} \to \Omega^i_{B/K} \otimes_B B_{\mathfrak{p}}
$$

for all i, and hence

$$
\Omega^i_{B/K} \otimes_B B_{\mathfrak{p}} \subseteq \Omega^i_{A/K} \otimes_A (\gamma^{-1}) \cdot B_{\mathfrak{p}}.
$$

But the "differential characterization of the different" [45, III, §7, Proposition 14] shows that (γ) is the different of $B_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$, and hence, by the *definition* of the inverse different, that the image of $\Omega^i_{B/K} \otimes_B B_{\mathfrak{p}}$ under (6.1.5) is contained in $\Omega^i_{A/K} \otimes_A A_{\mathfrak{p}}$, as desired.

Finally, we use Lemma 6.1.2 with respect to the exterior differentials d (which *are* maps of abelian sheaves, so the Lemma applies) to get the compatibility of the trace map on differential i-forms with the d-maps by restricting to the étale locus of f in Y, where such compatibility is obvious by the construction of tr_f. ■

Observe that the construction of tr_f and the proof of Theorem 6.1.1 for a finite flat morphism f of smooth rigid spaces over K carries over *mutatis mutandi* to the case of a finite flat morphism $f : X \to Y$ of smooth schemes over *any* field K of characteristic zero; in this situation, we will denote the corresponding trace morphism by tr_f also. Recall that in §2.3 we introduced another trace map (2.3.2) on algebraic de Rham complexes (denoted tr_f in that section) by using Grothendieck's theory of the trace. As we remarked in §2.3, these two trace maps coincide, as this may checked over the étale locus of f in Y, where it follows from how Grothendieck's trace map is defined in the finite étale case. Thus, our "abuse of notation" in denoting both of these maps by tr_f is justified.

We now wish to show that when K is a complete nonarchimedean field of characteristic zero, the trace morphisms we have constructed are compatible with the rigid analytification functor. More precisely, suppose that $f : X \to Y$ is a finite flat morphism of smooth schemes over a complete nonarchimedean field K of characteristic zero. Using the fact that finite pushforward of coherent sheaves and the formation of the sheaf of differentials commute with analytification, the analytification of the $\mathscr{O}_Y$-linear

$$\mathrm{tr}_f : f_*\Omega^i_{X/K} \to \Omega^i_{Y/K}$$

is an $\mathscr{O}_{Y^{\mathrm{an}}}$-linear morphism

$$(\mathrm{tr}_f)^{\mathrm{an}} : f^{\mathrm{an}}_*\Omega^i_{X^{\mathrm{an}}/K} \to \Omega^i_{Y^{\mathrm{an}}/K}.$$

On the other hand, associated to the finite flat morphism f^{an} is an $\mathscr{O}_{Y^{\mathrm{an}}}$-linear trace morphism

$$\mathrm{tr}_{f^{\mathrm{an}}} : f^{\mathrm{an}}_*\Omega^i_{X^{\mathrm{an}}/K} \to \Omega^i_{Y^{\mathrm{an}}/K}.$$

Proposition 6.1.3. *The two morphisms* $(\mathrm{tr}_f)^{\mathrm{an}}$ *and* $\mathrm{tr}_{f^{\mathrm{an}}}$ *are equal.*

Proof. By Lemma 6.1.2, it is enough to show that $(\mathrm{tr}_f)^{\mathrm{an}}$ and $\mathrm{tr}_{f^{\mathrm{an}}}$ agree over the étale locus U of f^{an} in Y^{an}. Using the fact that that U *is* the analytification of the étale locus of f, as can be seen by working with completed local rings on X, we may therefore assume that f and f^{an} are étale. By the construction of tr_f and $\mathrm{tr}_{f^{\mathrm{an}}}$, we are then reduced to showing the equality of the two morphisms

$$(\mathrm{tr}_f)^{\mathrm{an}}, \mathrm{tr}_{f^{\mathrm{an}}} : f^{\mathrm{an}}_* \mathscr{O}_{X^{\mathrm{an}}} \to \mathscr{O}_{Y^{\mathrm{an}}}$$

in degree zero.

To check such equality, we may work at the completed stalks at points of Y^{an}. Observe that we have canonical isomorphisms

$$\prod\nolimits_{x \in f^{-1}(y)} \mathscr{O}^{\wedge}_{X,x} \xrightarrow{\simeq} (f_* \mathscr{O}_X)^{\wedge}_y \tag{6.1.7}$$

for $y \in Y$ and

$$\prod\nolimits_{x \in (f^{\mathrm{an}})^{-1}(y)} \mathscr{O}^{\wedge}_{X^{\mathrm{an}},x} \xrightarrow{\simeq} (f^{\mathrm{an}}_* \mathscr{O}_{X^{\mathrm{an}}})^{\wedge}_y \tag{6.1.8}$$

for $y \in Y^{\mathrm{an}}$. Identifying the points of Y^{an} with the closed points of Y, it is therefore enough to show that for any closed point $y \in Y$ and any closed point $x \in X$ mapping to Y, the natural trace maps

$$(\mathrm{tr}_f)^{\wedge} : \mathscr{O}^{\wedge}_{X,x} \to \mathscr{O}^{\wedge}_{Y,y}$$

$$(\mathrm{tr}_{f^{\mathrm{an}}})^{\wedge} : \mathscr{O}^{\wedge}_{X^{\mathrm{an}},x} \to \mathscr{O}^{\wedge}_{Y^{\mathrm{an}},y}$$

induce the same map after making the canonical identifications

$$\mathscr{O}^{\wedge}_{X,x} \overset{\simeq}{\to} \mathscr{O}^{\wedge}_{X^{\mathrm{an}},x} \qquad\qquad \mathscr{O}^{\wedge}_{Y,y} \overset{\simeq}{\to} \mathscr{O}^{\wedge}_{Y^{\mathrm{an}},y}$$

(which are compatible with the isomorphisms (6.1.7) and (6.1.8)). But this is clear from the construction of tr_f and $\mathrm{tr}_{f^{\mathrm{an}}}$, as both maps induce the natural trace map

$$\mathrm{Tr}_{\mathscr{O}^{\wedge}_{X,x}/\mathscr{O}^{\wedge}_{Y,y}} : \mathscr{O}^{\wedge}_{X,x} \to \mathscr{O}^{\wedge}_{Y,y}$$

on completions. ■

For later use, let us record here the following easy result:

Proposition 6.1.4. *Let $f : X \to Y$ be a finite flat morphism of smooth rigid spaces over K. Then the composite map*

$$\Omega^i_{Y/K} \xrightarrow{f^*} f_*\Omega^i_{X/K} \xrightarrow{\mathrm{tr}_f} \Omega^i_{Y/K} \tag{6.1.9}$$

is multiplication by $\deg f$.

Proof. The claim may be checked over the étale locus of f in Y, and we are thus reduced to showing that the composite map in degree zero

$$\mathscr{O}_Y \xrightarrow{f^*} f_*\mathscr{O}_X \xrightarrow{\mathrm{tr}_f} \mathscr{O}_Y$$

is multiplication by $\deg f$, which follows immediately from the definition of tr_f in degree zero. ■

Corollary 6.1.5. *Let $f : X \to Y$ be a finite flat morphism of smooth rigid spaces over K. Then for each i, the map* tr_f *induces a K-linear map of vector spaces*

$$f_* : H^i_{\mathrm{dR}}(X/K) \to H^i_{\mathrm{dR}}(Y/K).$$

Moreover, when f has constant degree (e.g. Y is connected) the composite $f_* \circ f^*$ of f_* with the usual pullback map on cohomology f^* is multiplication by $\deg f$.

Proof. By Theorem 6.1.1, tr_f gives a K-linear trace map of complexes $\mathrm{tr}_f : f_*\Omega^\bullet_{X/K} \to \Omega^\bullet_{Y/K}$. We claim that the natural map of complexes

$$f_*\Omega^\bullet_{X/K} \to \mathbf{R}f_*\Omega^\bullet_{X/K}$$

is a quasi-isomorphism. This is equivalent to the vanishing of $R^i f_*\Omega^j_{X/K}$ for all $j \geq 0$ and all $i > 0$. But since f is finite, such vanishing is easily seen to hold by repeating the usual argument that the higher direct image functors of a finite morphism vanish on coherent sheaves in the scheme case (and by using Kiehl's "Theorem B" [29, 2.4.2]). Thus, using Remark A.1.2 (i.e. the Leray spectral sequence) to ensure that the natural map

$$\mathbf{H}^i(Y, \mathbf{R}f_*\Omega^\bullet_{X/K}) \to \mathbf{H}^i(X, \Omega^\bullet_{X/K})$$

is an isomorphism, we obtain the desired map:

$$\mathbf{H}^i(X, \Omega^\bullet_{X/K}) \xleftarrow{\simeq} \mathbf{H}^i(Y, \mathbf{R}f_*\Omega^\bullet_{X/K}) \simeq \mathbf{H}^i(Y, f_*\Omega^\bullet_{X/K}) \xrightarrow{\mathbf{H}^i(\mathrm{tr}_f)} \mathbf{H}^i(Y, \Omega^\bullet_{Y/K}) .$$

The second assertion of the corollary is an immediate consequence of Proposition 6.1.4. ■

An immediate consequence of Corollary 6.1.5 and the compatibility of tr_f with analytification in Proposition 6.1.3 is:

Corollary 6.1.6. *Suppose that $f : X \to Y$ is a finite flat morphism of smooth schemes over K. Then for all i the diagram*

$$\begin{array}{ccc} H^i_{\mathrm{dR}}(X_K/K) & \xrightarrow{(5.1.1)} & H^i_{\mathrm{dR}}(X_K^{\mathrm{an}}/K) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*^{\mathrm{an}}} \\ H^i_{\mathrm{dR}}(Y_K/K) & \xrightarrow[(5.1.1)]{} & H^i_{\mathrm{dR}}(Y_K^{\mathrm{an}}/K) \end{array}$$

commutes.

Now suppose R is a complete discrete valuation ring of mixed characteristic $(0, p)$ with field of fractions K and residue field k, and suppose $w : X' \to X$ is a finite morphism of k-schemes. We wish to show that in certain favorable situations, one obtains a "trace" morphism on rigid cohomology

$$w_* : H^i_{\mathrm{rig}}(X'/K) \to H^i_{\mathrm{rig}}(X/K).$$

In general, it seems difficult to associate a trace morphism on rigid cohomology to an arbitrary finite flat map $w : X' \to X$ of k-schemes. The fundamental problem is that, while it is possible (under our assumptions) to find embeddings of X' and X into topologically finite type formal R-schemes $\mathscr{X}$ and $\mathscr{X}'$ with a map $u : \mathscr{X}' \to \mathscr{X}$ lifting w, it is not at all clear that that this can be done with u finite flat. Indeed, the construction of u uses a fiber product argument, and projection maps from a fiber product are rarely finite.

Thus, in order to construct trace a morphisms attached to a finite k-map w as above, we will need to suppose that X and X' admit embeddings as above with a map u that is finite and flat. We therefore suppose that there is a commutative diagram

(6.1.10)
$$\begin{tikzcd}
X' \arrow[r, "j_{X'}"] \arrow[d, "w"'] & \overline{X}' \arrow[r, "i_{X'}"] \arrow[d, "v"] & \mathscr{X}' \arrow[d, "u"] \\
X \arrow[r, "j_X"] \arrow[dr] & \overline{X} \arrow[r, "i_X"] \arrow[d] & \mathscr{X} \arrow[d] \\
& \operatorname{Spec} k \arrow[r] & \operatorname{Spf} R
\end{tikzcd}$$

where $\overline{X}'$ and $\overline{X}$ are proper k-schemes, $j_{X'}$, j_X are open immersions, and $i_{X'}$, i_X are closed immersions into proper formal $\operatorname{Spf} R$-schemes $\mathscr{X}'$ and $\mathscr{X}$ which are smooth at the points of X' and X respectively. We further suppose that u is finite and flat and that X' and $\overline{X'}$ are the full preimages of X and $\overline{X}$ under u, respectively (in fact, since the trace map is "additive" we really only need that X and $\overline{X}$ are open and closed in their preimages; however, we won't use this).

We then obtain a finite flat morphism of rigid spaces $u^{\mathrm{rig}} : \mathscr{X}'^{\mathrm{rig}} \to \mathscr{X}^{\mathrm{rig}}$ which restricts to a finite and flat morphism of rigid spaces

$$u^{\mathrm{rig}} : (u^{\mathrm{rig}})^{-1}\left(]\overline{X}[_{\mathscr{X}}\right) \to]\overline{X}[_{\mathscr{X}};$$

since $]\overline{X}'[_{\mathscr{X}'} = (u^{\mathrm{rig}})^{-1}\left(]\overline{X}[_{\mathscr{X}}\right)$, this is a finite flat morphism

$$f :]\overline{X}'[_{\mathscr{X}'} \to]\overline{X}[_{\mathscr{X}}.$$

By Theorem 6.1.1, we get a morphism of complexes

$$\mathrm{tr}_f : f_*\Omega^\bullet_{]\overline{X}'[_{\mathscr{X}'}/K} \to \Omega^\bullet_{]\overline{X}[_{\mathscr{X}}/K}, \tag{6.1.11}$$

and hence, by Corollary 6.1.5, a trace map on cohomology

$$f_* : H^i_{\mathrm{dR}}(]\overline{X}'[_{\mathscr{X}'}/K) \to H^i_{\mathrm{dR}}(]\overline{X}[_{\mathscr{X}}/K). \tag{6.1.12}$$

In order to get a trace morphism on rigid cohomology that is compatible with the natural map (5.3.3) from de Rham cohomology to rigid cohomology, we need:

Lemma 6.1.7. *Suppose given the diagram* (6.1.10) *with u finite flat and X' and $\overline{X}'$ the full preimages under u of X and $\overline{X}$ in $\mathscr{X}'$, respectively, and denote by f the restriction of u^{rig} to $]\overline{X}'[_{\mathscr{X}'}$. Let $\mathscr{F}$ and $\mathscr{G}$ be abelian sheaves on $]\overline{X}'[_{\mathscr{X}'}$ and $]\overline{X}[_{\mathscr{X}}$, respectively, and suppose given a morphism of abelian sheaves on $]\overline{X}[_{\mathscr{X}}$*

$$f_*\mathscr{F} \to \mathscr{G}.$$

Then there is a canonical morphism of sheaves on $]\overline{X}[_{\mathscr{X}}$

$$f_*j^\dagger_{X'}\mathscr{F} \to j^\dagger_X\mathscr{G}$$

making the diagram

$$\begin{tikzcd} f_*\mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\ f_*j^\dagger_{X'}\mathscr{F} \arrow[r] & j^\dagger_X\mathscr{G} \end{tikzcd} \tag{6.1.13}$$

commute.

Proof. By Lemma B.1.1, the inductive system $f^{-1}(V)$ as V ranges over all strict open neighborhoods of $]X[_{\mathscr{X}}$ in $]\overline{X}[_{\mathscr{X}}$ is cofinal among strict open neighborhoods of $]X'[_{\mathscr{X}'}$ in $]\overline{X}'[_{\mathscr{X}'}$, so

$$f_* j^\dagger_{X'} \mathscr{F} = f_* \varinjlim_{V'} j_{V'*} j^{-1}_{V'} \mathscr{F} = f_* \varinjlim_{V} j_{f^{-1}(V)*} j^{-1}_{f^{-1}(V)} \mathscr{F}.$$

Since $f = u^{\mathrm{rig}}\big|_{]\overline{X}'[_{\mathscr{X}'}}$ is a finite morphism of rigid spaces, Lemma B.1.2 applies. Thus, f_* commutes with the direct limits above, whence the sheaf $f_* j^\dagger_{X'} \mathscr{F}$ is equal to

$$\varinjlim_{V} f_* j_{f^{-1}(V)*} j^{-1}_{f^{-1}(V)} \mathscr{F} = \varinjlim_{V} j_{V*} f_* j^{-1}_{f^{-1}(V)} \mathscr{F} = \varinjlim_{V} j_{V*} j^{-1}_{V} f_* \mathscr{F},$$

where the last equality results from the fact that $j_{f^{-1}(V)}$ is an open immersion. Thus,

$$f_* j^\dagger_{X'} \mathscr{F} = j^\dagger_X f_* \mathscr{F}, \tag{6.1.14}$$

and hence everything follows from the naturality of $\mathrm{id} \to j^\dagger_X$. ∎

Let us continue to suppose that we have a diagram (6.1.10) and keep the notation and hypotheses of Lemma 6.1.7. Then for every i we obtain a trace map

$$\mathrm{tr}_f : f_* j^\dagger_{X'} \Omega^i_{]\overline{X}'[_{\mathscr{X}'}/K} \to j^\dagger_X \Omega^i_{]\overline{X}[_{\mathscr{X}}/K},$$

and we claim that these maps give a morphism of overconvergent de Rham complexes. Since tr_f is a map of de Rham complexes, we know that for each i the diagram

$$\begin{tikzcd}
f_* \Omega^i_{]\overline{X}'[_{\mathscr{X}'}/K} \arrow[r, "f_*(d)"] \arrow[d, "\mathrm{tr}_f"] & f_* \Omega^{i+1}_{]\overline{X}'[_{\mathscr{X}'}/K} \arrow[d, "\mathrm{tr}_f"] \\
\Omega^i_{]\overline{X}[_{\mathscr{X}}/K} \arrow[r, "d"] & \Omega^{i+1}_{]\overline{X}[_{\mathscr{X}}/K}
\end{tikzcd}$$

commutes. Applying the functor $j^\dagger_X$ and using (6.1.14) gives our claim. Thanks to the commutativity of (6.1.13), we therefore have a commutative diagram of complexes

$$\begin{tikzcd}
f_* \Omega^\bullet_{]\overline{X}'[_{\mathscr{X}'}/K} \arrow[r, "\mathrm{tr}_f"] \arrow[d] & \Omega^\bullet_{]\overline{X}[_{\mathscr{X}}/K} \arrow[d] \\
f_* j^\dagger_{X'} \Omega^\bullet_{]\overline{X}'[_{\mathscr{X}'}/K} \arrow[r, "\mathrm{tr}_f"'] & j^\dagger_X \Omega^\bullet_{]\overline{X}[_{\mathscr{X}}/K}
\end{tikzcd} \tag{6.1.15}$$

Since $f\ :]\overline{X}'[_{\mathscr{X}'} \to]\overline{X}[_{\mathscr{X}}$ is finite, we have $R^i f_* \mathscr{E} = 0$ for any coherent $\mathscr{O}_{]\overline{X}'[_{\mathscr{X}'}}$-module $\mathscr{E}$ and all $i > 0$, so the natural map of functors $f_* \to \mathbf{R}f_*$ is a quasi-isomorphism on complexes of bounded below coherent $\mathscr{O}_{]\overline{X}'[_{\mathscr{X}'}}$-modules. In particular, the natural map

$$f_* \Omega^\bullet_{]\overline{X}'[_{\mathscr{X}'}/K} \to \mathbf{R}f_* \Omega^\bullet_{]\overline{X}'[_{\mathscr{X}'}/K}$$

is a quasi-isomorphism (cf. the proof of Corollary 6.1.5). We would like to say that the same holds when the de Rham complexes above are replaced by their "overconvergent" counterparts. However, as $j^\dagger$ will almost never take a coherent sheaf to a coherent sheaf, the above argument does not work. Instead, we appeal to Corollary B.1.4, which ensures that $j^\dagger \mathscr{E}$ is f_*-acyclic whenever $\mathscr{E}$ is. More precisely, the sheaves $\Omega^j_{]\overline{X}'[_{\mathscr{X}'}/K}$ *are* f_*-acyclic, so by Corollary B.1.4 we have

$$R^i f_* j^\dagger_{X'} \Omega^j_{]\overline{X}'[_{\mathscr{X}'}/K} = 0$$

for all $i > 0$ and all j. Thus, the natural map

$$f_* j^\dagger_{X'} \Omega^\bullet_{]\overline{X}'[_{\mathscr{X}'}/K} \to \mathbf{R}f_* j^\dagger_{X'} \Omega^\bullet_{]\overline{X}'[_{\mathscr{X}'}/K}$$

is a quasi-isomorphism. With these identifications, applying the functor $\mathbf{R}\Gamma(]\overline{X}[_{\mathscr{X}}, \cdot)$ to (6.1.15), using Remark A.1.2, and recalling that the map (5.3.3) is defined by (5.3.2), we have proved:

Theorem 6.1.8. *Let $w : X' \to X$ be a finite flat morphism of finite type k-schemes that fits into a commutative diagram* (6.1.10) *which satisfies the hypotheses of Lemma* 6.1.7*. Then for each i there is a trace map*

$$w_* : H^i_{\mathrm{rig}}(X') \to H^i_{\mathrm{rig}}(X)$$

such that the diagram

$$\begin{tikzcd} H^i_{\mathrm{dR}}(]\overline{X}'[_{\mathscr{X}'}/K) \arrow[r, "f_*", "(6.1.12)"'] \arrow[d] & H^i_{\mathrm{dR}}(]\overline{X}[_{\mathscr{X}}/K) \arrow[d] \\ H^i_{\mathrm{rig}}(X') \arrow[r, "w_*"'] & H^i_{\mathrm{rig}}(X) \end{tikzcd}$$

commutes, where the vertical maps are the natural maps (5.3.3).

By Proposition 6.1.4, we know that the composite map

$$\Omega^{\bullet}_{]\overline{X}[_{\mathscr{X}}/K} \xrightarrow{f^*} f_*\Omega^{\bullet}_{]\overline{X}'[_{\mathscr{X}'}/K} \xrightarrow{\mathrm{tr}_f} \Omega^{\bullet}_{]\overline{X}[/K}$$

is multiplication by $\deg f$. Applying the additive functor $j^{\dagger}_X$ and using (6.1.14), we find:

Corollary 6.1.9. *With the hypotheses of Theorem* 6.1.8*, if the finite map* u^{rig} *has constant degree then the composite*

$$H^i_{\mathrm{rig}}(X/K) \xrightarrow{w^*} H^i_{\mathrm{rig}}(X'/K) \xrightarrow{w_*} H^i_{\mathrm{rig}}(X/K)$$

is multiplication by $\deg f = \deg w$.

6.2 Correspondences

In this section, using our results on trace morphisms for maps of curves, we study the actions of a correspondence on several cohomology groups associated to a proper admissible curve.

Suppose that X_K and Y_K are smooth and proper geometrically connected curves over K that admit admissible models X and Y over R. Further suppose that T is a correspondence given by a pair of finite (flat) morphisms $\pi_1, \pi_2 : Y_K \rightrightarrows X_K$. We define T acting as a K-linear endomorphism of Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$ via $(\pi_1)_* \circ \pi_2^*$. By abuse of notation we will again write T to denote the endomorphism $(\pi_1)_* \circ \pi_2^*$ of the Hodge filtration.

Theorem 6.2.1. *The endomorphism T of the Hodge filtration of X_K preserves the canonical integral structure furnished by the admissible model X and the exact sequence* (3.4.1).

Proof. This is an immediate consequence of Theorem 3.4.5 (2). ■

Now suppose that the correspondence T extends to the integral level; i.e., that we have a pair of finite flat morphisms of admissible curves

$$\pi_1, \pi_2 : Y \rightrightarrows X.$$

Denote by $\widehat{X}$ the formal completion of X along the closed fiber X_0. Further, suppose that U is an open subscheme of the smooth locus of X_0, and denote by $\overline{U}$ the closure of U in X_0. Suppose that $\pi_2(\pi_1^{-1}U) \subseteq U$ and $\pi_2(\pi_1^{-1}\overline{U}) \subseteq \overline{U}$, so for $i = 1, 2$ we have a diagram

$$\begin{tikzcd} \pi_1^{-1}U \arrow[r] \arrow[d, "\pi_i"'] & \pi_1^{-1}\overline{U} \arrow[r] \arrow[d, "\pi_i"] & \widehat{Y} \arrow[d, "\pi_i"] \\ U \arrow[r] & \overline{U} \arrow[r] & \widehat{X} \end{tikzcd} .$$

By Theorem 6.1.8 we have a trace map

$$(\pi_1)_* : H^1_{\mathrm{rig}}(\pi^{-1}U/K) \to H^1_{\mathrm{rig}}(U/K).$$

Since we also have a pullback map

$$\pi_2^* : H^1_{\mathrm{rig}}(U/K) \to H^1_{\mathrm{rig}}(\pi^{-1}U/K),$$

we get an endomorphism $T := (\pi_1)_* \circ \pi_2^*$ of $H^1_{\mathrm{rig}}(U/K)$.

Theorem 6.2.2. *The natural map $H^1_{\mathrm{dR}}(X_K/K) \to H^1_{\mathrm{rig}}(U/K)$ that is the composite*

$$H^1_{\mathrm{dR}}(X_K/K) \xrightarrow{(5.2.2)} H^1_{\mathrm{dR}}(\widehat{X}^{\mathrm{rig}}) \xrightarrow{(5.3.1)} H^1_{\mathrm{rig}}(U/K) \tag{6.2.1}$$

is T-equivariant.

Proof. This follows from the T-equivariance of (5.2.2), which is a consequence of Corollary 6.1.6 and Proposition 5.2.1, and the T-equivariance of (5.3.1), which is deduced from Theorem 6.1.8. ∎

The T-equivariance of the map (6.2.1) will turn out to be very useful. Indeed, this map allows one to study the K-vector space $H^1_{\mathrm{dR}}(X_K/K)$ together with its action by any algebra of correspondences by means of characteristic p methods (like Frobenius, which one does *not* necessarily have acting on all of $H^1_{\mathrm{dR}}(X_K/K)$). The flexibility to specify U is also very useful, as it allows one to exploit the geometry of the curve X.

CHAPTER VII

Jacobians

Let K be a field of characteristic zero. In the next three chapters, we wish to give an alternate construction of the integral structure studied in chapter III using Jacobians and Néron models. In order to do this, we will first need to relate the Hodge filtration of the first de Rham cohomology of a smooth and proper curve X over K to that of its Jacobian J. We therefore begin in §7.1 by reviewing some facts about the Hodge filtration of a general abelian variety over K. Ultimately, our goal is to show that there is a canonical isomorphism between the Hodge filtrations of $H^1_{\mathrm{dR}}(X/K)$ and $H^1_{\mathrm{dR}}(J/K)$. We will also want to give a description of the self-duality of the Hodge filtration of X induced by cup product (see §2.2) in terms of the Hodge filtration of $H^1_{\mathrm{dR}}(J/K)$. To do this, we will need to spend some time discussing duality on the de Rham cohomology of a general abelian variety; this is addressed in §7.2. We remark that apart from being interesting in its own right, our discussion of duality on the de Rham cohomology of an abelian variety will be crucial for later proofs. Finally, we wish to give a description of the functoriality of the Hodge filtration of $H^1_{\mathrm{dR}}(X/K)$ via pullback and trace in finite morphisms of smooth curves purely in terms of the Jacobian J and the Hodge filtration of $H^1_{\mathrm{dR}}(J/K)$. We will do this using Albanese and Picard functoriality, which we review in §7.3, and will make critical use of the results in §7.2.

