ERRATUM TO BREUIL–KISIN MODULES VIA CRYSTALLINE COHOMOLOGY

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In [1, Lemma 5.8 (3)], we made the following claim:

Lemma 1. *Let M be a finitely generated A_{inf}-module. The map $M \to W(F) \otimes_{A_{\mathrm{inf}}} M$ is injective if and only if M has no u-torsion.*

This assertion is false: the A_{inf}-module $M = A_{\mathrm{inf}}/E(u)A_{\mathrm{inf}} \simeq \mathscr{O}_{\mathbf{C}_p}$ is u-torsion free as $u \in A_{\mathrm{inf}}$ acts on $M \simeq \mathscr{O}_{\mathbf{C}_p}$ as multiplication by π, but one has $M \otimes_{A_{\mathrm{inf}}} W(F) = 0$ as $E(u)$ is a unit in $W(F)$. In particular, $M \to W(F) \otimes_{A_{\mathrm{inf}}} M$ is not injective.

A corrected version of this Lemma, which will be sufficient for our applications, is:

Lemma 2. *Let M be a finitely presented A_{inf}-module with $M[\frac{1}{p}]$ finite and free over $A_{\mathrm{inf}}[\frac{1}{p}]$. If M is u-torsion free then the map $M \to W(F) \otimes_{A_{\mathrm{inf}}} M$ is injective.*

Proof. For ease of notation, if N is any A_{inf}-module, we will write ι_N for the natural map $N \to N \otimes_{A_{\mathrm{inf}}} W(F)$. As $M[\frac{1}{p}]$ is finite and free over $A_{\mathrm{inf}}[\frac{1}{p}]$, one has an exact sequence as in [1, (5.8)], which may be split as two short exact sequences:

$$0 \longrightarrow M_{\mathrm{tor}} \longrightarrow M \longrightarrow M' \longrightarrow 0 \tag{1a}$$

$$0 \longrightarrow M' \longrightarrow M_{\mathrm{free}} \longrightarrow \overline{M} \longrightarrow 0 \tag{1b}$$

where M_{tor} is killed by p^n for some n, and M_{free} is a finite free A_{inf}-module. Since $W(F)$ is flat over A_{inf} thanks to [1, Lemma 5.8 (1)], the sequences (1a) and (1b) remain short exact after extending scalars to $W(F)$. It follows that to prove ι_M is injective, it suffices to prove that $\iota_{M'}$ and $\iota_{M_{\mathrm{tor}}}$ are both injective. Now $\iota_{M_{\mathrm{free}}}$ is certainly injective as M_{free} is free over A_{inf}, so $\iota_{M'}$ is injective thanks to (1b) and what we have observed above. It remains to prove that $\iota_{M_{\mathrm{tor}}}$ is injective, so without loss of generality we may reduce to the case that M is killed by p^n for some n. We will show that ι_M is injective by induction on n. First suppose $n = 1$. Then M is finitely presented as a module over the valuation ring $R = A_{\mathrm{inf}}/pA_{\mathrm{inf}}$, whence it is a direct sum of a finite free R-module and a module of the form $\bigoplus_{i=1}^m R/a_iR$ with $a_i \in R$ nonzero [2, Tag 0ASU]; *cf.* the proof of [1, Lemma 5.10]. As M is assumed to be u-torsion free, we must have $a_i \in R^\times$ for all i, and M is finite and free over R. It follows at once that $\iota_M : M \to M \otimes_R F$ is

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injective. Now suppose that M is killed by p^n for some $n > 1$ and consider the exact sequence

$$0 \longrightarrow M[p] \longrightarrow M \longrightarrow M' \longrightarrow 0, \tag{2}$$

where $M[p] = \{x \in M \ : \ px = 0\}$. We claim that M' has no u-torsion. Indeed, if $y \in M$ has $uy \in M[p]$ then $puy = u(py) = 0$. Since M has no u-torsion, this forces $py = 0$ and $y \in M[p]$, so that M' is u-torsion free. Since $M[p]$ is also u-torsion free (being a submodule of a u-torsion free module) and both $M[p]$ and M' are killed by p^{n-1}, our inductive hypothesis gives that $\iota_{M[p]}$ and $\iota_{M'}$ are injective, and it follows that ι_M is injective as well. □

Replacing Lemma 1 with its corrected version Lemma 2 necessitates that we modify [1, Lemma 5.10] and its proof. First, the hypotheses (2) that M' is u-torsion free must be strengthened to:

Hypothesis (2′). M' is a finitely presented A_{inf}-module that is u-torsion free with $M'[\frac{1}{p}]$ finite and free over $A_{\mathrm{inf}}[\frac{1}{p}]$.

Then the following changes must be made in the proof of [1, Lemma 5.10]. To begin with, we no longer assume that f is surjective, whence the three rows in the large commutative diagram may no longer be right exact. This has no impact on the remainder of the argument, as such surjectivity is never used in the proof. In other words, removing the first sentence of the proof of [1, Lemma 5.10] "Replacing M' by $f(M)$, we may assume that f is surjective and hence also that M' is finitely generated," the word "then" from the following sentence, and the zeroes (together with the arrows that point to them) in the rightmost column of the large diagram results in a correct proof of the modified statement.

The sole application of [1, Lemma 5.8 (3)] is to the proof of Proposition 5.7 of *loc. cit.*, in which we only need the corrected version Lemma 2 since the modules M to which we apply the Lemma are all finitely presented A_{inf}-modules that are u-torsion free with $M[\frac{1}{p}]$ finite and free over $A_{\mathrm{inf}}[\frac{1}{p}]$. In particular, no further changes to [1] are necessary.

References

[1] Cais, Bryden and Liu, Tong. *Breuil-Kisin modules via crystalline cohomology*, Trans. Amer. Math. Soc. **371** (2019), 1199–1230.

[2] The Stacks project authors. *The Stacks project*, `https://stacks.math.columbia.edu` (2019).

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