7.1 The Hodge filtration of an abelian variety

Let A be an abelian variety over the field K and denote by ${}^tA$ the dual abelian variety over K. As is well-known, the Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^p(A, \Omega^q) \implies H^{p+q}_{\mathrm{dR}}(A/K)$$

of A degenerates at the E_1-stage (see [41, I, Remark 2] for an analytic proof in the case that $\mathrm{char}(K) = 0$ and [18] for an algebraic proof in any characteristic). In particular there is a natural short exact sequence of finite dimensional K-vector spaces

$$0 \longrightarrow H^0(A, \Omega^1_{A/K}) \longrightarrow H^1_{\mathrm{dR}}(A/K) \longrightarrow H^1(A, \mathscr{O}_A) \longrightarrow 0 \tag{7.1.1}$$

that we call the *Hodge filtration* of $H^1_{\mathrm{dR}}(A/K)$, or simply the *Hodge filtration* of A.

There is an alternate description of the flanking vector spaces in (7.1.1) that we will need later. For any commutative group scheme G over any base scheme S with unit section $e : S \to G$, we can consider the $\mathscr{O}_S$-module

$$\omega_G := e^*\Omega^1_{G/S} \tag{7.1.2}$$

on S. Any section of ω_G over S can be *uniquely* propagated to an invariant differential form $\omega \in \Gamma(G, \Omega^1_{G/S})$ by [9, §4.2 Proposition 1], so the canonical map from the sheaf of invariant differentials to ω_G is an isomorphism and $\Omega^1_{G/S} \simeq \pi^*\omega_G$, where $\pi : G \to S$ is the structure morphism. Consequently, we will often refer to ω_G as the sheaf of invariant differentials.

Lemma 7.1.1. *If A is an abelian scheme over S, then every global section of $\Omega^1_{A/S}$ is invariant. Consequently, the obvious canonical map*

$$\Gamma(A, \Omega^1_{A/S}) \to \Gamma(S, \omega_A) \tag{7.1.3}$$

is an isomorphism.

Proof. Denote by $\pi : A \to S$ the structure morphism. Since A is a smooth S-group and $\pi_*\mathscr{O}_A \simeq \mathscr{O}_S$ universally, $\pi_*\Omega^1_{A/S}$ is a vector group whose formation commutes with arbitrary base change. We deduce that $\mathbf{Aut}_S(\pi_*\Omega^1_{A/S})$ is an affine S-scheme of finite type (Zariski-locally it is GL_n over S), so the "translation map"

$$A \to \mathbf{Aut}_S(\pi_*\Omega^1_{A/S}) \tag{7.1.4}$$

is constant on geometric fibers (as any map from a connected projective variety to an affine variety is constant). Since it clearly takes the origin to the identity automorphism, we conclude from [40, Corollary 6.2] that (7.1.4) is zero. This gives our first assertion, and the second follows immediately from the discussion above. ∎

Lemma 7.1.1 gives a different description of the left term in (7.1.1). As for the right term, we have:

Lemma 7.1.2. *Let $S = \operatorname{Spec} B$ be any affine scheme and A an abelian scheme over S. There is a natural isomorphism of B-modules*

$$H^1(A, \mathscr{O}_A) \to \mathrm{Lie}({}^tA). \tag{7.1.5}$$

Proof. See §8.1 for a brief review of Lie algebras. The lemma is standard (see, for example, Lemma 1.3 (b) of [32]) and may be proved along the same lines as Proposition 9.1.1 below. ∎

Remark 7.1.3. The alternative descriptions in (7.1.3) and (7.1.5) of the flanking terms in the Hodge filtration of an abelian scheme are in some ways the "correct" descriptions. Indeed, in chapter VIII we will generalize the short exact sequence (7.1.1) to the case that A is the Néron model over a Dedekind base S of an abelian scheme over some dense open in the base. In this generalizaion, the flanking terms will be ω_A and $\mathrm{Lie}({}^tA)$ (where this latter ${}^tA$ is to be interpreted as the Néron model of the dual abelian scheme); the groups $H^1(A, \mathscr{O}_A)$ and $H^0(A, \Omega^1_{A/S})$ will be of no use, as they can fail to be finitely generated and can contain a lot of torsion (as A need not be proper).

Observe that this alternative description of the flanking terms in (7.1.1) sets up a canonical duality between $H^1(A, \mathscr{O}_A)$ and $H^0({}^tA, \Omega^1_{{}^tA/K})$ via the canonical evaluation pairing

$$\omega_{{}^tA} \times \mathrm{Lie}({}^tA) \to K. \tag{7.1.6}$$

In the next section, we will recall how this pairing on the flanking terms of the Hodge filtration of $H^1_{\mathrm{dR}}(A/K)$ can be extended to a duality on $H^1_{\mathrm{dR}}(A/K)$.

7.2 Duality and de Rham cohomology

There is a canonical K-bilinear pairing

$$(\cdot,\cdot)_A : H^1_{\mathrm{dR}}(A/K)^\vee \times H^1_{\mathrm{dR}}({}^tA/K)^\vee \to K \tag{7.2.1}$$

defined as in [6, 5.1.3], and hence a functorial homomorphism

$$\psi_A : H^1_{\mathrm{dR}}({}^tA/K)^\vee \to H^1_{\mathrm{dR}}(A/K). \tag{7.2.2}$$

Proposition 7.2.1. *The map ψ_A induces a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & H^1({}^tA, \mathscr{O}_{{}^tA})^\vee \arrow[r] \arrow[d, "\psi_A^1"'] & H^1_{\mathrm{dR}}({}^tA/K)^\vee \arrow[r] \arrow[d, "\psi_A"] & H^0({}^tA, \Omega^1_{{}^tA/K})^\vee \arrow[r] \arrow[d, "\psi_A^0"] & 0 \\ 0 \arrow[r] & H^0(A, \Omega^1_{A/K}) \arrow[r] & H^1_{\mathrm{dR}}(A/K) \arrow[r] & H^1(A, \mathscr{O}_A) \arrow[r] & 0 \end{tikzcd} \tag{7.2.3}$$

Proof. This is [6, Lemme 5.1.4]. ∎

Proposition 7.2.2. *The map ψ_A is compatible with duality in the following sense: for any morphism of abelian varieties $\varphi : A \to B$ with dual morphism ${}^t\varphi : {}^tB \to {}^tA$, the diagrams*

$$\begin{tikzcd} H^1_{\mathrm{dR}}(A/K)^\vee \arrow[r, "\psi_A^\vee"] \arrow[d, "(\varphi^*)^\vee"'] & H^1_{\mathrm{dR}}({}^tA/K) \arrow[d, "{}^t\varphi^*"] \\ H^1_{\mathrm{dR}}(B/K)^\vee \arrow[r, "\psi_B^\vee"'] & H^1_{\mathrm{dR}}({}^tB/K) \end{tikzcd} \tag{7.2.4}$$

and

$$\begin{tikzcd} H^1_{\mathrm{dR}}({}^tA/K)^\vee \arrow[r, "\psi_A"] & H^1_{\mathrm{dR}}(A/K) \\ H^1_{\mathrm{dR}}({}^tB/K)^\vee \arrow[r, "\psi_B"'] \arrow[u, "{}^t\varphi^{*\vee}"] & H^1_{\mathrm{dR}}(B/K) \arrow[u, "\varphi^*"'] \end{tikzcd} \tag{7.2.5}$$

commute. Moreover, if $j : A \to {}^{tt}A$ denotes the canonical isomorphism of A with its double dual, then the diagrams

$$\begin{tikzcd} H^1_{\mathrm{dR}}({}^tA/K)^\vee \arrow[r, "\psi_A"] \arrow[d, equal] & H^1_{\mathrm{dR}}(A/K) \\ H^1_{\mathrm{dR}}({}^tA/K)^\vee \arrow[r, "-\psi_{{}^tA}^\vee"'] & H^1_{\mathrm{dR}}({}^{tt}A/K) \arrow[u, "\simeq"', "j^*"] \end{tikzcd} \tag{7.2.6}$$

and

$$\begin{tikzcd} H^1({}^tA, \mathscr{O}_{{}^tA})^\vee \arrow[r, "\psi_A^1"] \arrow[d, equal] & H^0(A, \Omega^1_{A/K}) \\ H^1({}^tA, \mathscr{O}_{{}^tA})^\vee \arrow[r, "-(\psi_{{}^tA}^0)^\vee"'] & H^0({}^{tt}A, \Omega^1_{{}^{tt}A/K}) \arrow[u, "\simeq"', "j^*"] \end{tikzcd} \tag{7.2.7}$$

commute.

Proof. The first statement follows easily from the definition of ${}^t\varphi$ (see [6, 5.1.3.3]) while the second is [6, Lemme 5.1.5]. ■

We now have two different pairings between the vector spaces $H^1(A, \mathscr{O}_A) \simeq \mathrm{Lie}({}^tA)$ and $H^0({}^tA, \Omega^1_{{}^tA/K}) \simeq \omega_{{}^tA}$; one given by the pairing (7.1.6), and one given by the map ψ_A^0. We would like to know that they coincide.

Lemma 7.2.3. *Under the identifications of* (7.1.3) *and* (7.1.5)*, the maps* ψ_A^1 *and* ψ_A^0 *coincide with the natural maps arising from the canonical evaluation pairing (7.1.6).In other words, the diagram*

$$\begin{tikzcd} H^0({}^tA, \Omega^1_{{}^tA/K})^\vee \arrow[d, "\psi_A^0"'] & \omega_{{}^tA}^\vee \arrow[l, "(7.1.3)^\vee"', "\simeq"] \arrow[d] \\ H^1(A, \mathscr{O}_A) \arrow[r, "\simeq", "(7.1.5)"'] & \mathrm{Lie}({}^tA) \end{tikzcd} \tag{7.2.8}$$

commutes.

Proof. See the proof of Théorème 5.1.6 of [6]. ∎

Corollary 7.2.4. *The map of short exact sequences of K-vector spaces* (7.2.3) *is an isomorphism.*

Proof. This is Théorème 5.1.6 of [6]. To prove it, we note that the map ψ_A^0 is an isomorphism by the commutativity of (7.2.8) and the fact that the canonical map $\omega_{{}^tA}^\vee \to \mathrm{Lie}({}^tA)$ is an isomorphism. The commutativity of (7.2.7) then ensures that ψ_A^1 is an isomorphism, whence ψ_A is an isomorphism by (7.2.3). ∎

If the abelian variety A is equipped with a polarization

$$\varphi : A \longrightarrow {}^tA \tag{7.2.9}$$

corresponding to a line bundle $\mathscr{L}$ on $A \times A$, then we may use φ to define a map of short exact sequences of K-vector spaces

$$\begin{tikzcd} 0 \arrow[r] & H^1(A, \mathscr{O}_A)^\vee \arrow[r] \arrow[d, "(\varphi^*)^\vee"'] & H^1_{\mathrm{dR}}(A/K)^\vee \arrow[r] \arrow[d, "(\varphi^*)^\vee"] & H^0(A, \Omega^1_{A/K})^\vee \arrow[r] \arrow[d, "(\varphi^*)^\vee"] & 0 \\ 0 \arrow[r] & H^1({}^tA, \mathscr{O}_{{}^tA})^\vee \arrow[r] & H^1_{\mathrm{dR}}({}^tA/K)^\vee \arrow[r] & H^0({}^tA, \Omega^1_{{}^tA/K})^\vee \arrow[r] & 0 \end{tikzcd}$$

and hence, combining this with (7.2.3), a morphism

$$\begin{tikzcd} 0 \arrow[r] & H^1(A,\mathscr{O}_A)^\vee \arrow[r] \arrow[d, "\varphi_A^1"'] & H^1_{\mathrm{dR}}(A/K)^\vee \arrow[r] \arrow[d, "\varphi_A"] & H^0(A,\Omega^1_{A/K})^\vee \arrow[r] \arrow[d, "\varphi_A^0"] & 0 \\ 0 \arrow[r] & H^0(A,\Omega^1_{A/K}) \arrow[r] & H^1_{\mathrm{dR}}(A/K) \arrow[r] & H^1(A,\mathscr{O}_A) \arrow[r] & 0 \end{tikzcd} \tag{7.2.10}$$

Proposition 7.2.5. *The morphism of exact sequences* (7.2.10) *is the negative of its K-linear dual; in other words, the relations $\varphi_A^\vee = -\varphi_A$ and $(\varphi_A^1)^\vee = -\varphi_A^0$ hold.*

Proof. We form the diagram

$$\begin{tikzcd} H^1_{\mathrm{dR}}(A/K)^\vee \arrow[r, "-\psi_A^\vee"] \arrow[d, "(\varphi^*)^\vee"'] & H^1_{\mathrm{dR}}({}^tA/K) \arrow[d, "{}^t\varphi^*"] \\ H^1_{\mathrm{dR}}({}^tA/K)^\vee \arrow[r, "-\psi_{{}^tA}^\vee"] \arrow[d, "\psi_A"'] & H^1_{\mathrm{dR}}({}^{tt}A/K) \arrow[d, "\simeq"', "j^*"] \\ H^1_{\mathrm{dR}}(A/K) \arrow[r, equal] & H^1_{\mathrm{dR}}(A/K) \end{tikzcd}$$

with the indicated maps. Observe that the bottom square commutes by virtue of the commutativity of (7.2.6) in Proposition 7.2.2, and the top square commutes because of the commutativity of (7.2.4) in the same proposition. We claim that the composite map

$$j^* {}^t\varphi^* : H^1_{\mathrm{dR}}({}^tA/K) \to H^1_{\mathrm{dR}}(A/K)$$

is equal to φ^*. To prove this, it obviously suffices to show the equality

$${}^t\varphi \circ j = \varphi$$

as morphisms $A \to {}^tA$.

Such equality is an easy consequence of definitions, working with Poincaré bundles as follows. Let $\mathscr{P}_A$ and $\mathscr{P}_{{}^tA}$ denote the Poincaré bundles on $A \times {}^tA$ and ${}^tA \times {}^{tt}A$, respectively, let

$$s : {}^tA \times A \to A \times {}^tA$$

denote the morphism switching the factors, and recall that the polarization φ corresponds to an invertible sheaf $\mathscr{L}$ on $A \times A$. The morphisms $j : A \to {}^{tt}A$, ${}^t\varphi : {}^{tt}A \to {}^tA$, and $\varphi : A \to {}^tA$ are each uniquely characterized by

$$
\begin{aligned}
&(1 \times j)^*(\mathscr{P}_{{}^tA}) = s^*(\mathscr{P}_A) \\
&(1 \times {}^t\varphi)^*(\mathscr{P}_A) = (\varphi \times 1)^*(\mathscr{P}_{{}^tA}) \\
&(1 \times \varphi)^*(\mathscr{P}_A) = \mathscr{L}.
\end{aligned}
$$

We therefore find

$$
\begin{aligned}
(1 \times ({}^t\varphi \circ j))^*(\mathscr{P}_A) &= (1 \times j)^*(1 \times {}^t\varphi)^*(\mathscr{P}_A) \\
&= (1 \times j)^*(\varphi \times 1)^*(\mathscr{P}_{{}^tA}) \\
&= (\varphi \times 1)^*(1 \times j)^*(\mathscr{P}_{{}^tA}) \\
&= (\varphi \times 1)^* s^*(\mathscr{P}_A) \\
&= (1 \times \varphi)^*(\mathscr{P}_A) \\
&= \mathscr{L},
\end{aligned}
$$

from which we conclude that ${}^t\varphi \circ j = \varphi$, as desired. We thus deduce that the diagram

$$
\begin{tikzcd}
H^1_{\mathrm{dR}}(A/K)^\vee \arrow[r, equal] \arrow[d, "(\varphi^*)^\vee"'] & H^1_{\mathrm{dR}}(A/K)^\vee \arrow[d, "-\psi_A^\vee"] \\
H^1_{\mathrm{dR}}({}^tA/K)^\vee \arrow[d, "\psi_A"'] & H^1_{\mathrm{dR}}({}^tA/K) \arrow[d, "\varphi^*"] \\
H^1_{\mathrm{dR}}(A/K) \arrow[r, equal] & H^1_{\mathrm{dR}}(A/K)
\end{tikzcd}
$$

commutes, which shows that $\varphi_A^\vee = -\varphi_A$. The proof that $(\varphi_A^1)^\vee = -\varphi_A^0$ holds follows exactly in the same way, using (7.2.7) in place of (7.2.6). ■

7.3 Albanese and Picard functoriality

Consider a smooth, proper, and geometrically connected curve X over K, and suppose for the moment that $X(K) \neq \emptyset$. Later we will explain how to remove this hypothesis. Let $J := \mathrm{Pic}^0_{X/K}$ be the Jacobian of X; it is an abelian variety over K. Since $X(K) \neq \emptyset$, we may view J as representing isomorphism classes of line bundles of degree zero (cf. §8.1, Proposition 4 and §9.4, Proposition 4 of [9]). Moreover, due to the fact that J is the Jacobian of a proper smooth curve, there is a canonical principal polarization

(7.3.1) $$\varphi : J \xrightarrow{\simeq} {}^tJ$$

(defined by the Θ-divisor; see [9, p. 261]), enabling us to canonically identify J with its dual.

For any $P \in X(K)$, we define the *Albanese morphism* associated to P

$$a_P : X \to J$$

by the functorial recipe $x \mapsto \mathscr{O}(x) \otimes \mathscr{O}(P)^{-1}$. The morphism a_P is universal in the following sense: any morphism $\psi : X \to A$ from X to an abelian variety that takes P to zero uniquely factors through a_P. In particular, if $f : X \to X'$ is any morphism of smooth proper and geometrically connected curves, and J' denotes the Jacobian of X', then there is a unique homomorphism of abelian varieties $J \to J'$ making the diagram

(7.3.2) $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a_P"'] & X' \arrow[d, "a_{f(P)}"] \\ J \arrow[r] & J' \end{tikzcd}$$

commute. As any two Albanese maps differ by a translation on the Jacobian, it is easy to see that the map $J \to J'$ does not depend on the initial choice of point

P, and so by fppf descent exists as a map even when $X(K) = \emptyset$. We denote the resulting map $J \to J'$ by $\mathrm{Alb}(f)$. Observe that this makes J into a covariant functor of X; we will refer to this functorial interpretation of J as *Albanese functoriality*. Of course, since $\mathrm{Pic}^0_{X/K}$ is a contravariant functor of X, we also have a homomorphism $\mathrm{Pic}^0(f) : J' \to J$ associated to $f : X \to X'$ by *Picard functoriality*.

The morphisms $\mathrm{Alb}(f)$ and $\mathrm{Pic}^0(f)$ are related by duality in the following way.

Proposition 7.3.1. *Let $f : X \to X'$ be as above. Under the canonical identifications* (7.3.1) *of J and J' with their duals, the morphisms* $\mathrm{Alb}(f)$ *and* $\mathrm{Pic}^0(f)$ *are dual. Precisely, if $\varphi : J \to {}^tJ$ and $\varphi' : J' \to {}^tJ'$ are the canonical principal polarizations then the composite map*

$$J' \xrightarrow[\varphi']{\simeq} {}^tJ' \xrightarrow{{}^t\mathrm{Alb}(f)} {}^tJ \xleftarrow[\varphi]{\simeq} J \tag{7.3.3}$$

coincides with the map $\mathrm{Pic}^0(f) : J' \to J$.

Proof. The equality of two maps may be checked after any extension of the base field, so we may suppose that K is algebraically closed, so in particular we can choose $P \in X(K)$. By the definition of $\mathrm{Alb}(f)$, we have a commutative diagram as in (7.3.2). We apply the functor Pic^0 to this diagram to get an induced diagram

$$\begin{tikzcd} J & J' \arrow[l, "\mathrm{Pic}^0(f)"'] \\ {}^tJ \arrow[u, "i_P", "\simeq"'] & {}^tJ' \arrow[l, "{}^t\mathrm{Alb}(f)"] \arrow[u, "\simeq", "i_{f(P)}"'] \end{tikzcd} \tag{7.3.4}$$

Due to [38, Lemma 6.9], the maps i_P and $i_{f(P)}$ are inverse to the isomorphisms $-\varphi : J \to {}^tJ$ and $-\varphi' : J' \to {}^tJ'$ respectively, so we conclude at once from (7.3.4) that (7.3.3) coincides with $\mathrm{Pic}^0(f)$, as desired. ■

To lighten the notational burden, henceforth we denote by $H(X)$ (respectively $H(J)$) the Hodge filtration exact sequence of $H^1_{\mathrm{dR}}(X/K)$ (respectively $H^1_{\mathrm{dR}}(J/K)$).

Since the polarization (7.3.1) gives a canonical isomorphism $\varphi : J \xrightarrow{\simeq} {}^tJ$, we can identify the Hodge filtrations of $H^1_{\mathrm{dR}}(J/K)$ and $H^1_{\mathrm{dR}}({}^tJ/K)$ via φ^*. Moreover, we see from Corollary 7.2.4 that the natural morphism of short exact sequences

$$H(J)^\vee \xrightarrow{\varphi_J} H(J) \tag{7.3.5}$$

as in (7.2.10) is an isomorphism.

Corollary 7.3.2. *Keep the same notation as in Proposition* 7.3.1*. Under the identifications* (7.3.5) *of the Hodge filtrations of* $H^1_{\mathrm{dR}}(J/K)$ *and* $H^1_{\mathrm{dR}}(J'/K)$ *with their* K*-linear duals, the dual of the map* $\mathrm{Alb}(f)^*$ *on Hodge filtrations is the map* $\mathrm{Pic}^0(f)^*$*. That is, the composite morphism of exact sequences*

$$H(J) \xleftarrow[\varphi_J]{\simeq} H(J)^\vee \xrightarrow{(\mathrm{Alb}(f)^*)^\vee} H(J')^\vee \xrightarrow[\varphi_{J'}]{\simeq} H(J')$$

coincides with $\mathrm{Pic}^0(f)^*$*.*

Proof. The corollary follows immediately from Propositions 7.2.2 and 7.3.1. ∎

7.4 Relation of Hodge filtrations

As before, let X be a smooth proper geometrically connected curve over K and let J be the Jacobian of X. Suppose for the moment that $X(K) \neq \emptyset$, so a choice of K-rational point $P \in X(K)$ gives us a morphism

$$a_P : X \to J.$$

Via the natural pullback map on de Rham complexes (that respects the canonical filtrations giving rise to the Hodge to de Rham spectral sequences), we obtain a map of exact sequences

$$H(J) \xrightarrow{a_P^*} H(X)\,. \tag{7.4.1}$$

Lemma 7.4.1. *The map* (7.4.1) *is independent of the choice of section* $P \in X(K)$ *and commutes with extension on* K*. Consequently, this map exists over* K *even when* $X(K) = \emptyset$.

Proof. Compatibility with field extension (with P fixed) is clear. Since any two choices P, $P' \in X(K)$ yield maps a_P, a'_P that differ by a translation on J by a point in $J(K)$, it suffices to show that translation by $J(K)$ acts trivially on the Hodge filtration of J. This is easy to see: each of the terms of the Hodge filtration exact sequence $f_*\Omega^1_{J/K}$, $\mathbf{R}^1 f_*\Omega^\bullet_{J/K}$ and $R^1 f_*\mathscr{O}_X$ is a vector group whose formation commutes with any base change, so denoting any one of them by $\mathscr{E}$, the translation action of J gives a morphism

$$J \to \mathbf{Aut}_K(\mathscr{E})$$

that we wish to show is zero. Since the target is affine of finite type over the base and the source is an abelian scheme, this map factors through a section of the target [40, Proposition 6.1]; as it takes zero to zero, it must be the zero map. The second statement of the lemma follows at once by Galois descent. ∎

Since the map (7.4.1) is independent of P, we will simply denote it by a^*. We know that cup product followed by trace gives a functorial perfect pairing

$$(\cdot, \cdot)_X : H^1_{\mathrm{dR}}(X/K) \times H^1_{\mathrm{dR}}(X/K) \to K$$

as in (2.2.4) that extends to an identification (2.2.5) of the Hodge filtration of $H^1_{\mathrm{dR}}(X/K)$ with its K-linear dual (Proposition 2.2.1). Further, we have a natural isomorphism (7.3.5) of the Hodge filtration of $H^1_{\mathrm{dR}}(J/K)$ with its K-linear dual. Since the map (7.4.1) is canonical, it is reasonable to ask how it behaves with respect to these dualities.

Proposition 7.4.2. *With respect to the natural identifications of the Hodge filtrations of $H^1_{\mathrm{dR}}(X/K)$ and $H^1_{\mathrm{dR}}(J/K)$ with their K-linear duals, the map (7.4.1) is self-dual. More precisely, we have a commutative diagram of short exact sequences of K-vector spaces*

(7.4.2)
$$\begin{tikzcd} H(J) \arrow[d, "a^*"'] & H(J)^\vee \arrow[l, "\simeq"', "\varphi_J"] \\ H(X) \arrow[r, "\simeq", "(2.2.5)"'] & H(X)^\vee \arrow[u, "(a^*)^\vee"'] \end{tikzcd}$$

Proof. It suffices to prove that the diagram

(7.4.3)
$$\begin{tikzcd} H^1_{\mathrm{dR}}(J/K) \arrow[d, "a^*"'] & H^1_{\mathrm{dR}}(J/K)^\vee \arrow[l, "\simeq"', "\varphi_J"] \\ H^1_{\mathrm{dR}}(X/K) \arrow[r, "\simeq", "(2.2.5)"'] & H^1(X^1_{\mathrm{dR}})^\vee \arrow[u, "(a^*)^\vee"'] \end{tikzcd}$$

commutes. Let $(\cdot,\cdot)_X$ denote the natural perfect pairing (2.2.4) on the first de Rham cohomology of X given by cup product followed by the trace map. As before, denote by $(\cdot,\cdot)_J$ the perfect pairing (7.2.1) on $H^1_{\mathrm{dR}}(J/K)^\vee \times H^1_{\mathrm{dR}}({}^tJ/K)^\vee$. Via the principal polarization $\varphi : J \xrightarrow{\simeq} {}^tJ$, we get a perfect pairing

$$\langle\cdot,\cdot\rangle_J : H^1_{\mathrm{dR}}(J/K)^\vee \times H^1_{\mathrm{dR}}(J/K)^\vee \to K$$

given by $\langle f,g\rangle_J := (f,\varphi^*(g))$. By definition, the map φ_J takes $g \in H^1_{\mathrm{dR}}(J/K)^\vee$ to the element v_g of $H^1_{\mathrm{dR}}(J/K)$ characterized by the following property: for every $f \in H^1_{\mathrm{dR}}(J/K)^\vee$, the equality

(7.4.4)
$$f(v_g) = \langle f,g\rangle_J$$

holds. Since the map $H^1_{\mathrm{dR}}(X/K) \to H^1_{\mathrm{dR}}(X/K)^\vee$ is given by

$$c \mapsto (c,\cdot)_X,$$

we see that to show (7.4.3) commutes, it suffices to show that for all $f \in H^1_{\mathrm{dR}}(J/K)^\vee$, we have

$$(a^* v_f, a^*(\cdot))_X = f(\cdot).$$

Using that φ_J is an isomorphism by Corollary 7.2.4 (cf. (7.3.5)), evaluating this at $v_g \in H^1_{\mathrm{dR}}(J/K)$ and using (7.4.4), we must show that for all $f, g \in H^1_{\mathrm{dR}}(J)^\vee$ we have

$$(a^* v_f, a^* v_g)_X = \langle f, g \rangle_J.$$

This is proved in [12, Theorem 5.1]. ■

We can now prove our main result in this section, which gives an interpretation of the Hodge filtration of $H^1_{\mathrm{dR}}(X/K)$ and its functoriality in pullback and trace by finite morphisms of curves purely in terms of J and the Hodge filtration of $H^1_{\mathrm{dR}}(J/K)$.

Theorem 7.4.3. *The map* (7.4.1) *is an isomorphism. Moreover, for any finite map of smooth proper and geometrically connected curves $f : X \to X'$, the isomorphism* (7.4.1) *intertwines the maps* $\mathrm{Alb}(f)^*$ *and* f^* *and the maps* $\mathrm{Pic}^0(f)^*$ *and* f_*.

Proof. By passing to an extension of K if need be, we may choose a point $P \in X(K)$ and use it to define a map $a : X \to J$. Thanks to the functoriality of the canonical identification

$$H^1(X, \mathscr{O}_X) \xrightarrow{\simeq} \mathrm{Lie}(\mathrm{Pic}^0_{X/K})$$

we have a commutative diagram

$$\begin{array}{ccc} H^1(J, \mathscr{O}_J) & \xrightarrow{\simeq} & \mathrm{Lie}(\mathrm{Pic}^0_{J/K}) \\ \downarrow{\scriptstyle a^*} & & \downarrow{\scriptstyle \mathrm{Lie}(\mathrm{Pic}^0(a))} \\ H^1(X, \mathscr{O}_X) & \xrightarrow{\simeq} & \mathrm{Lie}(\mathrm{Pic}^0_{X/K}), \end{array} \tag{7.4.5}$$

(see [32, Proposition 1.3 (c)] and compare with Proposition 9.1.1 below). By [38, Lemma 6.9], the map $\mathrm{Pic}^0(a) : {}^tJ \to J$ is the inverse of the negative of the principal polarization $\varphi : J \to J'$, so in particular it is an isomorphism. We conclude from (7.4.5) that

$$a^* : H^1(J, \mathscr{O}_J) \to H^1(X, \mathscr{O}_X)$$

is an isomorphism. Taking K-duals and applying Proposition 7.4.2, we see that the map

$$a^* : H^0(J, \Omega^1_{J/K}) \to H^0(X, \Omega^1_{X/K})$$

is also an isomorphism. Since we now know that the flanking vertical maps in (7.4.1) are isomorphisms, we conclude that all three vertical maps are, as desired.

Now let $f : X \to X'$ be any finite morphism of smooth proper geometrically connected curves over K, and as usual set $J' = \mathrm{Pic}^0_{X'/K}$. To see that the map (7.4.1) intertwines f^* and $\mathrm{Alb}(f)^*$, we note that the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a_P"'] & X' \arrow[d, "a_{f(P)}"] \\ J \arrow[r, "\mathrm{Alb}(f)"'] & J' \end{tikzcd}$$

yields a commutative diagram of short exact sequences of K-vector spaces

(7.4.6)
$$\begin{tikzcd} H(J') \arrow[r, "\mathrm{Alb}(f)^*"] \arrow[d, "a^*_{f(P)}"'] & H(J) \arrow[d, "a^*_P"] \\ H(X') \arrow[r, "f^*"'] & H(X) \end{tikzcd}$$

which gives the desired result. As for the compatibility of $\mathrm{Pic}^0(f)^*$ and f_*, we dualize (7.4.6) and use the identification of Proposition 7.4.2 to get a commutative diagram

$$\begin{tikzcd} H(J) \arrow[d, "a^*_P"'] & H(J)^\vee \arrow[l, "\simeq"'] \arrow[r, "(\mathrm{Alb}(f)^*)^\vee"] & H(J')^\vee \arrow[r, "\simeq"] & H(J') \arrow[d, "a^*_{f(P)}"] \\ H(X) \arrow[r, "\simeq"] & H(X)^\vee \arrow[u, "(a^*_P)^\vee"'] \arrow[r, "(f^*)^\vee"'] & H(X')^\vee \arrow[u, "(a^*_{f(P)})^\vee"'] & H(X') \arrow[l, "\simeq"'] \end{tikzcd}$$

Corollary 7.3.2 ensures that the top composite map coincides with $\mathrm{Pic}^0(f)^*$ while Theorem 3.4.5 (3) says that the bottom composite map agrees with f_*. This gives what we want. ■

Remark 7.4.4. When K is the fraction field of a discrete valuation ring R, the interpretation of the Hodge filtration of $H^1_{\mathrm{dR}}(X/K)$ in terms of J given by Theorem 7.4.3 forms the cornerstone of our later work using Néron models to equip the Hodge filtration of $H^1_{\mathrm{dR}}(X/K)$ with a canonical integral structure (without requiring that X have an admissible model over R!)

If T is a correspondence on X consisting of a pair of finite morphisms $\pi_1, \pi_2 : Y \rightrightarrows X$ with Y a smooth and proper geometrically connected curve, then we have seen in chapter II that T induces an endomorphism of the Hodge filtration of $H^1_{\mathrm{dR}}(X/K)$ via $(\pi_1)_* \circ \pi_2^*$. Observe that T induces an endomorphism of J via Albanese and Picard functoriality:

$$\mathrm{Alb}(\pi_2) \circ \mathrm{Pic}^0(\pi_1) : J \to J; \tag{7.4.7}$$

by abuse of notation, we will denote this endomorphism by T. Since the Hodge filtration $H(J)$ of $H^1_{\mathrm{dR}}(J/K)$ is functorial in J, we get an endomorphism of this short exact sequence that we again denote by T.

Corollary 7.4.5. *The isomorphism* (7.4.1) *is T-equivariant.*

Proof. Just observe that by definition, T acts on the Hodge filtration of $H^1_{\mathrm{dR}}(J/K)$ as

$$(\mathrm{Alb}(\pi_2) \circ \mathrm{Pic}^0(\pi_1))^* = \mathrm{Pic}(\pi_1)^* \circ \mathrm{Alb}(\pi_2)^*,$$

which, thanks to Theorem 7.4.3, is intertwined by (7.4.1) with $(\pi_1)_* \circ \pi_2^* = T$ on the Hodge filtration of $H^1_{\mathrm{dR}}(X/K)$. ■

CHAPTER VIII

Integral structures and Abelian varieties

Suppose R is a discrete valuation ring with field of fractions K. Let A_K be an abelian variety over K and let A/R be the Néron model of A_K [9]. In this section we will use A to equip the Hodge filtration (7.1.1) of $H^1_{\mathrm{dR}}(A_K/K)$ with a canonical integral structure: a short exact sequence of finite free R-modules that recovers (7.1.1) after extending scalars to K that is moreover functorial in A_K. When A is an abelian scheme, the finite free R-module $H^1_{\mathrm{dR}}(A/R)$ (together with its corresponding "integral" Hodge filtration) gives the desired integral structure, as this lattice is functorial in A_K by the Néron mapping property. When A is not an abelian scheme, the R-module $H^1_{\mathrm{dR}}(A/R)$ is of no use, as it is usually not even finitely generated over R (as A is not proper). In order to find a suitable replacement for the de Rham cohomology of A over R, we will use the theory of rigidified extensions as developed in [35]. Specifically, we show that there exists a smooth group scheme $\mathscr{E}xtrig(A, \mathbf{G}_m)$ over R whose Lie algebra (over R) $\mathrm{Lie}\,\mathscr{E}xtrig(A, \mathbf{G}_m)$ yields a functorial R-lattice in the finite dimensional K-vector space $H^1_{\mathrm{dR}}(A_K/K)$. When A is an abelian scheme, this R-lattice will be canonically isomorphic to $H^1_{\mathrm{dR}}(A/R)$.

When A_K is the Jacobian of a smooth proper curve X_K, the identification of Hodge filtrations in (7.4.1) allows us to use this integral structure on the Hodge filtration of A_K to equip the Hodge filtration of X_K with an integral structure that will moreover be functorial in finite morphisms of X_K (via Picard and Albanese functoriality of A_K). This gives a different construction of the type of integral structure considered in §III that does not require X_K to have an admissible model. When X_K does have an admissible model, we will prove under additional mild hypotheses that these two integral structures on the Hodge filtration are canonically identified via the isomorphism (7.4.1).

8.1 Lie algebras

We begin by recalling some standard facts about Lie algebras of arbitrary commutative group functors that will be of use to us in the sequel. We refer to [32, §1] as a concise source on these matters.

For any scheme T, we define $T[\epsilon] = T \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}(\mathbf{Z}[\epsilon]/(\epsilon^2))$; the projection $\rho : T[\epsilon] \to T$ induced by the inclusion $\mathbf{Z} \to \mathbf{Z}[\epsilon]/(\epsilon^2)$ is finite and faithfully flat with a canonical section $\iota : T \to T[\epsilon]$ induced by the ring map $\mathbf{Z}[\epsilon]/(\epsilon^2) \to \mathbf{Z}$ which sends ϵ to 0.

Fix a base scheme S and let G be a group functor on the category of flat S-schemes. The group functor $\mathscr{L}ie(G)$ on flat S-schemes T is defined by

$$\mathscr{L}ie(G)(T) = \ker\left(G(T[\epsilon]) \to G(T)\right). \tag{8.1.1}$$

Any section t of $\mathscr{O}_T$ induces an $\mathscr{O}_T$-linear endomorphism of $\mathscr{O}_T[\epsilon]/(\epsilon^2)$ by $\epsilon \mapsto t \cdot \epsilon$. This gives a morphism

$$u_t : T[\epsilon] \to T[\epsilon] \tag{8.1.2}$$

which satisfies $u_t \circ \iota = \iota$ and $\rho \circ u_t = \rho$, and hence makes $\mathscr{L}ie(G)$ into an $\mathscr{O}_S$-module. Clearly $\mathscr{L}ie(G)$ is a covariant functor of G; moreover it is easy to check that if G is a sheaf for some topology on flat S-schemes, so is $\mathscr{L}ie(G)$. If G is represented by a smooth S-group, then $\mathscr{L}ie(G)$ agrees with the traditional relative Lie algebra (as a sheaf of modules). We set $\mathrm{Lie}(G) = \mathscr{L}ie(G)(S)$.

8.2 Canonical extensions

Let us now recall the theory of rigidified extensions, and apply it to our situation of interest to construct the desired integral structure on the Hodge filtration of an abelian variety over K.

Fix a base scheme S and a pair of commutative S-group schemes F and G. A *rigidified extension* of F by G is an extension (of fppf sheaves of abelian groups) of F by G endowed with a section of S-pointed S-schemes along the first infinitesimal neighborhood of the identity $\mathrm{Inf}^1_S(F)$ of F:

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\ & & & \mathrm{Inf}^1_S(F) \arrow[ul] \arrow[u] & \end{tikzcd}$$

A morphism of rigidified extensions is a morphism of extensions respecting the rigidifications. We denote by $\mathrm{Extrig}(F, G)$ the set of isomorphism classes of rigidified extensions of F by G. This set is naturally equipped with the structure of a commutative group via Baer sum of underlying extensions and “Baer sum” of rigidifications [35] I, §2. This structure makes the functor on S-schemes

$$T \mapsto \mathrm{Extrig}(F_T, G_T)$$

a group functor which is contravariant in the first variable via pullback (fibered product) of rigidified extensions, and covariant in the second via pushout (fibered coproduct).

Now specialize to the case that S is a connected Dedekind scheme and suppose that A/S is the Néron model of an abelian scheme A_U/U over some dense open $U \subset S$. Denote by ${}^tA$ the Néron model of the dual abelian scheme ${}^tA_U$, and by ${}^tA^0$ the relative identity component of ${}^tA$; this is unaffected by shrinking U and is functorial in the generic fiber of A. We write ω_A for the vector group associated to the relative cotangent space along the identity of A (i.e. the vector group of invariant differentials as in (7.1.2)), and $\mathscr{E}xtrig(A, \mathbf{G}_m)$ for the fppf sheaf of abelian groups associated to the presheaf $T \mapsto \mathrm{Extrig}(A_T, \mathbf{G}_{mT})$ on the category of S-schemes.

We will need:

Lemma 8.2.1. *There is a natural isomorphism of fppf abelian sheaves on the category of smooth S-schemes*

$$\mathscr{E}xt(A, \mathbf{G}_m) \xrightarrow{\simeq} {}^tA^0 . \tag{8.2.1}$$

Proof. This follows from Proposition C.14 and III, Theorem 2.5 in [37]. ∎

Proposition 8.2.2. *The fppf sheaf $\mathscr{E}xtrig(A, \mathbf{G}_m)$ on the category of smooth S-schemes is represented by a smooth and separated S-group scheme (that we also denote $\mathscr{E}xtrig(A, \mathbf{G}_m)$ by a slight abuse of notation). Moreover, there is a natural exact sequence of smooth group schemes over S*

$$0 \longrightarrow \omega_A \longrightarrow \mathscr{E}xtrig(A, \mathbf{G}_m) \longrightarrow {}^tA^0 \longrightarrow 0 . \tag{8.2.2}$$

Proof. One first shows, exactly as in [35, I, §2.6] that there is an exact sequence of fppf sheaves on the category of smooth S-schemes

$$0 \longrightarrow \omega_A \longrightarrow \mathscr{E}xtrig(A, \mathbf{G}_m) \longrightarrow \mathscr{E}xt(A, \mathbf{G}_m) \longrightarrow 0 . \tag{8.2.3}$$

Since ω_A is a vector group, it is represented by a smooth affine group scheme, and Lemma 8.2.1 shows that $\mathscr{E}xt(A, \mathbf{G}_m)$ is represented by the smooth group scheme ${}^tA^0$

on the category of smooth S-schemes. The proof of [42], Proposition 17.4, which carries over from the situation considered there (fpqc topology on all S-schemes) to our situation (fppf topology on smooth S-schemes) since ω_A and ${}^tA^0$ are smooth, shows (via fppf descent) that the middle term in (8.2.3) is represented on the category of smooth S-schemes by a smooth group scheme (which we denote again by $\mathscr{E}xtrig(A, \mathbf{G}_m)$), and that there is an exact sequence (8.2.2) of smooth S-group schemes. The group scheme $\mathscr{E}xtrig(A, \mathbf{G}_m)$ is separated because it is affine over the separated ${}^tA^0$ (due to ω_A being S-affine). ∎

Warning 8.2.3. Although the fppf abelian sheaf $\mathscr{E}xtrig(A, \mathbf{G}_m)$ is represented by the smooth S-group scheme $\mathscr{E}xtrig(A, \mathbf{G}_m)$ on the category of smooth S-schemes, we cannot conclude that it is represented by this group scheme on any larger category of S-schemes.

Remark 8.2.4. Since ω_A is a vector group, there is a canonical isomorphism

$$\mathscr{L}ie(\omega_A) \simeq \omega_A,$$

and we will henceforth implicitly make this identification. Further, over any affine base S, the sheaf of $\mathscr{O}_S$-modules ω_A is the sheaf associated to the $\mathscr{O}_S(S)$-module $\Gamma(S, \omega_A)$ in the usual way; therefore, to keep notation economical in this situation, we will often simply write ω_A for $\Gamma(S, \omega_A)$ when it is clear from context which we mean.

Corollary 8.2.5. *With the same notation as Proposition* 8.2.2*, suppose that* $S = \operatorname{Spec} B$ *is affine and set* $L = \operatorname{Frac}(B)$*. Then there is an exact sequence of finite locally free* B*-modules*

$$0 \longrightarrow \omega_A \longrightarrow \operatorname{Lie} \mathscr{E}xtrig(A, \mathbf{G}_m) \longrightarrow \operatorname{Lie}({}^tA^0) \longrightarrow 0 \tag{8.2.4}$$

that is functorial in A_L.

Proof. Note that we are making the identifications of Remark 8.2.4. It is easy to see that the functor Lie is always left exact. Moreover, since ω_A is a smooth group, the map of smooth group schemes

$$\mathscr{E}xtrig(A, \mathbf{G}_m) \to {}^tA^0$$

is smooth, so the resulting map on Lie algebras is surjective (see [32, Proposition 1.1, (c)]). The functoriality of Lie and of the exact sequence (8.2.2), together with the Néron mapping property, show that (8.2.4) is functorial in A_L. ∎

Remark 8.2.6. If A is an abelian scheme over an arbitrary base scheme S, then one still has a natural isomorphism of fppf abelian sheaves on the category of smooth S-schemes

$$\mathscr{E}xt(A, \mathbf{G}_m) \xrightarrow{\simeq} {}^tA$$

as in (8.2.1), and hence an exact sequence of smooth groups schemes

$$0 \longrightarrow \omega_A \longrightarrow \mathscr{E}xtrig(A, \mathbf{G}_m) \longrightarrow {}^tA \longrightarrow 0 \tag{8.2.5}$$

over S as in (8.2.2).

When A/S is an abelian scheme, the exact sequence (8.2.4) is a familiar one (cf. [27, 5.6.2]):

Proposition 8.2.7. *Let $S = \operatorname{Spec} B$ be any affine scheme and A/S an abelian scheme. Then there is a canonical isomorphism of exact sequences of finite locally free B-modules*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_A & \longrightarrow & \operatorname{Lie}\mathscr{E}xtrig(A, \mathbf{G}_m) & \longrightarrow & \operatorname{Lie}({}^tA) & \longrightarrow & 0 \\ & & \simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq & & \\ 0 & \longrightarrow & H^0(A, \Omega^1_{A/S}) & \longrightarrow & H^1_{\mathrm{dR}}(A/S) & \longrightarrow & H^1(A, \mathscr{O}_A) & \longrightarrow & 0 \end{array} \tag{8.2.6}$$

which commutes with any affine base change on S. Here, the top row is the exact sequence of Lie algebras associated to (8.2.5) *and the bottom row is the Hodge filtration of $H^1_{\mathrm{dR}}(A/S)$.*

Proof. This follows from Chapter 1 of [35], especially 3.2.3 (a), 4.1.7, and 4.2.1. ∎

Corollary 8.2.8. *Let $S = \operatorname{Spec} B$ be a connected affine Dedekind scheme and A_L an abelian variety over the fraction field L of B with Néron model A over S. Then the exact sequence of finite locally free B-modules* (8.2.4) *yields a canonical integral structure on the Hodge filtration of $H^1_{\mathrm{dR}}(A_L/L)$ that is functorial in A_L.*

Proof. Due to Corollary 8.2.5, it suffices to show that tensoring (8.2.4) over B with L recovers the Hodge filtration of $H^1_{\mathrm{dR}}(A_L/L)$. Using the fact that the formation of (8.2.4) is compatible with Zariski localization on S, we find that tensoring (8.2.4) with L yields the exact sequence of L-vector spaces

$$0 \longrightarrow \omega_{A_L} \longrightarrow \operatorname{Lie} \mathscr{E}xtrig(A_L, \mathbf{G}_m) \longrightarrow \operatorname{Lie}({}^tA_L) \longrightarrow 0\ ,$$

which by Proposition 8.2.7 is canonically isomorphic to the Hodge filtration of $H^1_{\mathrm{dR}}(A_L/L)$. We conclude that (8.2.4) yields the desired integral structure. ∎

8.3 Comparison of integral structures

Let $S = \operatorname{Spec} R$ be the spectrum of a discrete valuation ring with field of fractions K and residue field k. Fix an admissible curve $f : X \to S$. If we view X as a model of its generic fiber then we can interpret the exact sequence (3.4.1) as endowing the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$ with an integral structure that is functorial with respect to finite morphisms in X_K (see Theorem 3.4.5).

Let J_K be the Jacobian of X_K and J the Néron model of J_K over S. Applying Corollary 8.2.8 to J_K, we obtain an integral structure on the Hodge filtration of $H^1_{\mathrm{dR}}(J_K/K)$ that is functorial in J_K. The identification of Hodge filtrations of $H^1_{\mathrm{dR}}(X_K/K)$ and $H^1_{\mathrm{dR}}(J_K/K)$ in Theorem 7.4.1 raises an important question: what is the relationship between the two integral structures we have constructed?

Theorem 8.3.1. *With notation as above, suppose further that the closed fiber X_k is generically smooth. Then there exists a unique isomorphism of exact sequences of free R-modules*

$$\begin{tikzcd} 0 \arrow[r] & \omega_J \arrow[r] \arrow[d] & \mathrm{Lie}\,\mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] \arrow[d] & \mathrm{Lie}({}^tJ^0) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(X, \omega_{X/S}) \arrow[r] & H^1(X) \arrow[r] & H^1(X, \mathscr{O}_X) \arrow[r] & 0 \end{tikzcd} \tag{8.3.1}$$

which recovers the one constructed in Theorem 7.4.1 *after tensoring over R with K and using the identification of Proposition* 8.2.7

Remark 8.3.2. Let T be any correspondence on X_K. We have seen in §6.2 (cf. Theorem 6.2.1) that T acts as an endomorphism of the bottom row of (8.3.1). Furthermore, T acts as an endomorphism of J_K by Albanese and Picard functoriality as in §VII, and so by the functoriality aspect of Corollary 8.2.8 we see that T acts as an endomorphism of the top row of (8.3.1). The fact that the morphism of exact sequences in (8.3.1) is T-equivariant is guaranteed by Theorem 7.4.3, as such compatibility may be checked after extending scalars to K (since the exact sequences of (8.3.1) consist of free R-modules).

As the proof of Theorem 8.3.1 is rather involved, it will occupy the next section. In the remainder of this section, we will outline the proof.

Since the exact sequence (8.2.4) is obtained from the exact sequence of smooth groups (8.2.2) by applying the functor Lie, we first construct an exact sequence of group functors on S-schemes whose associated sequence of Lie algebras is the canonical sequence

$$0 \longrightarrow H^0(X, \omega_{X/S}) \longrightarrow H^1(X) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow 0\,, \tag{8.3.2}$$

from Proposition 3.4.3, so in particular this recovers the bottom row of (8.3.1).

To do this, for any S-scheme T we introduce the notion of a "regular connection" on a line bundle $\mathscr{L}$ on X_T in Definition 9.2.1: this is an $\mathscr{O}_T$-linear morphism $\nabla : \mathscr{L} \to \mathscr{L} \otimes \omega_{X_T/T}$ satisfying the Leibnitz rule with respect to the base change of the $\mathscr{O}_S$-linear map $d : \mathscr{O}_X \to \omega_{X/S}$ of Theorem 3.4.1. We then study the fppf sheaf on S-schemes $\mathrm{Pic}^{\natural}_{X/S}$ associated to the presheaf on S-schemes T given by

$$T \mapsto \left\{ \begin{array}{l} \text{The group of isomorphism classes of} \\ \text{invertible sheaves on } X_T \text{ equipped} \\ \text{with a regular connection} \end{array} \right\}.$$

In particular, we show that there is an exact sequence of fppf sheaves of abelian groups on S-schemes

$$0 \longrightarrow f_*\omega_{X/S} \longrightarrow \mathrm{Pic}^{\natural}_{X/S} \longrightarrow \mathrm{Pic}_{X/S}\,, \tag{8.3.3}$$

and by using Čech theory to describe these functors hypercohomologically we show that the functor Lie applied to (8.3.3) yields the exact sequence (8.3.2).

Our task is then to define a morphism of exact sequences of group sheaves for the fppf topology on smooth S-schemes

$$\begin{tikzcd} 0 \arrow[r] & \omega_J \arrow[r] \arrow[d] & \mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] \arrow[d] & \mathscr{E}xt(J, \mathbf{G}_m) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & f_*\omega_{X/S} \arrow[r] & \mathrm{Pic}^{\natural}_{X/S} \arrow[r] & \mathrm{Pic}_{X/S} & \end{tikzcd} \tag{8.3.4}$$

We do this as follows. Let $i : U \hookrightarrow X$ denote the S-smooth locus of X, and suppose for now that there is a section $x \in X_K(K)$ (such a section always exists after passing to a finite unramified extension of K since X_k has nonempty smooth locus). Then x furnishes a morphism $j_K : X_K \to J_K$ which extends to a morphism $j : U \to J$ by the Néron mapping property. For any smooth S-scheme T, we prove that an element of $\mathrm{Extrig}(J_T, \mathbf{G}_{mT})$ gives rise to an invertible sheaf $\mathscr{M}$ on J_T with connection, and hence, by pullback of this data along $j_T : U_T \to J_T$, an invertible sheaf on U_T with connection. Now we critically use our hypothesis that X_k is generically smooth: indeed, this ensures that the complement of U_T in the normal scheme X_T has codimension at least two, which enables us to show that the data of invertible sheaf with connection on U_T is equivalent to the data of an invertible sheaf on X_T with regular connection. In this way, we obtain the desired map of fppf sheaves on smooth S-schemes $\mathscr{E}xtrig(J, \mathbf{G}_m) \to \mathrm{Pic}^{\natural}_{X/S}$. By repeating this procedure without the rigidification, we easily define a compatible map $\mathscr{E}xt(J, \mathbf{G}_m) \to \mathrm{Pic}_{X/S}$. The left vertical map $\omega_J \to f_*\omega_{X/S}$ in (8.3.1) is similarly defined by pullback of differentials along $j : U \to J$, using that the natural map $\omega_{X/S} \to i_*\Omega^1_{U/S}$ is an isomorphism (cf. Lemma 3.3.7).

Since the exact sequence (8.2.3) of fppf abelian sheaves on the category of smooth S-schemes coincides with the exact sequence (8.2.2) of smooth group schemes on the category of smooth S-schemes, we get a morphism of exact sequences of fppf sheaves on smooth S-schemes

(8.3.5)
$$\begin{tikzcd}
0 \arrow[r] & \omega_J \arrow[r] \arrow[d] & \mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] \arrow[d] & {}^tA^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\omega_{X/S} \arrow[r] & \mathrm{Pic}^{\natural}_{X/S} \arrow[r] & \mathrm{Pic}_{X/S} &
\end{tikzcd}$$

Now for any smooth group scheme G and fppf abelian sheaf $\mathscr{F}$, the natural map "restriction to the smooth site"

$$\mathrm{Hom}_S(G, \mathscr{F}) \to \mathrm{Hom}_{S_{\mathrm{sm}}}(G, \mathscr{F})$$

is bijective thanks to Yoneda's lemma (with homomorphisms as fppf sheaves of abelian groups). In particular, since the terms in the top row of (8.3.5) are all smooth group schemes, the morphism (8.3.5) of fppf sheaves on smooth S-schemes induces a corresponding morphism of fppf sheaves on all S-schemes, and hence a map (8.3.1) on Lie algebras over S (as dual number points).

To show that the resulting morphism (8.3.1) is an isomorphism, we use work of Raynaud relating $\mathrm{Pic}^0_{X/S}$ and J^0 to show that the right vertical map in (8.3.1) is an isomorphism. It will then follow from duality theory that the left vertical map is also an isomorphism, and thus the middle map is an isomorphism too. The fact that the resulting isomorphism recovers the one in Theorem 7.4.1 after extending scalars to K and using the identification of Proposition (8.2.7) amounts to the fact that (8.3.4) is defined by pullback along the map j defined by the section x.

CHAPTER IX

Proof of Theorem 8.3.1

In this chapter, we prove Theorem 8.3.1. We keep the notation from chapter 8.3.

9.1 Interpreting the Hodge flitration in terms of Lie algebras

Our first task is to construct an exact sequence of group functors on S-schemes whose associated sequence of Lie algebras is (8.3.2).

Let T be any S-scheme and denote by $f_T : X_T \to T$ the base change of $f : X \to S$. We know by Theorem 3.4.1 that the map $d : \mathscr{O}_{X_K} \to \Omega^1_{X_K/K}$ restricts to an $\mathscr{O}_S$-linear map $d : \mathscr{O}_X \to \omega_{X/S}$, and since the formation of these sheaves and the map d all commute with base change on S, we obtain a complex $d : \mathscr{O}_{X_T} \to \omega_{X_T/T}$ over $\mathscr{O}_T$ that we denote $\omega^\bullet_{X_T/T}$. We similarly have a complex $\omega^{\times,\bullet}_{X_T/T}$ given by the map $d\log : \mathscr{O}^\times_{X_T} \to \omega_{X_T/T}$, which carries a section f of $\mathscr{O}^\times_{X_T}$ to $f^{-1} \cdot df$. The evident filtration of $\omega^{\times,\bullet}_{X_T/T}$ gives rise to a canonical short exact sequence of (vertical) complexes

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathscr{O}^\times_{X_T} \arrow[r, "\mathrm{id}"] \arrow[d, "d\log"] & \mathscr{O}^\times_{X_T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega_{X_T/T} \arrow[r, "\mathrm{id}"] & \omega_{X_T/T} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

which we denote

$$0 \longrightarrow \tau_{\geq 1}(\omega^\bullet_{X_T/T}) \longrightarrow \omega^{\times,\bullet}_{X_T/T} \longrightarrow \mathscr{O}^{\times,\bullet}_{X_T} \longrightarrow 0\,. \tag{9.1.1}$$

Since X is cohomologically flat in dimension zero, the natural map $\mathscr{O}_T \to f_{T*}\mathscr{O}_{X_T}$ is an isomorphism (as this holds over S), so the map $d : f_{T*}\mathscr{O}_{X_T} \to f_{T*}\omega_{X_T/T}$ is zero. Thus the "boundary map" $d\log : f_{T*}\mathscr{O}_{X_T}^{\times} \to f_{T*}\omega_{X_T/T}$ is zero because f_{T*} is left exact (so $f_{T*}\mathscr{O}_{X_T}^{\times}$ is a subsheaf of $f_{T*}\mathscr{O}_{X_T}$). It follows that applying the functor $\mathbf{R}^1 f_{T*}$ to the exact sequence (9.1.1) yields the exact sequence

$$0 \longrightarrow f_{T*}\omega_{X_T/T} \longrightarrow \mathbf{R}^1 f_{T*}\omega_{X_T/T}^{\times,\bullet} \longrightarrow R^1 f_{T*}\mathscr{O}_{X_T}^{\times}\ . \tag{9.1.2}$$

We abuse notation by letting $f_*\omega_{X/S}$, $\mathbf{R}^1 f_*\omega_{X/S}^{\times,\bullet}$ and $R^1 f_*\mathscr{O}_X^{\times}$ respectively denote the group functors (visibly fppf sheaves) on S-schemes T given by

$$T \mapsto f_{T*}\omega_{X_T/T} \tag{9.1.3}$$

$$T \mapsto \mathbf{R}^1 f_{T*}\omega_{X_T/T}^{\times,\bullet} \tag{9.1.4}$$

$$T \mapsto R^1 f_{T*}\mathscr{O}_{X_T}^{\times}, \tag{9.1.5}$$

so (9.1.2) gives an exact sequence of fppf sheaves on S-schemes

$$0 \longrightarrow f_*\omega_{X/S} \longrightarrow \mathbf{R}^1 f_*\omega_{X/S}^{\times,\bullet} \longrightarrow R^1 f_*\mathscr{O}_X^{\times}\ . \tag{9.1.6}$$

Proposition 9.1.1. *There is a canonical isomorphism of exact sequences of free R-modules*

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\omega_{X/S}) \arrow[r] \arrow[d, "\simeq"] & H^1(X) \arrow[r] \arrow[d, "\simeq"] & H^1(X,\mathscr{O}_X) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathrm{Lie}(f_*\omega_{X/S}) \arrow[r] & \mathrm{Lie}(\mathbf{R}^1 f_*\omega_{X/S}^{\times,\bullet}) \arrow[r] & \mathrm{Lie}(R^1 f_*\mathscr{O}_X^{\times}) &
\end{tikzcd} \tag{9.1.7}$$

where the top row is the exact sequence (3.4.1) *and the bottom row is the exact sequence of Lie algebras attached to* (9.1.6).

Proof. Recall that the exact sequence (3.4.1) results from the Hodge to de Rham spectral sequence attached to evident filtration of $\omega_{X/S}^{\bullet}$; cf. the proof of Proposition

3.4.3. Or, what is the same thing, the exact sequence (3.4.1) results from applying the functor $\mathbf{R}^1 f_*$ to the exact sequence of complexes

$$0 \longrightarrow \tau_{\geq 1}(\omega^\bullet_{X/S}) \longrightarrow \omega^\bullet_{X/S} \longrightarrow \mathscr{O}^\bullet_X \longrightarrow 0$$

defined by

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathscr{O}_X \arrow[r, "\mathrm{id}"] \arrow[d, "d"] & \mathscr{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega_{X/S} \arrow[r, "\mathrm{id}"] & \omega_{X/S} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

We have a canonically split exact sequence of filtered complexes

$$\begin{tikzcd}
0 \arrow[r] & \omega^\bullet_{X/S} \arrow[r] & \omega^{\times,\bullet}_{X_{S[\epsilon]}/S[\epsilon]} \arrow[r, shift left] & \omega^{\times,\bullet}_{X/S} \arrow[l, shift left] \arrow[r] & 0
\end{tikzcd} \tag{9.1.8}$$

defined by:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X \arrow[r, "h\mapsto 1+\epsilon h"] \arrow[d, "d"] & \mathscr{O}^\times_{X_{S[\epsilon]}} \arrow[r, shift left, "\iota^*"] \arrow[d, "d\log"] & \mathscr{O}^\times_X \arrow[l, shift left, "\rho^*"] \arrow[r] \arrow[d, "d\log"] & 1 \\
0 \arrow[r] & \omega_{X/S} \arrow[r, "\cdot\epsilon"'] & \omega_{X_{S[\epsilon]}/S[\epsilon]} \arrow[r, shift left, "\iota^*"] & \omega_{X/S} \arrow[l, shift left, "\rho^*"] \arrow[r] & 0
\end{tikzcd}$$

i.e. we have a commutative diagram of complexes with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tau_{\geq 1}(\omega^\bullet_{X/S}) \arrow[r] \arrow[d] & \tau_{\geq 1}(\omega^{\times,\bullet}_{X_{S[\epsilon]}/S[\epsilon]}) \arrow[r, shift left] \arrow[d] & \tau_{\geq 1}(\omega^\bullet_{X/S}) \arrow[l, shift left] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega^\bullet_{X/S} \arrow[r] \arrow[d] & \omega^{\times,\bullet}_{X_{S[\epsilon]}/S[\epsilon]} \arrow[r, shift left] \arrow[d] & \omega^\bullet_{X/S} \arrow[l, shift left] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}^\bullet_X \arrow[r] \arrow[d] & \mathscr{O}^{\times,\bullet}_{X_{S[\epsilon]}} \arrow[r, shift left] \arrow[d] & \mathscr{O}^{\times,\bullet}_X \arrow[l, shift left] \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{9.1.9}$$

We apply the functor $\mathbf{R}^1 f_*$ to (9.1.9) to obtain the commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & f_*\omega_{X/S} \arrow[r] \arrow[d] & f_*\omega_{X_{S[\epsilon]}/S[\epsilon]} \arrow[r] \arrow[d] & f_*\omega_{X/S} \arrow[d] \\
0 \arrow[r] & \mathbf{R}^1 f_*\omega^\bullet_{X/S} \arrow[r] \arrow[d] & \mathbf{R}^1 f_*\omega^{\times,\bullet}_{X_{S[\epsilon]}/S[\epsilon]} \arrow[r] \arrow[d] & \mathbf{R}^1 f_*\omega^\bullet_{X/S} \arrow[d] \\
0 \arrow[r] & R^1 f_*\mathscr{O}_X \arrow[r] & R^1 f_*\mathscr{O}^\times_{X_{S[\epsilon]}} \arrow[r] & R^1 f_*\mathscr{O}^\times_X
\end{tikzcd}
\tag{9.1.10}
$$

where the zeroes in the left column result from the splitting (i.e. $\mathbf{R}^1 f_*$ is left exact on *split* short exact sequences). We conclude from (8.1.1) and (9.1.3)–(9.1.5) that we have an isomorphism of exact sequences of abelian groups as in (9.1.7). It remains to show that this is in fact an R-linear isomorphism. Recall from §8.1 that for any group functor G on S-schemes, the multiplication on $\mathrm{Lie}(G)$ by $\mathscr{O}_S(S)$ is induced by the functoriality of G from the map $\mathscr{O}_S(S) \to \mathrm{End}_S(S[\epsilon])$ sending $s \in \mathscr{O}_S(S)$ to the map u_s of (8.1.2). Thus, the fact that the map (9.1.7) defined by (9.1.10) is a map of R-modules amounts to the assertion that for any $s \in \mathscr{O}_S(S)$ the following diagram of filtered complexes with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & \omega^\bullet_{X/S} \arrow[r] \arrow[d, "\cdot s"] & \omega^{\times,\bullet}_{X_{S[\epsilon]}/S[\epsilon]} \arrow[r] \arrow[d, "u_s^*"] & \omega^{\times,\bullet}_{X/S} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \omega^\bullet_{X/S} \arrow[r] & \omega^{\times,\bullet}_{X_{S[\epsilon]}/S[\epsilon]} \arrow[r] & \omega^{\times,\bullet}_{X/S} \arrow[r] & 0
\end{tikzcd}
\tag{9.1.11}
$$

commutes. This assertion is easily checked. ■

Remark 9.1.2. Observe that the proof of Proposition 9.1.1 is of a purely formal nature, so replacing S by an arbitrary S-scheme T throughout shows that we in fact have an isomorphism of exact sequences of fppf sheaves on S-schemes (utilizing our aforementioned abuse of notation):

$$\begin{tikzcd}
0 \arrow[r] & f_*\omega_{X/S} \arrow[r] \arrow[d, "\simeq"] & \mathbf{R}^1 f_*\omega^\bullet_{X/S} \arrow[r] \arrow[d, "\simeq"] & R^1 f_*\mathscr{O}_X \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathscr{L}ie(f_*\omega_{X/S}) \arrow[r] & \mathscr{L}ie(\mathbf{R}^1 f_*\omega^{\times,\bullet}_{X/S}) \arrow[r] & \mathscr{L}ie(R^1 f_*\mathscr{O}_X^\times) &
\end{tikzcd}$$

We will not use this fact.

9.2 Connections and relative Picard functors

In order to relate the exact sequence of fppf sheaves of abelian groups (9.1.6) to the exact sequence of smooth group schemes

$$0 \longrightarrow \omega_J \longrightarrow \mathscr{E}xtrig(J, \mathbf{G}_m) \longrightarrow {}^tJ^0 \longrightarrow 0\ ,$$

we need to reinterpret the former in a more geometric way. We will do this by introducing a generalization of the relative Picard functor.

We first recall the notion of a *connection*. Let T be any scheme, and Y any T-scheme. Denote by $\Delta^1(Y/T)$ the first infinitesimal neighborhood of the diagonal morphism $Y \to Y \times_T Y$. Composition with the projection maps $\mathrm{pr}_i : Y \times_T Y \rightrightarrows Y$ yields morphisms $p_i : \Delta^1(Y/T) \rightrightarrows Y$. Recall [35, I, §3.1] that for any $\mathscr{O}_Y$-module $\mathscr{E}$, a *connection* on $\mathscr{E}$ over T is an isomorphism of $\mathscr{O}_{\Delta^1(Y/T)}$-modules

$$p_1^*\mathscr{E} \simeq p_2^*\mathscr{E}$$

which pulls back to the identity over Y. This data is equivalent to an $\mathscr{O}_T$-linear homomorphism

$$\nabla' : \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_Y} \Omega^1_{Y/T}$$

satisfying the usual Leibnitz rule [35, I, §3.1]. A connection is said to be *integrable* if its curvature (which is an element of $\Gamma(Y, \Omega^2_{Y/T})$) is zero [35, I, §3.1.4].

Now let $S = \operatorname{Spec} R$ be as before and suppose that $f : X \to S$ is an admissible curve. Let T be any S-scheme and denote by $f_T : X_T \to T$ the base change of f. Thanks to Theorem 3.4.1, we have an $\mathscr{O}_T$-linear map $d : \mathscr{O}_{X_T} \to \omega_{X_T/T}$. Let $\mathscr{L}$ be any invertible sheaf on X_T.

Definition 9.2.1. A *regular connection* on $\mathscr{L}$ is an $\mathscr{O}_T$-linear morphism

$$\nabla : \mathscr{L} \to \mathscr{L} \otimes_{\mathscr{O}_{X_T}} \omega_{X_T/T}$$

satisfying the Leibnitz rule:

$$\nabla(h\eta) = \eta \otimes dh + h\nabla(\eta)$$

for any sections h of $\mathscr{O}_{X_T}$ and η of $\omega_{X_T/T}$.

Define the contravariant functors from the category of S-schemes to sets

$$P^{\natural}_{X/S} : T \mapsto \left\{ \begin{array}{l} \text{Isomorphism classes of invertible} \\ \text{sheaves on } X_T \text{ equipped with a} \\ \text{regular connection} \end{array} \right\}$$

$$P_{X/S} : T \mapsto \left\{ \begin{array}{l} \text{Isomorphism classes of invertible} \\ \text{sheaves on } X_T \end{array} \right\}$$

Given two line bundles with regular connection $(\mathscr{L}, \nabla)$ and $(\mathscr{L}, \nabla')$, the tensor product $\mathscr{L} \otimes \mathscr{L}'$ is equipped with the "tensor product regular connection"

$$\nabla \otimes \nabla' : \mathscr{L} \otimes \mathscr{L}' \longrightarrow \mathscr{L} \otimes \mathscr{L}' \otimes \omega$$

given for sections α of $\mathscr{L}$ and β of $\mathscr{L}'$ by

$$\nabla \otimes \nabla'(\alpha \otimes \beta) = \alpha \otimes \nabla'(\beta) + \beta \otimes \nabla(\alpha).$$

This makes the functors $P^{\natural}_{X/S}$ and $P_{X/S}$ group functors. As usual, we denote by $\operatorname{Pic}_{X/S}$ the sheafification of $P_{X/S}$ for the fppf topology and we write $\operatorname{Pic}^{\natural}_{X/S}$ for the fppf sheaf associated to $P^{\natural}_{X/S}$.

There is an exact sequence of group functors on the category of S-schemes

$$f_*\omega_{X/S} \longrightarrow P^\natural_{X/S} \longrightarrow P_{X/S} \tag{9.2.1}$$

that is defined as follows. The first map associates to a section ω of $f_{T*}\omega_{X_T/T}$ over T the pair $(\mathscr{O}_{X_T}, \nabla)$, with ∇ the connection on $\mathscr{O}_{X_T}$ determined by

$$\nabla(s) = s \otimes \omega + ds,$$

and the second map is the "forget the connection map" which takes a pair $(\mathscr{L}, \nabla)$ to the invertible sheaf $\mathscr{L}$.

Our "geometric" interpretation of the exact sequence (9.1.6) is:

Proposition 9.2.2. *There is an isomorphism of exact sequences of fppf sheaves of abelian groups on the category of S-schemes*

$$\begin{tikzcd} f_*\omega_{X/S} \arrow[r] \arrow[d, equal] & \mathrm{Pic}^\natural_{X/S} \arrow[r] \arrow[d] & \mathrm{Pic}_{X/S} \arrow[d] \\ f_*\omega_{X/S} \arrow[r] & \mathbf{R}^1 f_*\omega^{\times,\bullet}_{X/S} \arrow[r] & R^1 f_*\mathscr{O}_X^\times \end{tikzcd} \tag{9.2.2}$$

where the lower exact sequence is (9.1.6)*.*

Proof. It is clearly enough to define an isomorphism of exact sequences of groups

$$\begin{tikzcd} \Gamma(X_T, \omega_{X_T/T}) \arrow[r] \arrow[d, equal] & P^\natural_{X/S}(T) \arrow[r] \arrow[d] & P_{X/S}(T) \arrow[d] \\ \Gamma(X_T, \omega_{X_T/T}) \arrow[r] & \mathbf{H}^1(X_T, \omega^{\times,\bullet}_{X_T/T}) \arrow[r] & H^1(X_T, \mathscr{O}^\times_{X_T}) \end{tikzcd} \tag{9.2.3}$$

functorially in affine S-schemes T; we will do this using Čech theory (see Appendix A and [19, 0_{III}, §12.4.5] for the Čech-theoretic description of hypercohomology).

For $(\mathscr{L}, \nabla) \in P^\natural_{X/S}(T)$, let $\{U_i\}$ be a Zariski open cover of X_T which trivializes

$\mathscr{L}$ via

$$\mathscr{O}_{U_i} \xrightarrow{\simeq} \mathscr{L}\big|_{U_i}$$

$$1 \mapsto s_i$$

with $s_i \in \Gamma(U_i, \mathscr{L})$. Denote by $f_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}_{X_T}^{\times})$ the transition functions, which satisfy $s_i = f_{ij}s_j$. Because of the Leibnitz rule, the map $\nabla\big|_{U_i}$ is completely determined by its value on s_i, and there exists a unique $\omega_i \in \Gamma(U_i, \omega_{X_T/T})$ satisfying $\nabla(s_i) = s_i \otimes \omega_i$. Feeding the equality $s_i = f_{ij}s_i$ into ∇, we see that

$$\omega_i - \omega_j = df_{ij}/f_{ij}.$$

We thus obtain a Čech hyper 1-cocycle for the complex $\omega_{X_T/T}^{\times,\bullet}$:

$$(\omega_i, df_{ij}/f_{ij}) \in C^0(\{U_i\}, \omega_{X_T/T}) \oplus C^1(\{U_i\}, \mathscr{O}_{X_T}^{\times}) = C^1(\{U_i\}, \omega_{X_T/T}^{\times,\bullet}).$$

Given another trivialization

$$\mathscr{O}_{V_{i'}} \xrightarrow{\simeq} \mathscr{L}\big|_{V_{i'}}$$

$$1 \mapsto t_{i'}$$

over an open covering $\{V_{i'}\}$, we must have

$$s_i = \xi_{ii'}t_{i'}$$

over $U_i \cap V_{i'}$, for some $\xi_{ii'} \in \Gamma(U_i \cap V_{i'}, \mathscr{O}_{X_T}^{\times})$. Defining $\eta_{i'}$ by $\nabla(t_{i'}) = t_{i'} \otimes \eta_{i'}$, we calculate over $U_i \cap V_{i'}$

$$s_i \otimes \omega_i = \nabla(s_i) = \nabla(\xi_{ii'}t_{i'}) = t_{i'} \otimes d\xi_{ii'} + \xi_{ii'}t_{i'} \otimes_{\eta_i} = s_i \otimes d\log(\xi_{ii'}) + s_i \otimes \eta_i,$$

from which we conclude that $\omega_i - \eta_{i'} = d\log(\xi_{ii'})$ over $U_i \cap V_{i'}$. It follows that the two Čech hyper 1-cocycles determined by the trivializations of $\mathscr{L}$ over $\{V_{i'}\}$ and $\{U_i\}$

differ by a hyper coboundary when viewed as hyper 1-cocycles for the common refining open cover $\{U_i \cap V_{i'}\}$. Therefore, we obtain a well-defined cohomology class in $\check{\mathbf{H}}^1(\{U_i \cap V_{i'}\}, \omega^{\times,\bullet}_{X_T/T})$, and hence (under the natural isomorphism from Čech hypercohomology to hypercohomology in degree 1) an element of $\mathbf{H}^1(X_T, \omega^{\times,\bullet}_{X_T/T})$. Reversing this procedure, any degree 1 hypercohomology class determines an invertible sheaf with regular connection (well-defined up to isomorphism), and this gives the desired isomorphism in the middle column of (9.2.3).

The classical isomorphism $P_{X/S}(T) \to H^1(X_T, \mathscr{O}^{\times}_{X_T})$ is defined similarly in terms of Čech theory by ignoring the connection, and the left vertical map in (9.2.3) is simply the identity. To see that (9.2.3) commutes, we use the fact that the isomorphism from Čech hypercohomology to derived-functor hypercohomology in degrees 0 and 1 is δ-functorial to describe the maps in the lower row of (9.2.3) Čech-theoretically, whence the claimed commutativity is an easy calculation. ■

9.3 Definition of the map

Combining the identifications (9.1.7) and (9.2.2), we see that the exact sequence of R-modules

$$0 \longrightarrow H^0(X, \omega_{X/S}) \longrightarrow H^1(X) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow 0$$

of (3.4.1) is canonically isomorphic to the exact sequence of Lie algebras attached to the exact sequence of fppf sheaves of abelian groups on the category of S-schemes

$$0 \longrightarrow \omega_{X/S} \longrightarrow \mathrm{Pic}^{\natural}_{X/S} \longrightarrow \mathrm{Pic}_{X/S} .$$

Recall that our strategy to relate the integral structure on the Hodge filtration of $H^1_{\mathrm{dR}}(J_K/K)$ furnished by the exact sequence of Lie algebras

$$0 \longrightarrow \omega_J \longrightarrow \mathrm{Lie}\,\mathscr{E}xtrig(J, \mathbf{G}_m) \longrightarrow \mathrm{Lie}\,\mathscr{E}xt(J, \mathbf{G}_m) \longrightarrow 0$$

as in (8.2.4) to the integral structure (3.4.1) on the Hodge filtration of X_K is to define a map of exact sequences of fppf sheaves of abelian groups on the category of S-schemes

$$
\begin{tikzcd}
0 \arrow[r] & \omega_J \arrow[r] \arrow[d] & \mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] \arrow[d] & {}^tJ^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\omega_{X/S} \arrow[r] & \mathrm{Pic}^\natural_{X/S} \arrow[r] & \mathrm{Pic}_{X/S} &
\end{tikzcd}
\tag{9.3.1}
$$

and then apply the functor Lie. We first make the map (9.3.1) on the category of smooth S-schemes, where we use the fact (Proposition 8.2.2) that the top row coincides with the exact sequence of fppf abelian sheaves

$$0 \longrightarrow \omega_J \longrightarrow \mathscr{E}xtrig(J, \mathbf{G}_m) \longrightarrow \mathscr{E}xt(J, \mathbf{G}_m) \longrightarrow 0$$

on smooth test objects over S. That is, we will begin by defining a map of fppf abelian sheaves on the category of smooth S-schemes

$$
\begin{tikzcd}
0 \arrow[r] & \omega_J \arrow[r] \arrow[d] & \mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] \arrow[d] & \mathscr{E}xt(J, \mathbf{G}_m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\omega_{X/S} \arrow[r] & \mathrm{Pic}^\natural_{X/S} \arrow[r] & \mathrm{Pic}_{X/S} &
\end{tikzcd}
\tag{9.3.2}
$$

In this section, we will define the map (9.3.2).

For now, suppose there exists a section $x \in X(K)$; we will later reduce the general situation to this one. Denote by $j_K : X_K \to J_K$ the morphism from X_K to its Jacobian determined by x. The Néron mapping property gives an S-morphism $j : U \to J$ from the S-smooth locus U of X to J that recovers j_K on K-fibers.

For any smooth affine S-scheme T, we will show that "pullback along j" defines a map of exact sequences of groups

$$
\begin{tikzcd}
0 \arrow[r] & \Gamma(T, \omega_{J_T}) \arrow[r] \arrow[d] & \mathrm{Extrig}(J_T, \mathbf{G}_{mT}) \arrow[r] \arrow[d] & \mathrm{Ext}(J_T, \mathbf{G}_{mT}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(T, f_{T*}\omega_{X_T/T}) \arrow[r] & P^\natural_{X/S}(T) \arrow[r] & P_{X/S}(T) &
\end{tikzcd}
\tag{9.3.3}
$$

By sheafifying with respect to the fppf topology on smooth S-schemes, we will obtain the desired map (9.3.2).

By hypothesis, X is normal and the complement of U in X consists of points of codimension at least two (it is here that we critically use the hypothesis that the closed fiber X_k is generically smooth). Since $T \to S$ is smooth, the base change X_T is also normal (as it is smooth over the normal scheme X; see part (ii) of the Corollary to Theorem 23.9 in [34]). Since $T \to S$ is in particular flat, the complement of U_T in X_T has codimension at least 2 because codimension can only increase under flat base change [34, Theorem 15.1]. It follows that pushforward of line bundles via the inclusion $i_T : U_T \hookrightarrow X_T$ induces an isomorphism of abelian groups $P_{U/S}(T) \to P_{X/S}(T)$, so to give an invertible sheaf on X_T is the same as giving an invertible sheaf on U_T. Moreover, by Lemma 3.3.7 the canonical map $\omega_{X/S} \to i_*\Omega^1_{U/S}$ as in (3.3.8) is an isomorphism; since this map and the sheaves in question are compatible with base change, the canonical map $\omega_{X_T/T} \to i_{T*}\Omega^1_{U_T/T}$ is an isomorphism (or one could repeat the proof of Lemma 3.3.7). Therefore, giving a regular connection on an invertible sheaf $\mathscr{L}$ on X_T is equivalent to giving a connection on $\mathscr{L}\big|_{U_T}$. Thus, to construct the desired map, we will associate to any element of $\mathrm{Extrig}(J_T, \mathbf{G}_{mT})$ an invertible sheaf on U_T with connection in a manner that is Zariski-local on T, and so globalizes from the case of affine T.

Let

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{G}_{mT} \arrow[r] & E \arrow[r] & J_T \arrow[r] & 0 \\ & & & \mathrm{Inf}^1_T(J_T) \arrow[ul, "\sigma"] \arrow[u, "\iota"'] & \end{tikzcd} \tag{9.3.4}$$

be a representative of an element of $\mathrm{Extrig}(J_T, \mathbf{G}_{mT})$ and let $\mathscr{M}$ be the invertible sheaf on J_T corresponding to the $\mathbf{G}_{mT}$-torsor E (as in [35, I, 3.1.2 (a)] and [42, 17.6]).

We will show that the rigidification σ of E yields a connection on $\mathscr{M}$; pulling this data back along $j_T : U_T \to J_T$ yields the desired invertible sheaf with connection on U_T.

We want to give an isomorphism of $\mathscr{O}_{\Delta^1(J_T/T)}$-modules

$$p_1{}^*\mathscr{L} \simeq p_2{}^*\mathscr{L}$$

pulling back to the identity on J_T. The rigidification $\sigma : \mathrm{Inf}^1_T(J_T) \to E$ gives a section (and hence a splitting) of the pullback of E to $\mathrm{Inf}^1_T(J_T)$. In this way, the pullback $\iota^*\mathscr{M}$ of $\mathscr{M}$ to $\mathrm{Inf}^1_T(J_T)$ is canonically trivialized:

$$\theta : \mathscr{O}_{\mathrm{Inf}^1_T(J_T)} \to \iota^*\mathscr{M}.$$

Moreover, since σ is a morphism of T-pointed T-schemes, the pullback of θ via the identity section of $\mathrm{Inf}^1_T(J_T)$ recovers the canonical trivialization of $e_J^*\mathscr{M}$ coming from the identity section of E.

Denote by $s : J_T \times_T J_T \to J_T$ the subtraction map, which is determined by $s = \mathrm{pr}_2 - \mathrm{pr}_1$, with pr_i projection onto the ith factor.

Lemma 9.3.1. *There is a unique isomorphism of invertible sheaves on $J_T \times_T J_T$*

$$\rho : s^*\mathscr{M} \xrightarrow{\simeq} \mathrm{pr}_1^*\,\mathscr{M}^{-1} \otimes \mathrm{pr}_2^*\,\mathscr{M} \tag{9.3.5}$$

pulling back along the diagonal $J_T \to J_T \times_T J_T$ to the inverse of the canonical trivialization

$$\mathscr{O}_{J_T} \to \pi_J^* e_J^* \mathscr{M}.$$

Proof. Since $\mathrm{Ext}(\cdot, \mathbf{G}_m)$ is an *additive* functor [46, VII, Proposition 1 (d)], we have

$$s^*E = (p_2 - p_1)^*E \simeq \mathrm{pr}_2^*\,E - \mathrm{pr}_1^*\,E.$$

Since the natural map

$$\alpha : \mathrm{Ext}(J_T, \mathbf{G}_{mT}) \to H^1(J_T, \mathscr{O}_{J_T}^{\times})$$

is a functorial homomorphism [42, 17.6], we obtain an isomorphism of invertible sheaves

$$s^*\mathscr{M} \xrightarrow{\simeq} \mathrm{pr}_1^*\mathscr{M}^{-1} \otimes \mathrm{pr}_2^*\mathscr{M}\ . \tag{9.3.6}$$

Pulling back along the diagonal $J_T \to J_T \times_T J_T$ gives an isomorphism

$$\pi_J^* e^* \mathscr{M} \to \mathscr{O}_{J_T}.$$

This isomorphism differs from the inverse of the canonical one by a unit in $\Gamma(J_T, \mathscr{O}_{J_T})$. Since the natural isomorphism $\pi_{J*}\mathscr{O}_J \simeq \mathscr{O}_S$ persists after the flat base change to T, we have $\Gamma(J_T, \mathscr{O}_{J_T}) = \Gamma(T, \mathscr{O}_T)$, so we can modify (9.3.6) by a unique unit in $\Gamma(T, \mathscr{O}_T)$ to achieve the desired isomorphism. ■

Now $s : J_T \times_T J_T \to J_T$ kills the diagonal, so the composite

$$\Delta^1(J_T/T) \xrightarrow{\delta} J_T \times_T J_T \xrightarrow{s} J_T$$

factors through $\mathrm{Inf}^1_T(J_T) \hookrightarrow J_T$ by the definition of Inf^1_T. Thus, there is a commutative diagram

$$\begin{tikzcd} \Delta^1(J_T/T) \arrow[r, "\delta"] \arrow[d, "\tau"'] & J_T \times_T J_T \arrow[d, "s"] \\ \mathrm{Inf}^1_T(J_T) \arrow[r, "\iota"'] \arrow[dr] & J_T \arrow[d] \\ & T \end{tikzcd} \tag{9.3.7}$$

In particular, pulling the trivialization θ back along τ yields an isomorphism

$$\tau^*(\theta) : \mathscr{O}_{\Delta^1(J_T/T)} \xrightarrow{\simeq} \tau^*\iota^*\mathscr{M} = \delta^* s^* \mathscr{M}.$$

Since ι is monic and $s \circ \delta \circ \Delta = e_J \circ \pi_J$, we have that $\tau \circ \Delta = e_{\mathrm{Inf}^1} \circ \pi_J$, and hence the pullback

$$\Delta^* \tau^*(\theta) : \mathscr{O}_{J_T/T} \longrightarrow \Delta^* \delta^* s^* \mathscr{M} = \pi_J^* e_J^* \mathscr{M}$$

is equal to the pullback $\pi_J^* e_{\mathrm{Inf}^1}^*(\theta)$, which we know is the canonical trivialization of $\pi_J^* e_J^* \mathscr{M}$.

Composing $\tau^*(\theta)$ with the pullback via δ of the isomorphism ρ of Lemma 9.3.1, we obtain an isomorphism

(9.3.8)

$$\mathscr{O}_{\Delta^1(J_T/T)} \xrightarrow{\tau^*(\theta)} \delta^* s^* \mathscr{M} \xrightarrow{\quad \delta^*(\rho) \quad} \delta^* \operatorname{pr}_1^* \mathscr{M} \otimes \delta^* \operatorname{pr}_2^* \mathscr{M}^{-1} = p_1^* \mathscr{M} \otimes p_2^* \mathscr{M}^{-1} ,$$

which pulls back via Δ to the composite

$$\mathscr{O}_{J_T} \xrightarrow{\Delta^* \tau^*(\theta)} \pi_J^* e_J^* \mathscr{M} \xrightarrow{\Delta^* \delta^*(\rho)} \mathscr{O}_{J_T}. \tag{9.3.9}$$

But we know that $\Delta^* \tau^*(\theta)$ is the canonical trivialization of $\pi_J^* e_J^* \mathscr{M}$ and by Lemma 9.3.1, $\Delta^* \delta^*(\rho)$ is the inverse of the canonical trivialization, so (9.3.9) is the identity, as desired.

Thus, we have shown that the rigidified extension (9.3.4) gives an invertible sheaf $\mathscr{M}$ on J_T with connection. Pulling back $\mathscr{M}$ and the connection on $\mathscr{M}$ along $j_T : U_T \to J_T$, we obtain an invertible sheaf with connection on U_T. Thus, as explained above, we get a unique invertible sheaf with regular connection on X_T, and this defines the desired map

$$\operatorname{Extrig}(J_T, \mathbf{G}_{mT}) \to P^\natural_{X/S}(T).$$

By tracing through the definition of this map, it is straightforward to check that it is in fact a map of groups (which amounts to the fact that pullback and tensor product of sheaves commute).

In a similar way (ignoring the connection in the above construction) we obtain a compatible map $\mathrm{Ext}(J_T, \mathbf{G}_{mT}) \to P_{X/S}(T)$ and hence a commutative square

$$\begin{tikzcd}
\mathrm{Extrig}(J_T, \mathbf{G}_{mT}) \arrow[r] \arrow[d] & \mathrm{Ext}(J_T, \mathbf{G}_{mT}) \arrow[d] \\
P^{\natural}_{X/S}(T) \arrow[r] & P_{X/S}(T)
\end{tikzcd}$$

It follows that the composite map

$$\Gamma(T, \omega_{J_T}) \longrightarrow \mathrm{Extrig}(J_T, \mathbf{G}_{mT}) \longrightarrow P^{\natural}_{X/S}(T) \longrightarrow P_{X/S}(T)$$

is zero, so there is a unique map $\Gamma(T, \omega_{J_T}) \to \Gamma(T, f_{T*}\omega_{X_T/T})$ yielding a commutative diagram (9.3.3). Sheafifying with respect to the fppf topology on the category of smooth S-schemes, we get the desired map (9.3.2).

9.4 Completion of the proof

First, let us explain how to reduce to the case that $\in X_K(K) \neq \emptyset$. Since the problem is to show that the map (7.4.1) restricts to a map of integral structures whose formation commutes with local-étale extensions of discrete valuation rings, it is harmless to make such an extension on R. Moreover, since X_k has nonempty smooth locus, we may make such an extension to get a section of (the base change of) $X_K \to \mathrm{Spec}\, K$. Now the base change of an admissible curve by an étale morphism is again admissible by Lemma 3.1.13, and since the formation of the smooth locus commutes with étale base change, we see that our hypotheses on the existence of an admissible model of X_K with generically smooth closed fiber are preserved by such base change. Without loss of generality, we may therefore suppose there exists a section $x \in X_K(K)$ and use x to define a morphism $j_K : X_K \to J_K$.

Combining the diagrams (9.3.2) and (9.2.2), we obtain a morphism of exact sequences of fppf sheaves of abelian groups on the category of smooth S-schemes

$$
\begin{tikzcd}
0 \arrow[r] & \omega_J \arrow[r] \arrow[d] & \mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] \arrow[d] & \mathscr{E}xt(J, \mathbf{G}_m) \arrow[r] \arrow[d] & 0\,. \\
0 \arrow[r] & f_*\omega_{X/S} \arrow[r] & \mathbf{R}^1 f_*\omega_{X/S}^{\times,\bullet} \arrow[r] & R^1 f_*\mathscr{O}_X^{\times} &
\end{tikzcd}
\tag{9.4.1}
$$

We wish to apply the functor Lie to (9.4.1), and use the identifications of §9.1. However, the scheme $S[\epsilon]$ is *not* S-smooth, so we cannot simply use the functorial characterization of Lie. To sidestep this problem, we use Yoneda's lemma and the fact that the fppf abelian sheaves in the top row of (9.4.1) are represented by smooth S-schemes on the category of smooth S-schemes (by Lemma 8.2.1 and Proposition 8.2.2) as follows.

More generally, suppose that $\mathscr{F}$ and $\mathscr{G}$ are sheaves for some topology on S-schemes and that we have a map of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ on smooth S-schemes. If $\mathscr{F}$ is represented by a smooth S-scheme on the category of all S-schemes, then φ extends to a map of sheaves on all S-schemes T by composing a T-point $t : T \to \mathscr{F}$ with the canonical map of sheaves on S-schemes $\varphi(\mathscr{F})(\mathrm{id}) : \mathscr{F} \to \mathscr{G}$. Similarly, under this same hypothesis on $\mathscr{F}$, given two maps $\varphi, \varphi' : \mathscr{F} \rightrightarrows \mathscr{G}$ of sheaves on all S-schemes that agree on smooth S-schemes, we have for any S-scheme T and any $t : T \to \mathscr{F}$

$$\varphi(T)(t) = \varphi(\mathscr{F})(\mathrm{id})(t) = \varphi'(\mathscr{F})(\mathrm{id})(t) = \varphi'(T)(t),$$

so $\varphi = \varphi'$ on all S-schemes T. More concisely, the representability of $\mathscr{F}$ by a smooth S-scheme implies via Yoneda's lemma as above that the canonical "restriction" map

$$\mathrm{Hom}_S(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{S_{\mathrm{sm}}}(\mathscr{F}, \mathscr{G})$$

is bijective.

We conclude that the commutative diagram (9.4.1) extends uniquely to a commutative diagram of fppf sheaves of abelian groups on the category of S-schemes,

$$\begin{tikzcd}
0 \arrow[r] & \omega_J \arrow[r] \arrow[d] & \mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] \arrow[d] & {}^tJ^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*\omega_{X/S} \arrow[r] & \mathrm{Pic}^\natural_{X/S} \arrow[r] & \mathrm{Pic}_{X/S} &
\end{tikzcd}$$

so in particular a map of Lie algebras (where we have used Proposition 9.1.1):

$$\begin{tikzcd}
0 \arrow[r] & \omega_J \arrow[r] \arrow[d] & \mathrm{Lie}\,\mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] \arrow[d] & \mathrm{Lie}({}^tJ^0) \arrow[r] \arrow[d] & 0\,. \\
0 \arrow[r] & H^0(X, \omega_{X/S}) \arrow[r] & H^1(X) \arrow[r] & H^1(X, \mathscr{O}_X) \arrow[r] & 0
\end{tikzcd} \tag{9.4.2}$$

If we tensor the diagram (9.4.2) over R with K and use Corollary 8.2.8 (i.e. the identification (8.2.7)) then we get a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^0(J_K, \Omega^1_{J_K/K}) \arrow[r] \arrow[d] & H^1_{\mathrm{dR}}(J_K/K) \arrow[r] \arrow[d] & H^1(J_K, \mathscr{O}_{J_K}) \arrow[r] \arrow[d] & 0\,, \\
0 \arrow[r] & H^0(X_K, \Omega^1_{X_K/K}) \arrow[r] & H^1_{\mathrm{dR}}(X_K/K) \arrow[r] & H^1(X_K, \mathscr{O}_{X_K}) \arrow[r] & 0
\end{tikzcd}$$

and we claim that this map agrees with (7.4.1). This amounts to the assertion that the diagram

$$\begin{tikzcd}
\mathrm{Lie}\,\mathscr{E}xtrig(J_K, \mathbf{G}_m) \arrow[r, "\simeq", "(8.2.6)"'] \arrow[d] & H^1_{\mathrm{dR}}(J_K/K) \arrow[d, "j_K^*"] \\
\mathrm{Lie}\,\mathrm{Pic}^\natural_{X_K/K} \arrow[r, "\simeq"] & H^1_{\mathrm{dR}}(X_K/K)
\end{tikzcd}$$

commutes, which is an easy consequence of how the horizontal maps are defined (see [35], 3.2.3, 4.1.7, and 4.2.1 for the top) and the fact that the vertical maps are both given by "pullback along $j_K : X_K \to J_K$."

It remains to show that (9.4.2) is an isomorphism. We will first show that the right vertical map is an isomorphism, and then use duality to prove that the left vertical map is an isomorphism. It will then follow that the middle map in (9.4.2) is an isomorphism as well.

To prove that the map $\operatorname{Lie} \mathscr{E}xt(J, \mathbf{G}_m) \to H^1(X, \mathscr{O}_X)$ is an isomorphism, we must relate the Néron model J and the group functor $\operatorname{Pic}_{X/S}$. Recall that for any group functor G on the category of S-schemes, the fiber G_t of G at a point t of an S-scheme T is the group functor on $\operatorname{Spec} k(t)$-schemes given by viewing $k(t)$-schemes as T-schemes via $\operatorname{Spec} k(t) \to T$. If G has representable fibers, we denote by G^0 the subfunctor of G defined by

$$G^0(T) = \{a \in G(T) \ : \ a_t \in G^0_t(\operatorname{Spec} k(t)) \text{ for all } t \in T\},$$

where G^0_t is the connected component of the identity in G_t; see [43, 3.2 (d)]. In particular, since the fibers $\operatorname{Pic}_{X_s/k(s)}$ are representable [9, §8.2, Theorem 3], we may define $\operatorname{Pic}^0_{X/S}$ as above. Concretely, this is the subfunctor of $\operatorname{Pic}_{X/S}$ consisting of those line bundles whose restriction to each fiber X_s has partial degree zero on every irreducible component of $X_s \times_{k(s)} \overline{k(s)}$ (cf. [9, §9.3, Corollary 13]).

Since the closed fiber X_k of X is generically smooth by hypothesis, the greatest common divisor of the geometric multiplicities of the components of X_k must be 1, so we may apply [9, §9.4 Theorem 2 (b)] to conclude that $\operatorname{Pic}^0_{X/S}$ is a separated S-scheme that is moreover smooth by [9, §8.4, Proposition 2] . The Néron mapping property of J thus yields a morphism

$$\operatorname{Pic}^0_{X/S} \to J^0 \tag{9.4.3}$$

extending the canonical isomorphism $\operatorname{Pic}^0_{X_K/K} \xrightarrow{\simeq} J_K$.

Proposition 9.4.1. *The morphism* (9.4.3) *is an isomorphism.*

Proof. This follows at once from [9, §9.7, Theorem 1], which shows that (9.4.3) is an isomorphism if and only if X has rational singularities. ∎

Via the map $j_K : X_K \to J_K$ determined by the section $x \in X_K(K)$, we have an isomorphism of abelian varieties $\mathrm{Pic}^0_{J_K/K} \to \mathrm{Pic}^0_{X_K/K}$ which extends to an isomorphism of S-groups

$$ {}^tJ^0 \xrightarrow{\simeq} J^0 \tag{9.4.4}$$

via the Néron mapping property. We thus obtain an isomorphism of S-group schemes

$$ {}^tJ^0 \xrightarrow[(9.4.4)]{\simeq} J^0 \xleftarrow[(9.4.3)]{\simeq} \mathrm{Pic}^0_{X/S} $$

and hence an isomorphism of Lie algebras

$$ \mathrm{Lie}({}^tJ^0) \xrightarrow{\simeq} \mathrm{Lie}\,\mathrm{Pic}^0_{X/S}. \tag{9.4.5}$$

Composing (9.4.5) with the canonical isomorphisms of R-modules

$$ \mathrm{Lie}\,\mathrm{Pic}^0_{X/S} \xrightarrow{\simeq} \mathrm{Lie}\,\mathrm{Pic}_{X/S} $$

and

$$ \mathrm{Lie}\,\mathrm{Pic}_{X/S} \xrightarrow{\simeq} H^1(X, \mathscr{O}_X) $$

(cf. [9], §8.4 Theorem 1), we obtain a natural isomorphism

$$ \mathrm{Lie}\,{}^tJ^0 \xrightarrow{\simeq} H^1(X, \mathscr{O}_X) \tag{9.4.6}$$

of finite free R-modules. We claim that this map coincides with the right vertical map in (8.3.1). This may be checked over K, where it is clear as the composite map (9.4.5) on the level of (isomorphism classes of) invertible sheaves is simply pullback by the map $j_K : X_K \to J_K$ determined by the section x. Thus, the right vertical map in (8.3.1) is an isomorphism.

To show that the left vertical map in (8.3.1) is an isomorphism, we proceed as follows. The free R-modules $\mathrm{Lie}({}^tJ^0)$ and ω_J are canonically R-dual, as are $H^0(X, \omega_{X/S})$

and $H^1(X, \mathscr{O}_X)$ (by Grothendieck duality (3.3.4)) and we claim that the left vertical map in (8.3.1) is the R-dual of the inverse of the right vertical map, and is therefore also an isomorphism. As the R-modules in question are all free, it suffices to check such duality after extending scalars to K. Since the map (8.3.1) recovers (7.4.1) after tensoring over R with K and using the identification (8.2.7), the desired duality follows at once from Proposition 7.4.2. Since the left and right maps of (8.3.1) are isomorphisms, the middle one is too.

Remark 9.4.2. It is likely that the hypothesis that X have generically smooth closed fiber can be omitted. Indeed, we have used the hypothesis that X have generically smooth closed fiber as a "crutch" to pass to the smooth locus U of X and use the Néron mapping property of J to get a map $U \to J$ allowing us to relate constructions on J to constructions on X. However, using the functorial characterization of J, [32] shows that there is *always* a map

$$\mathrm{Lie}({}^tJ^0) \to H^1(X, \mathscr{O}_X)$$

that is moreover an isomorphism in our situation (to be precise, [32] assumes that X is regular, but we may bypass this assumption by virtue of our hypothesis that X has rational singularities). The author believes that it is similarly possible to define a map

$$\mathrm{Lie}\,\mathscr{E}xtrig(J, \mathbf{G}_m) \to H^1(X)$$

(that is moreover an isomorphism) without requiring that X have generically smooth closed fiber. To construct such a map in this generality would require answering a question raised in [35, I, §5], which asks for a "functorial characterization" (i.e. a universal mapping property) of the canonical extension of a Néron model.

CHAPTER X

Compatibility theorems and application to companion forms

In this chapter we state and prove two compatibility theorems between the structures and cohomology theories we have studied. Our motivation in proving these theorems is that they resolve the "unchecked compatibilities" in Gross' paper [22]; we explain this in §10.3.

We fix a discrete valuation ring R of mixed characteristic $(0, p)$ having fraction field K and perfect residue field k. As usual, we let e denote the absolute ramification index of R, and we suppose that $e < p$. Finally, we assume that R admits a lift σ of the Frobenius endomorphism of k, and we denote by σ_0 the canonical lift of Frobenius on k to the Witt ring $W(k)$ (so σ restricted to $W(k)$ is σ_0).

10.1 Grothendieck's isomorphism

For later use we recall the statement of *Grothendieck's isomorphism.* This is a natural isomorphism which relates the crystalline cohomology of an abelian variety over a perfect field to the Dieudonné module of its p-divisible group in a manner that is compatible with "Hodge filtrations."

Let D_k be the *Dieudonné ring*; it is the associative $W(k)$-algebra generated by two indeterminates F and V satisfying the relations

1. $FV = VF = p$

2. $F\lambda = \lambda^{\sigma_0} F$

3. $\lambda V = V\lambda^{\sigma_0}$

for all $\lambda \in W(k)$. When $k = \mathbf{F}_p$, the Dieudonné ring is commutative (otherwise it is not).

For any abelian variety A over k, the first crystalline cohomology of A over $W(k)$ has a canonical structure of (finite) D_k-module as follows. By Proposition 4.3.2, we know that $H^1_{\text{cris}}(A/W(k))$ is a $W(k)$-module, and by Corollary 4.3.7, this $W(k)$-module is equipped with a canonical σ_0-linear (i.e. satisfying (2) above) endomorphism F. Thus, to prove our claim we need only equip it with an endomorphism V verifying (1) and (3).

To define V, we use the Verschiebung k-morphism

$$V_{A/k} : A^{(p)} \to A$$

as defined in [47, Exp. VII$_{\text{A}}$, §4.3]. This gives a $W(k)$-linear morphism

$$V^*_{A/k} : H^1_{\text{cris}}(A/W(k)) \to H^1_{\text{cris}}(A^{(p)}/W(k)),$$

which by base-change (Proposition 4.3.5) yields a σ_0^{-1}-linear (i.e. satisfying (3)) endomorphism V of $H^1_{\text{cris}}(A/W(k))$.

Property (1) above is a consequence of the classical identity $V_{A/k} \circ F_{A/K} = p \cdot \text{id}_A$ (see [47, Exp. VII$_{\text{A}}$, §4.3]) and the fact that $H^1_{\text{cris}}(A/W(k))$ is $W(k)$-flat (even free) by work of Mazur-Messing cited in the proof of Theorem 10.1.1 below.

For an abelian variety A over k, we denote by $A[p^\infty]$ its associated Barsotti-Tate group over k. Let $\mathbf{D}$ be the contravariant Dieudonné functor from the category of

Barsotti-Tate groups over k to the category of D_k-modules that are finite free over $W(k)$.

Theorem 10.1.1. *For any abelian variety A over k, there is a canonical isomorphism of D_k-modules*

$$H^1_{\mathrm{cris}}(A/W(k)) \simeq \mathbf{D}(A[p^\infty]) \tag{10.1.1}$$

which is natural in A. If A is the reduction of an abelian scheme $\mathscr{A}$ over R, then this isomorphism is compatible with Hodge filtrations in the sense that there is canonical isomorphism of exact sequences of free R-modules

$$\begin{tikzcd} 0 \arrow[r] & \omega_{\mathscr{A}[p^\infty]} \arrow[r] \arrow[d, "\simeq"] & \mathbf{D}(A[p^\infty]) \otimes R \arrow[r] \arrow[d, "\simeq"] & \mathrm{Lie}({}^t\mathscr{A}[p^\infty]) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \omega_{\mathscr{A}} \arrow[r] & H^1_{\mathrm{cris}}(A/W(k)) \otimes R \arrow[r] & \mathrm{Lie}({}^t\mathscr{A}) \arrow[r] & 0 \end{tikzcd} \tag{10.1.2}$$

that is natural in $\mathscr{A}$.

Proof. By the work of Mazur-Messing [35], there is a canonical isomorphism of $W(k)$-modules $H^1_{\mathrm{cris}}(A/W(k)) \simeq \mathbf{D}(A[p^\infty])$ that is natural in A and compatible with base change. The functoriality and compatibility with base change ensure the isomorphism is one of D_k-modules (due to how F and V on $\mathbf{D}(A[p^\infty])$ are defined).

The statement about Hodge filtrations also follows from [35]; see [35, Corollary 7.13] and [27, 5.6.13]. ∎

10.2 Main compatibility theorems

Let X_K be a smooth proper and geometrically connected curve over K that possesses an admissible model X over R, and suppose that the closed fiber X_k of X is generically smooth. We denote by J_K the Jacobian of X_K and by J the Néron model of J_K over R. Suppose that T is a correspondence on X_K given by a pair of finite (flat) morphisms of smooth proper and geometrically connected curves

$$\pi_1, \pi_2 : Y_K \rightrightarrows X_K,$$

and suppose moreover that Y_K admits an admissible model Y over R. As in (7.4.7), we have an endomorphism of J_K given by

$$\mathrm{Alb}(\pi_2) \circ \mathrm{Pic}^0(\pi_1) : J_K \to J_K, \tag{10.2.1}$$

and hence (by the Néron mapping property of J) an endomorphism of J; we will abuse notation (again) by referring to this endomorphism simply by T. We further suppose we are given an abelian scheme quotient B of J such that the endomorphism T passes to the quotient B (since B is the Néron model of its generic fiber B_K, it is equivalent to require that the endomorphism (10.2.1) of J_K pass to B_K).

As in §10.1, we denote by $\mathbf{D}$ the contravariant Dieudonné functor on the category of Barsotti-Tate groups over the perfect field k. If B_0 is the reduction of B, the endomorphism T of B yields an endomorphism of B_0 and hence of its associated Barsotti-Tate group $B_0[p^\infty]$; since $\mathbf{D}$ is a functor, we thus obtain an endomorphism of the Dieudonne module $\mathbf{D}(B_0[p^\infty])$, which we will yet again denote by T.

Thanks to Theorem 10.1.1, we have a natural isomorphism of $W(k)$-modules, compatible with the σ_0-semilinear Frobenius endomorphisms of both sides and respecting Hodge filtrations as in (10.1.2)

$$\mathbf{D}(B_0[p^\infty]) \xrightarrow[(10.1.1)]{\simeq} H^1_{\mathrm{cris}}(B_0/W(k)).$$

By base extension, we get a natural isomorphism of R-modules with σ-semilinear Frobenius endomorphisms

$$\mathbf{D}(B_0[p^\infty]) \otimes_{W(k)} R \xrightarrow[(10.1.1)]{\simeq} H^1_{\mathrm{cris}}(B_0/W(k)) \otimes_{W(k)} R$$

respecting Hodge filtrations. Due to the base change formula in crystalline cohomology (Proposition 4.3.5), we then get a natural isomorphism of R-modules

$$\mathbf{D}(B_0[p^\infty]) \otimes_{W(k)} R \xrightarrow{\simeq} H^1_{\mathrm{cris}}(B_0/R) \tag{10.2.2}$$

that is compatible with the σ-semilinear Frobenius endomorphisms on both sides and with Hodge filtrations. We thus have an isomorphism of short exact sequences of free R-modules

$$\begin{tikzcd} 0 \arrow[r] & \omega_{B[p^\infty]} \arrow[r] \arrow[d, "\simeq"] & \mathbf{D}(B_0[p^\infty]) \otimes R \arrow[r] \arrow[d, "\simeq"] & \mathrm{Lie}({}^tB[p^\infty]) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \omega_B \arrow[r] & H^1_{\mathrm{cris}}(B_0/R) \arrow[r] & \mathrm{Lie}({}^tB) \arrow[r] & 0 \end{tikzcd} \tag{10.2.3}$$

that is natural in B. Moreover, we have canonical isomorphisms of exact sequences of free R-modules

$$\begin{tikzcd} 0 \arrow[r] & \omega_B \arrow[r] \arrow[d, equal] & H^1_{\mathrm{cris}}(B_0/R) \arrow[r] \arrow[d, "\simeq", "(5.3.4)"'] & \mathrm{Lie}({}^tB) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \omega_B \arrow[r] \arrow[d, equal] & H^1_{\mathrm{dR}}(B/R) \arrow[r] \arrow[d, "\simeq", "(8.2.6)"'] & \mathrm{Lie}({}^tB) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \omega_B \arrow[r] & \mathrm{Lie}\,\mathscr{E}xtrig(B, \mathbf{G}_m) \arrow[r] & \mathrm{Lie}({}^tB) \arrow[r] & 0 \end{tikzcd} \tag{10.2.4}$$

that are natural in B, as well as a natural R-linear map

$$\begin{tikzcd} 0 \arrow[r] & \omega_B \arrow[r] \arrow[d] & \mathrm{Lie}\,\mathscr{E}xtrig(B, \mathbf{G}_m) \arrow[r] \arrow[d] & \mathrm{Lie}({}^tB) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \omega_J \arrow[r] & \mathrm{Lie}\,\mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] & \mathrm{Lie}({}^tJ) \arrow[r] & 0 \end{tikzcd} \tag{10.2.5}$$

obtained from the map $J \to B$ by the functoriality of the $\mathrm{Lie}\,\mathscr{E}xtrig$ exact sequence. Since X_k is generically smooth, Theorem 8.3.1 gives us an isomorphism of exact sequences of free R-modules

$$\begin{tikzcd} 0 \arrow[r] & \omega_J \arrow[r] \arrow[d, "\simeq"] & \mathrm{Lie}\,\mathscr{E}xtrig(J, \mathbf{G}_m) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Lie}({}^tJ) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & H^0(X, \omega_X) \arrow[r] & H^1(X) \arrow[r] & H^1(X, \mathscr{O}_X) \arrow[r] & 0 \end{tikzcd} \tag{10.2.6}$$

The composition of the maps (10.2.3)–(10.2.6) is an R-linear map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \omega_{B[p^\infty]} \arrow[r] \arrow[d] & \mathbf{D}(B_0[p^\infty]) \otimes R \arrow[r] \arrow[d] & \mathrm{Lie}({}^tB[p^\infty]) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(X, \omega_X) \arrow[r] & H^1(X) \arrow[r] & H^1(X, \mathscr{O}_X) \arrow[r] & 0 \end{tikzcd} \tag{10.2.7}$$

Recall that the correspondence T on X_K acts on the top row via the endomorphism T of B, and on the bottom by Theorem 6.2.1 (ultimately by Theorem 3.4.5 (2)).

Theorem 10.2.1. *The composite map* (10.2.7) *is T-equivariant.*

Proof. Due to Theorem 10.1.1, the map (10.1.2) is *natural*, and so T-equivariant because T acts on both rows via the endomorphism T of B. Similarly, the crystalline base change map of Proposition 4.3.5 is functorial, and so also T-compatible. Thus, the composite (10.2.3) is T-equivariant. The maps (5.3.4) and (8.2.6) are functorial by Theorems 5.3.2 and 8.2.7 respectively, so (10.2.4) is functorial in B, whence it is T-equivariant. On the other hand, the map (10.2.5) is T-equivariant because it is obtained via functoriality of the Lie Extrig exact sequence from the map $J \to B$, which is T-equivariant by hypothesis. The T-equivariance of (10.2.6) is the content of Remark 8.3.2. We conclude that the composite (10.2.7) of the maps (10.2.2)–(10.2.6) is T-equivariant, as claimed. ■

Now suppose that the correspondence T on X_K extends to the "integral level", i.e. to a pair of finite flat maps of admissible curves

$$\pi_1, \pi_2 : Y \rightrightarrows X$$

recovering the original correspondence on generic fibers. Let U be any open subscheme of the special fiber X_0 of X which is contained in X_0^{sm} and suppose that $\pi_2\left(\pi_1^{-1}U\right) \subseteq U$.

Denote by $\widehat{X}$ the formal completion of X along X_0. By hypothesis, the proper formal Spf R-scheme $\widehat{X}$ is smooth at the points of U, so we may use the open immersion $j : U \hookrightarrow X_0$ and the closed immersion $i : X_0 \hookrightarrow \widehat{X}$ to calculate the rigid cohomology $H^1_{\mathrm{rig}}(U/K)$. Since $\pi_2\left(\pi_1^{-1}U\right) \subseteq U$, we know that T acts as an endomorphism of $H^1_{\mathrm{rig}}(U/K)$ by Theorem 6.1.8 (cf. §6.2). Moreover, there is a natural map of K-vector spaces.

$$H^1_{\mathrm{dR}}(X_K/K) \xrightarrow[(5.2.2)]{\simeq} H^1_{\mathrm{dR}}(\widehat{X}^{\mathrm{rig}}/K) \xrightarrow[(5.3.1)]{} H^1_{\mathrm{rig}}(U/K)\ . \tag{10.2.8}$$

Theorem 10.2.2. *The map* (10.2.8) *is T-equivariant.*

Proof. This is Theorem 6.2.2. ∎

Viewing $H^1(X)$ as an R-lattice in $H^1_{\mathrm{dR}}(X_K/K)$ via Proposition 3.4.3, we obtain an R-linear map

$$\mathbf{D}(B_0[p^\infty]) \otimes R \xrightarrow{(10.2.7)} H^1(X) \hookrightarrow H^1_{\mathrm{dR}}(X_K/K) \xrightarrow{(10.2.8)} H^1_{\mathrm{rig}}(U/K) \tag{10.2.9}$$

which is T-equivariant by Theorems 10.2.1 and 10.2.2. By hypothesis, the Frobenius endomorphism of k admits a σ-semilinear lift to R and so induces σ-linear endomorphisms of $\mathbf{D}(B_0[p^\infty]) \otimes_{W(k)} R$ and $H^1_{\mathrm{rig}}(U/K)$ (see §10.1 and Corollary 4.2.8); as before, we denote each of these endomorphisms by F, so the left and right sides of (10.2.9) come equipped with a σ-semilinear endomorphism F.

Theorem 10.2.3. *The map* (10.2.9) *is F-equivariant.*

The proof of Theorem 10.2.3 is subtle because one in general does *not* expect the R-modules Lie $\mathscr{E}xtrig(J, \mathbf{G}_m)$, $H^1(X)$, or $H^1_{\mathrm{dR}}(X_K/K)$ to come equipped with a Frobenius endomorphism (though in the case that X has semistable reduction—equivalently J^0 is semi-abelian—see [11]). We will need the following lemma:

Lemma 10.2.4. *Let X be an admissible curve over S and let A be an abelian scheme over S. Then any morphism $f_K : X_K \to A_K$ uniquely extends to a morphism $f : X \to A$.*

Proof. Uniqueness is clear, since X is S-flat and A is S-separated. In the case that X is regular, any rational S-morphism $X \dashrightarrow A$ extends to a morphism $X \to A$ by [9, §8.4 Corollary 6]. We wish to reduce the general case to this one.

We first reduce to the case of algebraically closed residue field by the following standard trick. Let Γ_K denote the graph of $f_K : X_K \to A_K$ in $X_K \times A_K \simeq (X \times_S A)_K$, and let $\Gamma \subseteq X \times_S A$ be its (scheme-theoretic) closure. The existence of an S-map $X \to A$ extending f_K is equivalent to the first projection $\mathrm{pr}_1 : \Gamma \to X$ being an isomorphism, which may be checked after any fpqc base change (as such base change commutes with scheme-theoretic closure). Let R' be any unramified extension of R with algebraically closed residue field (e.g. $R' = R \otimes_{W(k)} W(\overline{k})$, where $\overline{k}$ is an algebraic closure of k and W is the Witt vector functor). Because $R \to R'$ is unramified, base change by the fpqc map $S' := \operatorname{Spec} R' \to S$ preserves normality and regularity, hence admissibility. Thus, we can base change to S' and it suffices to solve our problem there, and so may suppose that R has algebraically closed residue field.

Now we explain how to reduce the general case (with $k = \overline{k}$) to the case that X is regular. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities; clearly π is an isomorphism outside a finite set of closed points in the closed fiber of X. Since $\widetilde{X}$ is regular, we see that the given map $f_K : \widetilde{X}_K = X_K \to A_K$ extends to a unique morphism $\widetilde{f} : \widetilde{X} \to A$, and we wish to show that this morphism uniquely factors through π. For such a factorization to exist, it is certainly necessary that as a map of *topological* spaces $\widetilde{f}$ be constant on the fibers of π. In fact, since π is a proper and birational map of normal schemes, [19, II, Lemme 8.11.1] guarantees that such constancy on topological fibers is *sufficient* for the desired factorization to exist. Thus, it suffices to show that $\widetilde{f}$ is constant on the reduced schemes underlying the fibers of π.

Since π is its own Stein factorization, its fibers are geometrically connected. Thus, we just have to focus on the positive-dimensional fibers. We claim that the reduced schemes underlying the fibers of π with positive dimension are connected, and have each irreducible component isomorphic to $\mathbf{P}^1_k$. Granting this claim for a moment, the desired constancy on fibers follows easily because any k-morphism from $\mathbf{P}^1_k$ to an abelian variety is constant.

It remains to verify the claim. Let $x \in X$ be any k-point in the closed fiber over which the fiber of π is positive-dimensional and let $Z = \pi^{-1}(x)$ be the scheme-theoretic fiber. By Stein factorization, Z is a connected k-scheme of pure dimension one. As $R^2\pi_* = 0$ on coherent sheaves due to the theorem on formal functions, the functor $R^1\pi_*$ is right exact, so we have a surjection $R^1\pi_*\mathscr{O}_{\widetilde{X}} \twoheadrightarrow R^1\pi_*\mathscr{O}_Z$ coming from the surjection $\mathscr{O}_{\widetilde{X}} \twoheadrightarrow \mathscr{O}_Z$ realizing Z as a closed subscheme of $\widetilde{X}$. By the rational singularities hypothesis, we have $R^1\pi_*\mathscr{O}_{\widetilde{X}} = 0$ so $R^1\pi_*\mathscr{O}_Z = 0$. As this is the skyscraper sheaf concentrated at x associated to the k-vector space $H^1(Z, \mathscr{O}_Z)$, we deduce that $H^1(Z, \mathscr{O}_Z) = 0$. The theorem on formal functions again ensures that $H^1(X, \cdot)$ is right exact, so applying this functor to the surjection $\mathscr{O}_Z \twoheadrightarrow \mathscr{O}_{Z_{\mathrm{red}}}$ shows that $H^1(Z_{\mathrm{red}}, \mathscr{O}_{Z_{\mathrm{red}}}) = 0$. Since Z_{red} is reduced and pure one-dimensional over an algebraically closed field k, a standard argument shows that Z_{red} has each irreducible component isomorphic to $\mathbf{P}^1_k$, as claimed. ■

Proof of Theorem 10.2.3. The K-scheme X acquires a section after passing to some finite unramified extension K' since X_k has nonempty smooth locus by hypothesis. Since étale base change preserves normality and regularity, it also preserves admissibility. Moreover, the formation of rigid cohomology commutes with any finite extension of scalars [5, Proposition 1.8], and since the absolute ramification index of K' is also equal to e, extending scalars to K' does not alter any of our hypotheses. We may therefore assume that $X(K)$ is nonempty. We make a choice of K-rational point in $X(K)$ and use it to define a morphism $X_K \to J_K$ as in §7.3. Lemma 7.4.1 guarantees that the resulting map (7.4.1) on Hodge filtrations does not depend on this choice.

We therefore get a map

$$X_K \to B_K. \tag{10.2.10}$$

Now since B is an abelian scheme and X is admissible, Lemma 10.2.4 ensures that (10.2.10) uniquely extends to an S-morphism

$$f : X \to B. \tag{10.2.11}$$

Letting $\widehat{B}$ and $\widehat{X}$ be the formal completions of B and X along their special fibers, we thus have a morphism $\widehat{f} : \widehat{X} \to \widehat{B}$, which induces the commutative diagram

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & X_0 \arrow[r, hook] \arrow[d] & \widehat{X} \arrow[d, "\widehat{f}"] \\ B_0 \arrow[r, hook] & B_0 \arrow[r, hook] & \widehat{B} \end{tikzcd} \tag{10.2.12}$$

and hence pullback by $\widehat{f}^{\mathrm{rig}}$ gives a K-linear map

$$H^1_{\mathrm{rig}}(B_0/K) \longrightarrow H^1_{\mathrm{rig}}(U/K). \tag{10.2.13}$$

From the description of the functoriality of $H^\bullet_{\mathrm{rig}}$ in (4.2.9), we see that this map is equal to the natural map induced by the k-morphism $U \to B_0$ (see also the discussion preceeding Théorème 1.6 of [5]).

We claim that the diagram

(10.2.14)
$$\begin{tikzcd}
H^1_{\mathrm{cris}}(B_0/R)\otimes_R K \arrow[r, "\simeq", "(5.3.4)"'] & H^1_{\mathrm{dR}}(B_K/K) \arrow[r, "f_K^*"] \arrow[d, "\simeq"', "(5.2.2)"] & H^1_{\mathrm{dR}}(X_K/K) \arrow[d, "\simeq"', "(5.2.2)"] \\
H^1_{\mathrm{rig}}(B_0/K) \arrow[u, "(5.3.8)", "\simeq"'] \arrow[dr, "(10.2.13)"'] & H^1_{\mathrm{dR}}(\widehat{B}^{\mathrm{rig}}/K) \arrow[l, "\simeq"', "(5.3.1)"] \arrow[r, "(\widehat{f}^{\mathrm{rig}})^*"] & H^1_{\mathrm{dR}}(\widehat{X}^{\mathrm{rig}}/K) \arrow[dl, "(5.3.1)"] \\
& H^1_{\mathrm{rig}}(U/K) &
\end{tikzcd}$$

of natural maps commutes. For the right square, commutativity is an immediate consequence of the naturality of the map (5.2.2). The commutativity of the triangle likewise follows from the naturality of (5.3.1), together with the fact that (10.2.13) is pullback by $\widehat{f}^{\mathrm{rig}}$ via the diagram (10.2.12). Finally, the commutativity of the left square is assured by Lemma 5.4.1.

Now the map (10.2.9) is by definition the composite of the canonical isomorphism (10.2.2)

$$\mathbf{D}(B_0[p^\infty])\otimes_{W(k)} K \simeq H^1_{\mathrm{cris}}(B_0/R)\otimes_R K$$

with the upper and right outside edge of the diagram (10.2.14). Thanks to the commutativity of (10.2.14), this composite is equal to the composite of the above map with

(10.2.15)
$$H^1_{\mathrm{cris}}(B_0/R)\otimes_R K \xleftarrow[(5.3.8)]{\simeq} H^1_{\mathrm{rig}}(B_0/K) \xrightarrow[(10.2.13)]{} H^1_{\mathrm{rig}}(U/K)\,.$$

The map (5.3.8) is F-equivariant by Corollary 5.3.4. Now H^1_{rig} is a functor on k-schemes by Proposition 4.2.2, and the map (10.2.13) is induced by the k-map $U\to B_0$, which intertwines the relative Frobenius morphisms of U and B_0 because all maps of k-schemes commute with Frobenius. Using that rigid cohomology is functorial and compatible with finite flat base change (see §4.2), it follows that (10.2.13) is F-equivariant. We conclude that (10.2.15) is F-equivarient. As we noted at the beginning of this section, the map (10.2.2) is F-equivarient (this is part of the statement of Theorem 10.1.1). We conclude that the the composite (10.2.9) of the F-equivariant maps (10.2.2) and (10.2.15) is F-equivariant. ■

10.3 Applications to companion forms

We now wish to indicate how the type of compatibility theorems we have proved play a central role in Gross' proof of the companion form theorem [22, Theorem 13.10]. Rather than explain Gross' argument in detail, we will content ourselves with a sketch of the main ideas. To keep our discussion as clear as possible, we will axiomatize certain aspects of Gross' situation, and will avoid saying much about modular forms and modular curves.

We will suppose that the residue field k of R is finite of order $q = p^r$. Let K^{nr} be the completion of the maximal unramified extension of K, and denote by R^{nr} the valuation ring of K^{nr}; it is a complete discrete valuation ring that is unramified over R (so π is a uniformizer of R^{nr}).

Let X_K be a smooth proper and geometrically connected curve over K that has an admissible model X over R whose closed fiber X_k is generically smooth. We denote by J_K the Jacobian of X_K and by J the Néron model of J_K over R. Further, let A_K be the maximal quotient of J_K having good reduction over R; its Néron model A is an abelian scheme.

Let H be any commutative subring of the ring of correspondences on X that are defined over R. Then H acts on the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$ by pullback and trace, and this action preserves the integral structure on the Hodge filtration furnished by the admissible model X. Via Albanese and Picard functoriality, H also acts on J_K, and we will henceforth view H as a subring of $\mathrm{End}_K(J_K)$. An easy argument using the maximality of A_K shows that H acts on A_K as well. Observe that the Néron mapping properties of J and A ensure that the action of H on J_K and A_K extends to J and A respectively.

Fx a homomorphism

$$\Psi : H \to R^{\mathrm{nr}}.$$

The homomorphism Ψ determines a local continuous p-adic Galois representation $\rho_\Psi : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V)$, with V a K-vector space, as follows. Let $\mathfrak{m}$ be the maximal ideal that is the kernel of the composite map

$$H \xrightarrow{\Psi} R^{\mathrm{nr}} \twoheadrightarrow R^{\mathrm{nr}}/\pi R^{\mathrm{nr}} \tag{10.3.1}$$

from H to the residue field of R^{nr}. Let $H_{\mathfrak{m}} = \varprojlim_n H/\mathfrak{m}^n H$ be the $\mathfrak{m}$-adic completion of H and $H_p = \varprojlim_n H/p^n H = H \otimes_Z \mathbf{Z}_p$ the p-adic completion. Then $H_{\mathfrak{m}}$ is a direct factor of H_p; we denote by $\epsilon_{\mathfrak{m}} \in H_p$ the idempoent corresponding to the factor $H_{\mathfrak{m}}$. We will use this idempotent to decompose certain H_p-modules.

The Tate module

$$T_p J_K := \varprojlim_n J_K[p^n](\overline{K})$$

is naturally a $\mathbf{Z}_p$-module and comes equipped with an action of H (through J_K) and so in particular is an H_p-module. Since H is a subalgebra of the endomorphism ring of J_K over K, the action of H_p on T_pJ_K commutes with the natural Galois action of $\mathscr{G}_K := \mathrm{Gal}(\overline{K}/K)$. Thus, the $H_{\mathfrak{m}}$-module $\epsilon_{\mathfrak{m}} T_p J_K$ is naturally a $\mathscr{G}_K$-module. It therefore defines a p-divisible (i.e. Barsotti-Tate) group G_K over K with $T_p G_K = \epsilon_{\mathfrak{m}} T_p J_K$. We define $V := T_p G_K \otimes_{\mathbf{Z}_p} K$. This is a finite-dimensional K-vector space with a continuous action of $\mathscr{G}_K$, and so gives a continuous Galois representation ρ_Ψ. Observe that the action of $H_{\mathfrak{m}} \otimes K$ on V is through the K-linear homomorphism

$H_\mathfrak{m} \otimes_{\mathbf{Z}_p} K \to K$ deduced from Ψ.

In order to analyze the Galois representation ρ_Ψ, we wish to study the properties of the p-divisible group G_K. We will make the following simplifying assumptions. First, we suppose that $\mathrm{Hom}_{\mathbf{Z}_p}(H_\mathfrak{m}, \mathbf{Z}_p)$ is a free $H_\mathfrak{m}$-module of rank 1 (equivalently, we assume that $H_\mathfrak{m}$ is Gorenstein). Second, we suppose that $\epsilon_\mathfrak{m} J_K = \epsilon_\mathfrak{m} A_K$, so that G occurs as a p-divisible subgroup of the p-divisible group $A_K[p^\infty]$ of A_K. Since A_K extends to an abelian scheme A over R, the group G_K extends to a p-divisible group G over R. We let G_k denote the reduction of G over (the perfect field) k, and we further assume that G_k is ordinary.

By our assumption that G_k is ordinary, the maximal connected subgroup G_k^0 has no local-local component. Thus we have a splitting $G_k = G_k^m \times G_k^{\text{ét}}$ of G_k into its multiplicative and étale subgroups, and this splitting is reflected in an exact sequence of Barsotti-Tate groups over R with $H_\mathfrak{m}$-action (the connected étale sequence)

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0\,, \tag{10.3.2}$$

where G^0 is of multiplicative type. We will henceforth assume that the $H_\mathfrak{m}$-modules T_pG, T_pG^0, and $T_pG^{\text{ét}}$ are free. The exact sequence (10.3.2) gives a $\mathscr{G}_K$-stable filtration of $T_pG = T_pG_K$:

$$0 \longrightarrow T_pG^0 \longrightarrow T_pG \longrightarrow T_pG^{\text{ét}} \longrightarrow 0$$

which provides refined information about the representation of $\mathscr{G}_K$ on $T_pG \otimes K$.

An important invariant of the ordinary p-divisible group G is the invariant of Serre-Tate. This is a $\mathbf{Z}_p$-bilinear pairing

$$q'_G : T_pG_k \times T_p{}^t\!G_k \longrightarrow (1 + \pi R^{\text{nr}})$$

which satisfies

$$q'_G(h\alpha, \beta) = q'_G(\alpha, {}^t\!h\beta)$$

for all $h \in H_{\mathfrak{m}}$; here ${}^t h$ denotes the Cartier dual of $h : G_k \to G_k$. We may thus uniquely extend q'_G to an $H_{\mathfrak{m}}$-bilinear pairing

$$q_G : T_p G_k \times T_p {}^t G_k \longrightarrow \mathrm{Hom}_{\mathbf{Z}_p}(H_{\mathfrak{m}}, 1 + \pi R^{\mathrm{nr}})$$

by defining

$$q_G(\alpha, \beta)(h) := q'_G(h\alpha, \beta).$$

Since $e < p$, the ideal πR^{nr} of R^{nr} has divided powers, so the p-adic logarithm gives a $\mathbf{Z}_p$-linear continuous homomorphism

$$\log : (1 + \pi R^{\mathrm{nr}}) \to \pi R^{\mathrm{nr}}.$$

Thus, for any local étale extension R' of R we may define the $R' \otimes H_{\mathfrak{m}}$-bilinear map

$$\log q_G : (R' \otimes T_p G_k) \times (R' \otimes T_p {}^t G_k) \longrightarrow \mathrm{Hom}_{R'}(H_{\mathfrak{m}} \otimes R', \pi R^{\mathrm{nr}})$$

as the composite of q_G and log.

By work of Mazur-Messing (see [35, II, Corollary 7.13]) there is a canonical exact sequence of locally free R'-modules

$$0 \longrightarrow \omega_{GR'} \longrightarrow \mathbf{D}(G_k) \otimes_{W(k)} R' \longrightarrow \mathrm{Lie}({}^t G)_{R'} \longrightarrow 0 \tag{10.3.3}$$

that is natural in G. Here, $\omega_{GR'} := \omega_G \otimes_R R'$ and $\mathrm{Lie}({}^t G)_{R'} := \mathrm{Lie}({}^t G) \otimes_R R'$.

Since $H_{\mathfrak{m}}$ acts on G by endomorphisms, we see that $H_{\mathfrak{m}}$ acts R'-linearly on (10.3.3). The q-power Frobenius on $k = \mathbf{F}_q$ lifts to the identity on R and so by extension of scalars yields an R'-linear endomorphism F of $\mathbf{D}(G_k) \otimes R'$ that commutes with the $H_{\mathfrak{m}}$-action (as F is induced by the relative q-power Frobenius on G_k, and this map commutes with any endomorphism). We define

$$U_{R'} := \mathbf{D}(G_k^{\text{ét}}) \otimes R' \simeq \mathrm{Hom}_{R'}(T_p G_k \otimes_{\mathbf{Z}_p} R', R')$$

and

$$P_{R'} := \mathbf{D}(G_k^m) \otimes R' \simeq T_p {}^t\!G_k \otimes_{\mathbf{Z}_p} R'.$$

The splitting $G_k = G_k^m \times G_k^{\text{ét}}$ of G_k gives a decomposition

$$\mathbf{D}(G_k) \otimes R' = U_{R'} \oplus P_{R'}.$$

Moreover, the subspace $U_{R'}$ (respectively $P_{R'}$) is exactly the subspace of $\mathbf{D}(G_k) \otimes R'$ where F acts as an invertible (respectively a topologically nilpotent) endomorphism.

Lemma 10.3.1. *The composite of the inclusion $\omega_{G_{R'}} \to \mathbf{D}(G_k) \otimes R' = U_{R'} \oplus P_{R'}$ with the second projection $U_{R'} \oplus P_{R'} \to P_{R'}$ is an isomorphism of $H_{\mathfrak{m}}$-modules.*

Proof. First, we claim that the map $\omega_{G_{R'}} \to P_{R'}$ above is injective. It suffices to show that the intersection of $\omega_{G_{R'}}$ and $U_{R'}$ inside $\mathbf{D}(G_k) \otimes R'$ is zero. But the relative q-power Frobenius $F_{G_k/k} : G_k \to G_k^{(p)}$ kills ω_{G_k}, so since the formation of ω_G commutes with arbitrary base change, we see that $F\omega_{G_{R'}} \subseteq p\omega_{G_{R'}}$, and hence the intersection of $\omega_{G_{R'}}$ and $U_{R'}$ is zero since F acts as a unit on $U_{R'}$.

In order to show that $\omega_{G_{R'}} \to P_{R'}$ is surjective, it suffices to show that the reduction of this map is surjective (by Nakayama's Lemma). For dimension reasons (recall that G is ordinary), it is therefore enough to show that the reduction is injective. Since (10.3.3) shows that $\omega_{G_{R'}}$ is locally a direct summand of $\mathbf{D}(G_k) \otimes R'$, such injectivity is clear from the fact that the intersection of $\omega_{G_{R'}}$ and $U_{R'}$ is zero. ∎

Specializing Lemma 10.3.1 to the case $R' = R^{\text{nr}}$, we see that any

$$\beta \in R^{\text{nr}} \otimes T_p {}^t\!G_k \simeq P_{R^{\text{nr}}}$$

uniquely determines elements $p_\beta \in P_{R^{\text{nr}}}$ and $u_\beta \in U_{R^{\text{nr}}}$ and an invariant differential $\omega_\beta \in \omega_{G_{R^{\text{nr}}}}$ having the decomposition

$$\omega_\beta = u_\beta + p_\beta.$$

Similarly, any choice

$$\alpha \in R^{\mathrm{nr}} \otimes T_p G_k \simeq \mathrm{Hom}_{R^{\mathrm{nr}}}(U_{R^{\mathrm{nr}}}, R^{\mathrm{nr}})$$

determines an R^{nr}-linear homomorphism $\varphi_\alpha : U_{R^{\mathrm{nr}}} \to R^{\mathrm{nr}}$ which uniquely extends to an $R^{\mathrm{nr}} \otimes H_{\mathfrak{m}}$-linear map $\psi_\alpha : U_{R^{\mathrm{nr}}} \to \mathrm{Hom}_{R^{\mathrm{nr}}}(R^{\mathrm{nr}} \otimes H_{\mathfrak{m}}, R^{\mathrm{nr}})$ by the formula $\psi_\alpha(u)(h) = \psi_\alpha(hu)$.

Proposition 10.3.2 (Dwork)**.** *For any* $\alpha \in R^{\mathrm{nr}} \otimes T_p G_k$ *and* $\beta \in R^{\mathrm{nr}} \otimes T_p {}^t G_k$ *we have*

$$\log q_G(\alpha, \beta) = \psi_\alpha(u_\beta)$$

in $\mathrm{Hom}_{R^{\mathrm{nr}}}(R^{\mathrm{nr}} \otimes H_{\mathfrak{m}}, \pi R^{\mathrm{nr}})$.

Proof. See [22, Proposition 15.9] ■

Recall by our assumptions that $\mathrm{Hom}_{\mathbf{Z}_p}(H_{\mathfrak{m}}, \mathbf{Z}_p)$ and $T_p G_k = T_p G^{\text{ét}}$ are free $H_{\mathfrak{m}}$-modules. Let $h \in \mathrm{Hom}_{\mathbf{Z}_p}(H_{\mathfrak{m}}, \mathbf{Z}_p)$ be a basis and fix a basis $\alpha_1, \ldots, \alpha_g$ of the free $H_{\mathfrak{m}} \otimes R^{\mathrm{nr}}$-module $T_p G_k \otimes R^{\mathrm{nr}}$. For $1 \leq i \leq g$, let u_{α_i} be the unique element of $U_{R^{\mathrm{nr}}}$ satisfying

$$\psi_{\alpha_j}(u_{\alpha_i}) = \begin{cases} 1 \otimes h \text{ if } i = j \\ 0 \text{ otherwise} \end{cases} .$$

If we define $c(\alpha, \beta) \in \pi R^{\mathrm{nr}} \otimes H_{\mathfrak{m}}$ by the relation

$$\log q_G(\alpha, \beta) = c(\alpha, \beta) \cdot (1 \otimes h),$$

then as an immediate corollary of Proposition 10.3.2, we have the formula

$$\omega_\beta = \sum_{1 \leq i \leq g} c(\alpha_i, \beta) \cdot u_{\alpha_i} + p_\beta. \tag{10.3.4}$$

This enables us to "read off" the Serre-Tate invariant of G from the unit eigencomponents of invariant differentials on G.

Corollary 10.3.3. *Fix bases $\{\alpha_i\}_{1\le i\le g}$ and $\{\beta_j\}_{1\le j\le g}$ of the free $R^{\mathrm{nr}}\otimes H_{\mathfrak{m}}$-modules $R^{\mathrm{nr}}\otimes T_pG_k$ and $R^{\mathrm{nr}}\otimes T_p{}^tG_k$, respectively, and let I_G be the principal ideal of $R^{\mathrm{nr}}\otimes H_{\mathfrak{m}}$ generated by $\det(c(\alpha_i,\beta_j))$. Then I_G does not depend on the choices of bases.*

For any $\omega\in\omega_{GR'}$, denote by u_ω the F-unit eigencomponent of ω and let $V_{R'}$ be the $H_{\mathfrak{m}}\otimes R'$-submodule of $U_{R'}$ generated by the u_ω for all $\omega\in\omega_{GR'}$. Then I_G is the expansion of the ideal

$$\mathrm{Ann}_{R'\otimes H_{\mathfrak{m}}}\left(\wedge^g U_{R'}/\wedge^g V_{R'}\right) \tag{10.3.5}$$

of $R'\otimes H_{\mathfrak{m}}$ to $R^{\mathrm{nr}}\otimes H_{\mathfrak{m}}$.

Proof. The first part follows easily from the fact that $c(\alpha,\beta)$ is $R^{\mathrm{nr}}\otimes H_{\mathfrak{m}}$-bilinear (as $\log q_G$ is). When $R'=R^{\mathrm{nr}}$, the second is a restatement of (10.3.4). The general case follows from this one. ∎

So far, we have not used any of the results of this thesis in our discussion. Following Gross, we will now explain how to use Theorems 10.2.1–10.2.3 to "calculate" the invariants $c(\alpha,\beta)$ in terms of differentials on the curve X_K. We will then briefly indicate how Gross uses this to prove the companion form theorem.

Since X is admissible with generically smooth closed fiber, the composition of (10.2.3)–(10.2.6) is a diagram of exact sequences of free R-modules

$$\begin{tikzcd}
0 \arrow[r] & \omega_{A[p^\infty]} \arrow[r] \arrow[d, "\simeq"] & \mathbf{D}(A_k[p^\infty])\otimes R \arrow[r] \arrow[d, "\simeq"] & \mathrm{Lie}({}^tA[p^\infty]) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \omega_A \arrow[r] \arrow[d] & \mathrm{Lie}\,\mathscr{E}xtrig(A,\mathbf{G}_m) \arrow[r] \arrow[d] & \mathrm{Lie}({}^tA) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega_J \arrow[r] \arrow[d, "\simeq"] & \mathrm{Lie}\,\mathscr{E}xtrig(J,\mathbf{G}_m) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Lie}({}^tJ) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & H^0(X,\omega_X) \arrow[r] & H^1(X) \arrow[r] & H^1(X,\mathscr{O}_X) \arrow[r] & 0
\end{tikzcd} \tag{10.3.6}$$

that is H-compatible by Theorem 10.2.1.

Theorem 10.3.4. *For any complete local flat extension R' of R there is an isomorphism of exact sequences of $H_{\mathfrak{m}} \otimes R'$-modules*

$$\begin{tikzcd}
0 \arrow[r] & \omega_{G R'} \arrow[r] \arrow[d, "\simeq"] & \mathbf{D}(G_k) \otimes R' \arrow[r] \arrow[d, "\simeq"] & \mathrm{Lie}({}^tG)_{R'} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \epsilon_{\mathfrak{m}} H^0(X, \omega_{X/S})_{R'} \arrow[r] & \epsilon_{\mathfrak{m}} H^1(X)_{R'} \arrow[r] & \epsilon_{\mathfrak{m}} H^1(X, \mathscr{O}_X)_{R'} \arrow[r] & 0
\end{tikzcd}$$

Proof. For $R' = R$ this follows immediately from (10.3.6) by applying the idempotent $\epsilon_{\mathfrak{m}}$ and using our hypothesis that $\epsilon_{\mathfrak{m}} J = \epsilon_{\mathfrak{m}} A$. For general R', it follows by extension of scalars. ∎

Remark 10.3.5. This is the natural generalization of [22, Proposition 15.8].

It follows from Theorem 10.3.4 that the $H_{\mathfrak{m}} \otimes R'$-module $\epsilon_{\mathfrak{m}} H^1(X) \otimes R'$ is equipped with an R'-linear Frobenius endomorphism F that commutes with $H_{\mathfrak{m}}$ (by transport of structure from $\mathbf{D}(G_k) \otimes R'$). Thus, we have a decomposition

$$\epsilon_{\mathfrak{m}} H^1(X)_{R'} = U_{X,R'} \oplus P_{X,R'} \tag{10.3.7}$$

with $U_{X,R'}$ the sub $R' \otimes H_{\mathfrak{m}}$-module of $\epsilon_{\mathfrak{m}} H^1(X)_{R'}$ where F acts invertibly, and $P_{X,R'}$ the sub $R' \otimes H_{\mathfrak{m}}$-module where F acts as a topologically nilpotent endomorphism. Under the isomorphism $\epsilon_{\mathfrak{m}} H^1(X)_{R'} \simeq \mathbf{D}(G_k) \otimes R'$, the submodules $U_{X,R'}$ and $P_{X,R'}$ are identified with $U_{R'}$ and $P_{R'}$, respectively (by the definition of F on $\epsilon_{\mathfrak{m}} H^1(X)_{R'}$).

The homomorphism $\Psi : H \to R^{\mathrm{nr}}$ necessarily factors through some complete, local, flat R-algebra R_Ψ that is finite over R as a module, and hence gives an R_Ψ-linear homomorphism

$$H_{\mathfrak{m}} \otimes R_\Psi \to R_\Psi$$

that we also denote by Ψ. For any $H_{\mathfrak{m}} \otimes R_\Psi$-module M, we will write M^Ψ for the R_Ψ-submodule of M where $H_{\mathfrak{m}} \otimes R_\Psi$ acts through Ψ.

We will now make some further assumptions on the curve X in order to calculate the ideal I_G purely in terms of the cohomology of X. We suppose that there exists an open subscheme $U \subseteq X_k^{\mathrm{sm}}$ of the smooth locus of the closed fiber of X having schematic closure $\overline{U}$ in X_k and verifying the following properties:

1. There exists a set of correspondences $\mathscr{T}$ which generate H such that for each $T \in \mathscr{T}$ that is given by a pair of finite flat morphisms of admissible curves $\pi_1, \pi_2 : Y \rightrightarrows X$, we have $\pi_2\pi_1^{-1}(\overline{U}) \subseteq \overline{U}$ and $\pi_2\pi_1^{-1}(U) \subseteq U$.

2. There exists a correspondence $T_F \in \mathscr{T}$ given by a pair of finite flat morphisms of admissible curves $\pi_1, \pi_2 : Y \rightrightarrows X$ such that $\pi_2 : \pi_1^{-1}U \to U$ is an isomorphism, and under this identification, the map $\pi_1 : \pi_1^{-1}U \to U$ coincides with the relative q-power Frobenius. Moreover, we suppose that $\Psi(T_F) = q\xi$ with $\xi \in R_\Psi^\times$.

By our first assumption, each $T \in H$ acts on $H^1_{\mathrm{rig}}(U/K)$ as a K-linear endomorphism via §6.2. Let K_Ψ be the fraction field of R_Ψ and let

$$\eta : H^1_{\mathrm{dR}}(X_K/K_\Psi) \to H^1_{\mathrm{rig}}(U/K_\Psi)$$

be the map obtained from (10.2.8) by base change (using the isomorphism (4.2.10)); it is K_Ψ-linear by construction and T-equivariant for every $T \in H$ by Theorem 10.2.2.

As above, we define $H^1_{\mathrm{rig}}(U/K_\Psi)^\Psi$ as the subspace of $H^1_{\mathrm{rig}}(U/K_\Psi)$ where each $T \in \mathscr{T}$ acts as multiplication by the scalar $\Psi(T)$ (viewing T as an element of H). We obtain a K_Ψ-linear map on Ψ "eigenspaces"

$$\eta : H^1_{\mathrm{dR}}(X_K/K_\Psi)^\Psi \longrightarrow H^1_{\mathrm{rig}}(U/K_\Psi)^\Psi.$$

that is moreover T-equivariant for each $T \in H$.

Thanks to Corollary 4.2.8, we have a K-linear frobenius endomorphism F on $H^1_{\mathrm{rig}}(U/K)$, so by extension of scalars and base change in rigid cohomology, F extends to a K_Ψ-linear endomorphism of $H^1_{\mathrm{rig}}(U/K_\Psi)$, which we also denote by F. We claim that F stabilizes the subspace $H^1_{\mathrm{rig}}(U/K_\Psi)^\Psi$. Indeed, by our second assumption above and Corollary 6.1.9 we have $T_F F = q$, so if $v \in H^1_{\mathrm{rig}}(U/K_\Psi)^\Psi$ we see that

$$q \cdot v = T_F F(v) = F T_F(v) = \Psi(T_F) \cdot Fv$$

and it follows from our second assumption above that $Fv = \xi v$, and hence Fv is an element of $H^1_{\mathrm{rig}}(U/K_\Psi)^\Psi$, as desired. We therefore see that F restricts to a K_Ψ-linear endomorphism of the Ψ-"eigenspace" $H^1_{\mathrm{rig}}(U/K_\Psi)^\Psi$ that commutes with the action of each $T \in H$.

Theorem 10.3.6 (Gross)**.** *Assume that the restriction of η to the sub R_Ψ-module U^Ψ_{X,R_Ψ} of $H^1_{\mathrm{rig}}(U/K_\Psi)^\Psi$ is injective. Assume that U^Ψ_{X,R_Ψ} has rank g as an $R_\Psi \otimes H_{\mathfrak{m}}$-module and set*

$$I_\Psi := \mathrm{Ann}_{R_\Psi}\left(\frac{\wedge^g \eta\left(H^1(X)^\Psi_{R_\Psi}\right)}{\wedge^g \eta\left(H^0(X, \omega_{X/S})^\Psi_{R_\Psi}\right)}\right).$$

Then we have

$$I_\Psi \cdot R^{\mathrm{nr}} \subseteq \Psi(I_G).$$

Proof. Due to how $\mathfrak{m}$ is defined, we know that $H^1(X)^\Psi_{R_\Psi}$ is a sub R_Ψ-module of $\epsilon_{\mathfrak{m}} H^1(X)_{R_\Psi}$, and hence has an action of Frobenius F. Moreover, since F commutes with $H_{\mathfrak{m}}$, we see that U^Ψ_{X,R_Ψ} is exactly the subspace of $H^1(X)^\Psi_{R_\Psi}$ where F acts as a unit. Recall that the action of F on $H^1(X)^\Psi_{R_\Psi}$ is *defined* by transport of structure via the isomorphism $\epsilon_m H^1(X)_{R_\Psi} \simeq \mathbf{D}(G_k) \otimes R_\Psi$. Since the map (10.2.9) is F-equivariant by Theorem 10.2.3 and F acts on $H^1_{\mathrm{rig}}(U/K_\Psi)^\Psi$ as the unit ξ by the argument above, it follows that the restriction

$$\eta : H^1_{\mathrm{dR}}(X_K/K_\Psi)^\Psi \longrightarrow H^1_{\mathrm{rig}}(U/K_\Psi)^\Psi$$

kills P^Ψ_{X,R_Ψ}.

If we denote by V_{X,R_Ψ} the sub-$R_\Psi \otimes H_\mathfrak{m}$ module of U_{X,R_Ψ} generated by the F-unit eigencomponents of $\omega \in \epsilon_m H^0(X, \omega_{X/S})_{R_\Psi}$, then the isomorphism of Theorem 10.3.4 gives isomorphisms of $R_\Psi \otimes H_\mathfrak{m}$-modules

$$U_{X,R_\Psi} \simeq U_{R_\Psi} \subseteq \mathbf{D}(G_k) \otimes R_\Psi$$

and

$$V_{X,R_\Psi} \simeq V_{R_\Psi} \subseteq U_{R_\Psi}.$$

Thus, Corollary 10.3.3 tells us that the expansion of

$$\mathrm{Ann}_{R_\Psi \otimes H_\mathfrak{m}} \left(\wedge^g U_{X,R_\Psi} / \wedge^g V_{X,R_\Psi}\right).$$

to $R^{\mathrm{nr}} \otimes H_\mathfrak{m}$ is the ideal I_G. Since we have a commutative diagram of inclusions of free $R_\Psi \otimes H_\mathfrak{m}$-modules of rang g

$$\begin{tikzcd} U^\Psi_{X,R_\Psi} \arrow[r, hook] & U_{X,R_\Psi} \\ V^\Psi_{X,R_\Psi} \arrow[r, hook] \arrow[u, hook] & V_{X,R_\Psi} \arrow[u, hook] \end{tikzcd}$$

we have the inclusion of annhilator ideals

$$\mathrm{Ann}_{R_\Psi \otimes H_\mathfrak{m}} \left(\wedge^g U^\Psi_{X,R_\Psi} / \wedge^g V^\Psi_{X,R_\Psi}\right) \subseteq \mathrm{Ann}_{R_\Psi \otimes H_\mathfrak{m}} \left(\wedge^g U_{X,R_\Psi} / \wedge^g V_{X,R_\Psi}\right).$$

Since η is injective on U^Ψ_{X,R_Ψ}, the expansion of the $R_\Psi \otimes H_\mathfrak{m}$-ideal

$$\mathrm{Ann}_{R_\Psi \otimes H_\mathfrak{m}} \left(\wedge^g \eta\left(U^\Psi_{X,R_\Psi}\right) / \wedge^g \eta\left(V^\Psi_{X,R_\Psi}\right)\right)$$

to $R^{\mathrm{nr}} \otimes H_\mathfrak{m}$ is contained in I_G. But the restriction of η to Ψ-eigenspaces kills P^Ψ_{X,R_Ψ}, so we have

$$\eta\left(H^1(X)^\Psi_{R_\Psi}\right) = \eta\left(U_{X,R^\Psi_\Psi}\right)$$

and

$$\eta\left(H^0(X, \omega_{X/S})^\Psi_{R_\Psi}\right) = \eta\left(V_{X,R^\Psi_\Psi}\right),$$

and it follows that the expansion of

$$\mathrm{Ann}_{R_\Psi \otimes H_\mathfrak{m}} \left(\frac{\wedge^g \eta \left(H^1(X)^\Psi_{R_\Psi} \right)}{\wedge^g \eta \left(H^0(X, \omega_{X/S})^\Psi_{R_\Psi} \right)} \right)$$

to $R_\Psi \otimes H_\mathfrak{m}$ is contained in I_G. Since $R_\Psi \otimes H_\mathfrak{m}$ acts on Ψ-eigenspaces through Ψ, this gives the theorem. ■

Let us conclude with a brief discussion of how Gross uses Theorem 10.3.6 to prove the companion form theorem. Fix an integer N and a prime p not dividing N, and let $R = \mathbf{Z}_p[\zeta_p]$. Then the curve $X_K := X_1(Np)_{\mathbf{Q}_p(\zeta_p)}$ has an admissible model X over R whose closed fiber X_k consists of two irreducible components I and I' crossing transversally at a finite set of points Σ (the supersingular points in characteristic p) [22, §7]. We let H denote the Hecke algebra as defined in [22, §12]; it is a commutative subring of the ring of correspondences on X_K defined over $\mathbf{Q}_p(\zeta_p)$.

Let $f = \sum_n a_n q^n$ be a mod p normalized cuspidal eigenform for $\Gamma_1(N)$ with character ϵ. Assume that f has weight k satisfying $2 \leq k \leq p$, and that $a_p \neq 0$; when $k = p$ further assume that $a_p^2 \neq \epsilon(p)$. To keep our exposition simple, we will not treat the case $k = 2$, though this case does follow from Gross' work (and our compatibility theorems). By [22, Proposition 9.3], there exists a lifting of f to an eigenform F of weight 2 for $\Gamma_1(Np)$. Since F is an eigenform, it determines a homomorphism $\Psi : H \to R^{\mathrm{nr}}$; we let $\mathfrak{m}$ be the maximal ideal associated to this homomorphism as in 10.3.1. Keeping our notation from the beginning of this section, we let J_K be the Jacobian of X_K and A_K the maximal quotient of J_K having good reduction over R. We have $\epsilon_\mathfrak{m} J_K = \epsilon_\mathfrak{m} A_K$, so the p-divisible group G_K associated to $\mathfrak{m}$ extends to a p-divisible group G that is a subgroup of $A[p^\infty]$; moreover, G_k is ordinary [22, Proposition 12.9].

Under the assumptions on f, Gross shows that when the (global) mod p Galois representation attached to f (as in [22, §11]) is irreducible, T_pG^0, T_pG, and $T_pG^{\text{ét}}$ are free $R\otimes H_{\mathfrak{m}}$-modules of ranks 1, 2, and 1 respectively. Gross then shows (Proposition 13.2 and Proposition 14.7 of [22]) that if the local Galois representation associated to f (as in [22, §12]) splits, then the the ideal I_G of Corollary 10.3.3 is contained in $\mathfrak{m}\cdot(R^{\text{nr}}\otimes H_{\mathfrak{m}})$. Let F' be the eigenform of weight 2 and level Np defined as in [22, (9.10)], and let Ψ' be the associated homomorphism. By the definition of F', the maximal ideal associated to Ψ' is $\mathfrak{m}$. Moreover, the subscheme $U := I - \Sigma$ of X_k and the correspondence U_p satisfy the hypotheses (1) and (2) above. It follows from Theorem 10.3.6 that

$$I_{\Psi'} := \operatorname{Ann}_{R_{\Psi'}}\left(\frac{\eta\left(H^1(X)^{\Psi'}_{R_{\Psi'}}\right)}{\eta\left(H^0(X,\omega_{X/S})^{\Psi'}_{R_{\Psi'}}\right)}\right)$$

is contained in the ideal $\Psi'(I_G)$. Using this and the explicit description of the group $H^1_{\text{rig}}(U/K)$ as "overconvergent differentials on $V :=]U[_{\mathscr{X}^{\text{rig}}}\subseteq X_K^{\text{an}}$ modulo exact differentials" (which comes out of Čech theory), it follows that when the local Galois representation attached to f splits, the regular differential $\omega_{F'}\in H^0(X,\omega_{X/S})_{R_{\Psi'}}$ attached to F' satisfies

$$\omega_{F'} = \alpha\cdot v' + d\mathscr{F}$$

as (overconvergent) differentials on V. Here, v' is some overconvergent differential on V, $\mathscr{F}$ is a rigid analytic function on V with $|\mathscr{F}|\leq 1$, and $\alpha\in\pi\cdot R_{\Psi'}$. By working with expansions at ∞, Gross shows (see the proof of [22, Propostion 16.8]) that the reduction of $\mathscr{F}$, which is a function on $I-\Sigma$ and hence a mod p modular form of level N (by [22, Proposition 5.5]), is a companion form to f.

APPENDICES

APPENDIX A

Cohomology of topoi and ringed sites

As both crystalline cohomology and the cohomology of rigid spaces play an essential role in this work and neither is a ringed space theory in the classical sense, we recall how to define cohomology on a site. We have also used Čech theory to describe certain maps between the cohomology of sites, so we provide a reference for this as well. We refer primarily to [49], Exp. II, III, IV and [48], Exp V.

A.1 Generalities

Fix a site X and denote by $\operatorname{Cat} X$ the underlying category of X. Associated to X is the category $\mathscr{P}$ of presheaves of sets on X (that is, cofunctors from $\operatorname{Cat} X$ to the category of sets). We also have the topos $\mathscr{T}$ that is the category of sheaves of sets on X. This is the full subcategory of $\mathscr{P}$ consisting of those presheaves $\mathscr{F}$ that satisfy the sheaf axiom. The natural inclusion $i : \mathscr{T} \hookrightarrow \mathscr{P}$ is left exact (in the sense of left exactness in any category; i.e. it commutes with finite inverse limits) and has a right adjoint $\mathscr{F} \mapsto \mathscr{F}^{\#}$ called "sheafification" [49, Exp. II, 3.4]. The category $\mathscr{T}$ admits finite products and has a final object e, the sheafification of the presheaf that assigns $\{\emptyset\}$ to each object of $\operatorname{Cat}(X)$. The abelian group objects in $\mathscr{T}$ are precisely the abelian sheaves.

Let X and X' be two sites with associated topoi $\mathscr{T}$ and $\mathscr{T}'$. A *morphism of topoi* $u : \mathscr{T} \to \mathscr{T}'$ is a pair of functors $u_* : \mathscr{T} \to \mathscr{T}'$ and $u^* : \mathscr{T}' \to \mathscr{T}$, such that u^* is left adjoint to u_* and left exact; this implies by [49, Exp. IV, 3.1.2] that u^* is exact (in the sense of exactness in any category; i.e. it is left exact and preserves epimorphisms). A *morphism of sites* $X \to X'$ is a pair (f, u) consisting of a functor $f : \operatorname{Cat} X' \to \operatorname{Cat} X$ that is continuous (i.e. f takes covers to covers) and a morphism of topoi $u : \mathscr{T} \to \mathscr{T}'$ satisfying $u_*\mathscr{F} = \mathscr{F} \circ f$ for any $\mathscr{F} \in \mathscr{T}$ [49, Exp. IV, 4.9]. We note that the condition $u_*\mathscr{F} = \mathscr{F} \circ f$ shows that u is uniquely determined by f, if it exists.

For any object T of $\mathscr{T}$, we define the functor $\Gamma(T, \cdot)$ from $\mathscr{T}$ to the category of sets by $\Gamma(T, \cdot) = \operatorname{Hom}_{\mathscr{T}}(T, \cdot)$. If U is any object of $\operatorname{Cat} X$ and $\widetilde{U} \in \mathscr{T}$ denotes the sheafification of the presheaf $\operatorname{Hom}(\cdot, U)$, then Yoneda's Lemma shows that $\Gamma(\widetilde{U}, \mathscr{F}) = \mathscr{F}(U)$ for any $\mathscr{F} \in \mathscr{T}$. Clearly, for any morphism of topoi $u : \mathscr{T}' \to \mathscr{T}$ we have that u^* respects final objects: $u^*e = e'$. This follows from the fact that u^* commutes with finite inverse limits and the final object is the inverse limit over the empty category. Writing $\Gamma(\mathscr{T}, \cdot)$ for the functor $\Gamma(e, \cdot)$, for any $\mathscr{F}$ in $\mathscr{T}$ we have

$$\Gamma(\mathscr{T}, \mathscr{F}) = \varprojlim_{U \in \operatorname{Cat} X} \mathscr{F}(U).$$

In concrete terms, this just means that a global section of $\mathscr{F}$ is a compatible collection of sections $\mathscr{F}(U)$ for each $U \in \operatorname{Cat}(X)$; when X is a topological space and $\operatorname{Cat}(X)$ is the category of open sets of X (with final object X), this recovers the usual notion of global sections.

Let $\operatorname{Ab}(\mathscr{T})$ be the subcategory of abelian group objects in $\mathscr{T}$. The usual definitions of left exact and exact in the abelian category $\operatorname{Ab}(\mathscr{T})$ coincide with the more general ones in $\mathscr{T}$. Using Grothendieck's method of proof that the category of abelian sheaves on an arbitrary topological space has enough injectives [24, 1.10.1], one can show [1, II, 1.8] that $\mathrm{Ab}(\mathscr{T})$ has enough injectives. This enables us to define the right derived functors of any left exact functor; in particular, since the functor $\Gamma(\mathscr{T},\cdot)$ is left exact, we can define:

Definition A.1.1. For any object T of $\mathscr{T}$, the *ith cohomology functor of* T, denoted $H^i(T,\cdot)$, is the ith right-derived functor of $\Gamma(T,\cdot)$ on $\mathrm{Ab}(\mathscr{T})$.

Observe that $H^\bullet(T,\cdot)$ is a δ-functor. Moreover, there is a canonical extension of $H^i(T,\cdot)$ to the derived category of bounded below complexes in $\mathrm{Ab}(\mathscr{T})$, which we denote $\mathbf{H}^i(T,\cdot)$ (it is the ith hyper-derived functor of $\Gamma(T,\cdot)$). If $T=\widetilde{U}$ for an object U of $\mathrm{Cat}\,X$, we will often write $\mathbf{H}^i(U,\cdot)$ for $\mathbf{H}^i(T,\cdot)$.

Remark A.1.2. Let $u:\mathscr{T}'\to\mathscr{T}$ be any morphism of topoi, and set $F=\Gamma(\mathscr{T},\cdot)$ and $G=u_*$. The composite $F\circ G$ of these functors is

$$\Gamma(\mathscr{T},u_*(\cdot))=\mathrm{Hom}_{\mathscr{T}}(e,u_*(\cdot))=\mathrm{Hom}_{\mathscr{T}'}(u^*e,\cdot)=\mathrm{Hom}_{\mathscr{T}'}(e',\cdot)=\Gamma(\mathscr{T}',\cdot).$$

Since u_* has exact left adjoint u^* on the subcategory of abelian group objects, $u_*:\mathrm{Ab}(\mathscr{T}')\to\mathrm{Ab}(\mathscr{T})$ takes injectives to injectives, which are F-acyclic. The Leray spectral sequence for the composition of F and G then shows that $\mathbf{R}(F\circ G)=\mathbf{R}F\circ\mathbf{R}G$, or in other words, for any bounded below complex $\mathscr{F}^\bullet$ of abelian sheaves on $\mathscr{T}'$, the natural map

$$\mathbf{H}^\bullet(\mathscr{T}',\mathscr{F}^\bullet)\to\mathbf{H}^\bullet(\mathscr{T},\mathbf{R}u_*\mathscr{F}^\bullet)$$

is an isomorphism.

Now suppose that $\mathscr{F}^\bullet$ is a bounded below complex of abelian sheaves on X and consider a morphism of sites $X'\to X$ given by a functor $f:\mathrm{Cat}(X)\to\mathrm{Cat}(X')$ and a morphism of topoi $u:\mathscr{T}'\to\mathscr{T}$ satisfying $u_*\mathscr{F}=\mathscr{F}\circ f$. For any $T\in\mathscr{T}$ the canonical morphism $\mathscr{F}^\bullet \to u_* u^* \mathscr{F}^\bullet$ arising from the adjointness property of u^* and u_* induces a map

$$\text{(A.1.1)} \qquad \Gamma(T, \mathscr{F}^\bullet) = \mathrm{Hom}_{\mathscr{T}}(T, \mathscr{F}^\bullet) \to \mathrm{Hom}_{\mathscr{T}}(T, u_* u^* \mathscr{F}^\bullet) = \Gamma(f(T), u^* \mathscr{F}^\bullet).$$

Using the exactness of u^* and general properties of injective Cartan-Eilenberg resolutions as in [19, 0_{III}, 12.1.3.2] there is a canonical morphism

$$\text{(A.1.2)} \qquad \mathbf{H}^\bullet(T, \mathscr{F}^\bullet) \to \mathbf{H}^\bullet(f(T), u^* \mathscr{F}^\bullet)$$

recovering (A.1.1) in degree 0. Given a morphism of complexes $u^* \mathscr{F}^\bullet \to \mathscr{F}'^\bullet$ in $\mathscr{T}'$, composition of (A.1.2) with the map

$$\mathbf{H}^\bullet(f(T), u^* \mathscr{F}^\bullet) \to \mathbf{H}^\bullet(f(T), \mathscr{F}'^\bullet)$$

obtained by functoriality yields a natural map

$$\text{(A.1.3)} \qquad \mathbf{H}^\bullet(T, \mathscr{F}^\bullet) \to \mathbf{H}^\bullet(f(T), \mathscr{F}'^\bullet)$$

that we call *pullback* in hypercohomology (with respect to the given map of sites f).

A.2 Čech theory

We now wish to set up a suitable "Čech theory" for sites and in certain instances describe the morphism (A.1.3) in terms of this Čech theory. Fix a site X and a complex $\mathscr{F}^\bullet$ of abelian sheaves with $\mathscr{F}^j = 0$ for all $j < 0$. For any $U \in \mathrm{Cat}\, X$ and any covering $\mathscr{U} := \{U_i \to U\}_{i \in I}$ of U, we can form the "Čech double complex" $C^\bullet(\mathscr{U}, \mathscr{F}^\bullet)$ whose (i,j)-term is the group $C^i(\mathscr{U}, \mathscr{F}^j)$ of Čech i-cochains with respect to $\mathscr{U}$ having values in $\mathscr{F}^j$; this is functorial in $\mathscr{F}^\bullet$ *c.f.* [48, Exp. V, 2.3.3], and [19, 0_{III}, 12.4.5].

We define

$$\mathbf{H}^\bullet(\mathscr{U}, \mathscr{F}^\bullet) := H^\bullet_{\mathrm{tot}}(C^\bullet(\mathscr{U}, \mathscr{F}^\bullet)).$$

Using the mapping properties of Cartan-Eilenberg double complexes as in [19, 0_{III}, 12.4.5] yields a canonical morphism, functorial in $\mathscr{F}^\bullet$,

$$\mathbf{H}^\bullet(\mathscr{U}, \mathscr{F}^\bullet) \to \mathbf{H}^\bullet(U, \mathscr{F}^\bullet). \tag{A.2.1}$$

Proposition A.2.1. *Suppose that for all i and j one has $\mathbf{H}^n(U_i, \mathscr{F}^j) = 0$ for all $n > 0$. Then* (A.2.1) *is an isomorphism.*

Proof. In the case that X is a ringed (topological) space, this is [19, 0_{III}, Corollaire 12.4.7]. The proof given there carries over verbatim to our more general situation. ∎

If $X' \to X$ is a morphism of sites with $f : \operatorname{Cat} X \to \operatorname{Cat} X'$ the corresponding functor on underlying categories, and $u : \mathscr{T}' \to \mathscr{T}$ is a compatible morphism of topoi, the canonical map (A.1.1) induces a morphism

$$C^\bullet(\mathscr{U}, \mathscr{F}^\bullet) \to C^\bullet(\mathscr{U}', u^*\mathscr{F}^\bullet),$$

where $\mathscr{U}' = f(\mathscr{U})$ is the cover of $f(U)$ obtained by applying f to $\mathscr{U}$ (using the continuity of f to know that it takes covers to covers). We thus get a map on Čech cohomology

$$\mathbf{H}^\bullet(\mathscr{U}, \mathscr{F}^\bullet) \to \mathbf{H}^\bullet(\mathscr{U}', u^*\mathscr{F}^\bullet). \tag{A.2.2}$$

Proposition A.2.2. *The diagram*

$$\begin{tikzcd}
\mathbf{H}^\bullet(\mathscr{U}, \mathscr{F}^\bullet) \arrow[r, "(A.2.2)"] \arrow[d, "(A.2.1)"'] & \mathbf{H}^\bullet(\mathscr{U}', u^*\mathscr{F}^\bullet) \arrow[d, "(A.2.1)"] \\
\mathbf{H}^\bullet(U, \mathscr{F}^\bullet) \arrow[r, "(A.1.2)"'] & \mathbf{H}^\bullet(f(U), u^*\mathscr{F}^\bullet)
\end{tikzcd}$$

commutes.

Proof. The commutativity is proved in exactly the same way as in [19, 0_{III}, 12.1.4.2.]; see also [19, 0_{III}, 12.4.8]. ∎

Combining this with the natural map guaranteed by the functoriality of (A.2.1), we see that for a morphism of complexes $u^*\mathscr{F}^\bullet \to \mathscr{F}'^\bullet$ we obtain a commutative diagram

(A.2.3)
$$\begin{tikzcd}
\mathbf{H}^\bullet(\mathscr{U}, \mathscr{F}^\bullet) \arrow[r, "\text{(A.2.2)}"] \arrow[d] & \mathbf{H}^\bullet(\mathscr{U}', \mathscr{F}'^\bullet) \arrow[d] \\
\mathbf{H}^\bullet(U, \mathscr{F}^\bullet) \arrow[r, "\text{(A.1.3)}"'] & \mathbf{H}^\bullet(f(U), \mathscr{F}'^\bullet)
\end{tikzcd}$$

and when the vertical arrows are isomorphisms, this gives a Čech-theoretic description of the bottom map.

APPENDIX B

Technical results

B.1 The functor $j^\dagger$

We prove two results concerning the functor $j^\dagger$ that were used in §6.1. As usual, we fix a complete discrete valuation ring R with fraction field K of characteristic zero and residue field k of characteristic $p > 0$. We denote by π a uniformizer of R and by $|\cdot|$ the absolute value on K.

Lemma B.1.1. *Suppose given the diagram* (6.1.10)*, and set* $f = u^{\mathrm{rig}}$*. If* $\overline{X}'$ *is the full preimage of* $\overline{X}$ *and* X' *is the full preimage of* X *in* $\mathscr{X}'$*, then the inductive system* $f^{-1}(V)$ *as* V *ranges over all strict open neighborhoods of* $]X[$ *in* $]\overline{X}[$ *is cofinal among strict open neighborhoods of* $]X'[$ *in* $]\overline{X}'[$.

Proof. We use the explicit description of a fundamental system of strict open neighborhoods as in [3], 1.2.4, which we now summarize.

Fix a proper formal Spf R-scheme $\mathscr{P}$, a closed subscheme W of the special fiber $\mathscr{P}_0$, and an open subscheme U of W. Let $Z = W - U$ be the complement of U in W. For $\lambda \in |\overline{K}^\times|$ with $|\pi| \leq \lambda < 1$, we denote by $[W]_{\mathscr{P},\lambda}$ the closed tube of W in $\mathscr{P}^{\mathrm{rig}}$ or radius λ as defined in (4.2.2). We define the open tube of Z in $\mathscr{P}^{\mathrm{rig}}$ of radius λ, denoted $]Z[_{\mathscr{P},\lambda}$ exactly as in (4.2.2), except we replace the non-strict inequalities in that definition with strict ones. We note that since $\lambda \geq |\pi|$, these admissible open subsets of $\mathscr{P}^{\mathrm{rig}}$ are intrinsic to W and Z respectively (given λ).

Set $U_\lambda =]W[_{\mathscr{P}} -]Z[_{\mathscr{P},\lambda}$. For any sequences $\underline{\lambda} = \{\lambda_n\}$ and $\underline{\eta} = \{\eta_n\}$ of elements of $|\overline{K}^\times|$ with $|\pi| \leq \lambda_n,\ \eta_n$ for all n and which tend to 1 through strictly increasing values, put

$$V_n := [W]_{\eta_n} \cap U_{\lambda_n}$$

and set

$$V_{\underline{\lambda},\underline{\eta}} = \bigcup_n V_n.$$

It is not hard to show that $V_{\underline{\lambda},\underline{\eta}}$ is a strict open neighborhood of $]U[_{\mathscr{P}}$ in $]W[_{\mathscr{P}}$. For fixed $\underline{\eta}$ and variable $\underline{\lambda}$, the strict open neighborhoods $V_{\underline{\lambda},\underline{\eta}}$ form a fundamental system of strict open neighborhoods of $]U[_{\mathscr{P}}$ in $]W[_{\mathscr{P}}$; see [3, 1.2.4]

Now let $V_{\underline{\lambda},\underline{\eta}}$ be the fundamental system of strict open neighborhoods described above associated to $U = X$, $W = \overline{X}$, and let $V'_{\underline{\lambda},\underline{\eta}}$ be the system of strict open neighborhoods associated to $U = X'$, $W = \overline{X}'$. Then since for any $|\pi| \leq \lambda \leq 1$ and any locally closed subset U of the special fiber of $\mathscr{X}$ one has

$$f^{-1}(]U[_{\mathscr{X},\lambda}) =]u^{-1}(U)[_{\mathscr{X}',\lambda} \qquad\qquad f^{-1}([U]_{\mathscr{X},\lambda}) = [u^{-1}(U)]_{\mathscr{X}',\lambda},$$

the hypotheses that X' is the full preimage of X in $\mathscr{X}'$ and $\overline{X}'$ is the full preimage of $\overline{X}$ ensure that

$$f^{-1}(V_{\underline{\lambda},\underline{\eta}}) = V'_{\underline{\lambda},\underline{\eta}},$$

and this proves our claim. ■

Lemma B.1.2. *Let X be a quasi-compact and quasi-separated rigid space. Then the functors $H^i(X,\cdot)$ on abelian sheaves commute with filtered direct limits. Consequently, if $f : X \to Y$ is any quasi-compact and quasi-separated morphism of rigid spaces then the functors $R^i f_*$ commute with filtered inductive limits for all i.*

Proof. This is [3, (0.1.8)], and follows from [48, Exp. VI, 5.1–5.2]. ■

Remark B.1.3. Let X be a k-scheme and $j : X \hookrightarrow \overline{X}$ a compactification of X. Suppose further that we are given a closed immersion $\overline{X} \hookrightarrow \mathscr{P}$ of $\overline{X}$ into a topologically finite type formal R-scheme. Fix an abelian sheaf $\mathscr{E}$ on $]\overline{X}[_{\mathscr{P}}$. As a simple application of Lemma B.1.2, we can give a slightly more explicit description of the sections of $j^\dagger\mathscr{E}$ over any quasicompact open $W \subseteq]\overline{X}[_{\mathscr{P}}$. Indeed, by Lemma B.1.2 we have

$$\Gamma(W, j^\dagger\mathscr{E}) = H^0(W, j^\dagger\mathscr{E}) = \varinjlim_V H^0(W, j_{V*}j_V^{-1}\mathscr{E}) = \varinjlim_V \Gamma(W \cap V, \mathscr{E}).$$

Corollary B.1.4. *Suppose given the diagram* (6.1.10)*.If $\mathscr{E}$ is any abelian sheaf on $]\overline{X}'[_{\mathscr{X}'}$ that is u_*^{rig}-acyclic, then $j_{X'}^\dagger\mathscr{E}$ is u_*^{rig}-acyclic.*

Proof. We begin by showing that for any injective abelian sheaf $\mathscr{I}$ on $]\overline{X}'[_{\mathscr{X}'}$, the sheaf $j_{X'}^\dagger\mathscr{I}$ is u_*^{rig}-acyclic. Since u is proper, Lemma B.1.2 ensures that $R^iu_*^{\mathrm{rig}}$ commutes with filtered direct limits, so in particular

$$R^iu_*^{\mathrm{rig}}\left(j_{X'}^\dagger\mathscr{I}\right) = \varinjlim_V R^iu_*^{\mathrm{rig}}\left(j_{V*}j_V^{-1}\mathscr{I}\right),$$

with the limit taken over all strict open neighborhoods V of $]X'[$ in $]\overline{X}'[_{\mathscr{X}'}$ Now for each V, the functor j_V^{-1} takes injectives to injectives as V is open in $]\overline{X}'[$, and j_{V*} carries injectives to injectives because it has exact left adjoint j_V^{-1}. We therefore conclude that $j_{X'}^\dagger$ carries injectives to u_*^{rig}-acyclics.

Now suppose that $\mathscr{E}$ is any abelian sheaf on $]\overline{X}'[$ that is u_*^{rig}-acyclic, and let $\mathscr{I}^\bullet$ be an injective resolution of $\mathscr{E}$. Since $j_{X'}^\dagger$ is exact on abelian sheaves [3, 2.1.3] and takes injectives to u_*^{rig}-acyclics, we may calculate the higher direct images $R^iu_*^{\mathrm{rig}}\left(j_{X'}^\dagger\mathscr{I}\right)$ as the cohomology of the complex $u_*^{\mathrm{rig}}j_{X'}^\dagger\mathscr{I}^\bullet$. Now the proof of Lemma 6.1.7 shows that

$$u_*^{\mathrm{rig}}j_{X'}^\dagger\mathscr{I}^\bullet = j_X^\dagger u_*^{\mathrm{rig}}\mathscr{I}^\bullet,$$

and since $\mathscr{E}$ is u_*^{rig}-acyclic, we know that $u_*^{\mathrm{rig}}\mathscr{I}^\bullet$ is exact. Since $j_X^\dagger$ is an exact functor, we conclude that $u_*^{\mathrm{rig}}j_{X'}^\dagger\mathscr{I}^\bullet$ is exact and $R^i u_*^{\mathrm{rig}}\left(j_{X'}^\dagger\mathscr{I}\right) = 0$ for all $i > 0$, as desired. ■

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] M. Artin. Grothendieck topologies. Seminar notes, Harvard University, Cambridge, Mass., 1962. Notes on a seminar by M. Artin, spring 1962, Harvard Department of Mathematics.

[2] M. Artin. Lipman's proof of resolution of singularities for surfaces. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 267–287. Springer, New York, 1986.

[3] P. Berthelot. Cohomologie rigide et cohomologie rigide à supports propres: Première partie, 1996.

[4] Pierre Berthelot. *Cohomologie cristalline des schémas de caractéristique* $p > 0$. Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 407.

[5] Pierre Berthelot. Finitude et pureté cohomologique en cohomologie rigide. *Invent. Math.*, 128(2):329–377, 1997. With an appendix in English by Aise Johan de Jong.

[6] Pierre Berthelot, Lawrence Breen, and William Messing. *Théorie de Dieudonné cristalline. II*, volume 930 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[7] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J., 1978.

[8] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.

[9] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[10] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. Formal and rigid geometry. III. The relative maximum principle. *Math. Ann.*, 302(1):1–29, 1995.

[11] Robert Coleman and Adrian Iovita. The Frobenius and monodromy operators for curves and abelian varieties. *Duke Math. J.*, 97(1):171–215, 1999.

[12] Robert F. Coleman. Duality for the de Rham cohomology of an abelian scheme. *Ann. Inst. Fourier (Grenoble)*, 48(5):1379–1393, 1998.

[13] B. Conrad. Arithmetic moduli of generalized elliptic curves, 2005.

[14] Brian Conrad. Irreducible components of rigid spaces. *Ann. Inst. Fourier (Grenoble)*, 49(2):473–541, 1999.

[15] Brian Conrad. *Grothendieck duality and base change*, volume 1750 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[16] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 143–316. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973.

[17] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[18] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[19] J. Dieudonné and A. Grothendieck. Éléments de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.*, (4,8,11,17,20,24,28,37), 1960–7.

[20] Renée Elkik. Rationalité des singularités canoniques. *Invent. Math.*, 64(1):1–6, 1981.

[21] Jean Fresnel and Marius van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2004.

[22] Benedict H. Gross. A tameness criterion for Galois representations associated to modular forms (mod p). *Duke Math. J.*, 61(2):445–517, 1990.

[23] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

[24] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[25] Alexander Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA* 2). North-Holland Publishing Co., Amsterdam, 1968. Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2.

[26] Robin Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.

[27] Nicholas M. Katz. Crystalline cohomology, Dieudonné modules, and Jacobi sums. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 165–246. Tata Inst. Fundamental Res., Bombay, 1981.

[28] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[29] Reinhardt Kiehl. Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie. *Invent. Math.*, 2:256–273, 1967.

[30] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[31] Joseph Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.*, (36):195–279, 1969.

[32] Qing Liu, Dino Lorenzini, and Michel Raynaud. Néron models, Lie algebras, and reduction of curves of genus one. *Invent. Math.*, 157(3):455–518, 2004.

[33] W. Lütkebohmert. On compactification of schemes. *Manuscripta Math.*, 80(1):95–111, 1993.

[34] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[35] B. Mazur and William Messing. *Universal extensions and one dimensional crystalline cohomology*. Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 370.

[36] B. Mazur and K. A. Ribet. Two-dimensional representations in the arithmetic of modular curves. *Astérisque*, (196-197):6, 215–255 (1992), 1991. Courbes modulaires et courbes de Shimura (Orsay, 1987/1988).

[37] J. S. Milne. *Arithmetic duality theorems*, volume 1 of *Perspectives in Mathematics*. Academic Press Inc., Boston, MA, 1986.

[38] J. S. Milne. Jacobian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 167–212. Springer, New York, 1986.

[39] P. Monsky and G. Washnitzer. Formal cohomology. I. *Ann. of Math. (2)*, 88:181–217, 1968.

[40] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[41] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[42] F. Oort. *Commutative group schemes*, volume 15 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1966.

[43] M. Raynaud. Spécialisation du foncteur de Picard. *Inst. Hautes Études Sci. Publ. Math.*, (38):27–76, 1970.

[44] Michel Raynaud and Laurent Gruson. Critères de platitude et de projectivité. Techniques de "platification" d'un module. *Invent. Math.*, 13:1–89, 1971.

[45] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[46] Jean-Pierre Serre. *Groupes algébriques et corps de classes*. Publications de l'Institut Mathématique de l'Université de Nancago [Publications of the Mathematical Institute of the University of Nancago], 7. Hermann, Paris, second edition, 1984. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], 1264.

[47] *Schémas en groupes. I: Propriétés générales des schémas en groupes*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151. Springer-Verlag, Berlin, 1970.

[48] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269.

[49] *Théorie des topos et cohomologie étale des schémas. Tome 2*. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 270.

ABSTRACT

Correspondences, Integral Structures, and Compatibilities in p-adic Cohomology

by

Bryden R. Cais

Fix a discrete valuation ring R with fraction field K of characteristic zero and residue field k. In this thesis, we study the Hodge filtration of the first de Rham cohomology of a smooth, proper, and geometrically connected curve X_K over K and the endomorphisms of it induced by correspondences on the curve. Our investigations proceed along two courses: in one direction, we equip the Hodge filtration with a canonical integral structure, i.e. a short exact sequence of free R-modules that is functorial in finite K-morphisms of X_K and recovers the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$ after extending scalars to K. In the other direction, we suppose that k has characteristic $p > 0$ and employ techniques from rigid and formal geometry to study the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$—together with its canonical integral structure—via p-adic cohomology.

We give two different constructions of a canonical integral structure on the Hodge filtration of $H^1_{\mathrm{dR}}(X_K/K)$. One construction uses certain models of X_K over R and Grothendieck duality theory, while the other uses the Néron model of the Jacobian of X_K and the theory of rigidified extensions of Grothendieck and Mazur-Messing. We

will prove under mild hypotheses that these different constructions yield the same integral structure.

When k has characteristic $p > 0$, we can attach several different p-adic cohomology theories to X_K. We will study the comparison maps between these theories, and will be especially concerned with their interaction with endomorphisms induced by correspondences on X_K and Frobenius.

A significant impetus for writing this thesis comes from Gross' beautiful paper [22] on Galois representations. In that paper, Gross' main theorem is conditional on certain compatibilities between p-adic cohomology theories. As a consequence of our work, we resolve these compatibility issues. Our work concerning canonical integral structures moreover provides a reference for several results that play a key role in [22] but that we have been unable to find in the literature.