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Bounded Solutions of Perturbed Volterra Integrodifferential Systems

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1. PRELIMINARIES

Many recent papers (e.g., [1, 2, 4, 6–9] and the references cited therein) have dealt with the existence of bounded solutions of perturbed Volterra integrodifferential systems of the form

$$x'(t) = A(t)\,x(t) + \int_0^t B(t,s)\,x(s)\,ds + p(t)(x), \qquad t > 0. \tag{P}$$

(Here, the perturbation term $p(t)(x)$ is an operator about which more is specifically stated below.) Most of these papers consider (P) under boundedness or other various stability assumptions on the related linear system

$$y'(t) = A(t)\,y(t) + \int_0^t B(t,s)\,y(s)\,ds, \qquad t > 0. \tag{L}$$

More recently (P) has been studied under the weaker assumption that (L) is admissible with respect to certain pairs of spaces. The author [2] proved under such admissibility assumptions that locally the set of bounded solutions of (P) is homeomorphic to the set of bounded solutions of (L), provided p satisfies a Lipschitz-type condition in x. Other authors [1, 6] have obtained similar and related results using a different approach for a restricted class of systems ($A(s) = A$, $B(t,s) = B(t-s)$). Our purpose here is to extend our previous results in [2] to a broader class of perturbations; the expense of this generalization is less structure for the set of bounded solutions of (P) as related to the corresponding set for (L). This set will no longer necessarily be locally homeomorphic to that of (L), but nevertheless will be "at least as large;" we make this precise by using the notation of locally nonempty partitions with respect to an index set as defined below. Our main results appear in Theorem 1.

For simplicity we assume throughout that $A(t)$ and $B(t, s)$ are continuous $n \times n$ matrix valued functions defined on $t \geqslant 0$ and $t \geqslant s \geqslant 0$, respectively. This guarantees that (L) has, for each assigned initial condition $y(0) = y_0 \in R^n$, a unique solution defined for all $t \geqslant 0$. These continuity assumptions, however, could be considerably weakened, for all that is essentially needed below is the global existence and uniqueness of the initial value problem for (L), properties valid under less restrictive conditions on A and B [7]. By a solution of (P) we mean an absolutely continuous function $x(t)$: $R^+ \to R^n$ that reduces (P) to an identity for almost all $t > 0$.

The key concept we use is that of admissibility, by which we mean exactly that defined by Massera and Schäffer [5] for differential systems. Namely, we say (L) is *(B_1, B_2)-admissible* for Banach function spaces B_1 and B_2, if for every $g(t)$ in B_1, there exists *at least one* solution of the nonhomogeneous system

$$z'(t) = A(t)\,z(t) + \int_0^t B(t, s)\,z(s)\,ds + g(t), \qquad t > 0, \tag{NH}$$

in B_2. In this paper we will always take $B_2 = BC$, the Banach space of bounded, continuous functions of R^+ with norm $|z|_0 = \sup_{t\geqslant 0} |z(t)|$. We will also consider the Banach spaces L^p, $1 < p < \infty$, consisting of all functions measurable in $t \geqslant 0$ for which $|z|_p = (\int_0^\infty |z(t)|^p\,dt)^{1/p} < \infty$, and M^p, $1 < p < \infty$, consisting of all functions measurable in $t \geqslant 0$ for which $|z|_{m,p} = \sup_{t\geqslant 0}(\int_t^{t+1} |z(s)|^p\,ds)^{1/p} < \infty$. The latter spaces are frequently used in studying perturbation problems for differential systems. Note that L^p is a proper subset of M^p. For notational convenience, we will also denote BC by L^∞. The Banach space $BC = L^\infty$ is a subspace of the convex Fréchet space C of continuous functions for $t \geqslant 0$ under the (so called compact open) topology of uniform convergence on compact intervals. Note that BC has a stronger topology than that of C. For convenience, all functions are assumed to vanish for all $t < 0$.

Let T be any linear topological space and let Ω_1, Ω_2 be any subsets with Ω_2 having a topology equivalent to or stronger than that induced from T. We say Ω_1 has a partition with respect to Ω_2 if there exists a collection $\{K_\omega \mid \omega \in \Omega_2\}$ of subsets of T indexed by Ω_2 such that $\bigcup_{\Omega_2} K_\omega = \Omega_1$ and $K_\omega \cap K_{\omega'} = \varnothing$ for all $\omega, \omega' \in \Omega_2$, $\omega \neq \omega'$. A partition of Ω_1 with respect to Ω_2 is called *locally nonempty* if $0 \in \Omega_2$ and there exists a neighborhood U of 0 in the topology of Ω_2 such that $K_\omega \neq \varnothing$ for all $\omega \in U$. Notice that if Ω_1 has a locally nonempty partition then Ω_1 is nonempty since U always contains at least the element 0.

2. The Main Result

We need the following hypothesis on the perturbation term p, which, without loss of generality, we write as $p(t)(x) \equiv h(t)(x) + g(t)$, $h(t)(0) \equiv 0$ for $t \geqslant 0$. Let Y denote any one of the spaces L^p or M^p.

H1. There exists a constant r, $0 < r \leqslant \infty$, such that $p(t)(x)$ maps the ball $\{x \in BC: | x |_0 \leqslant r\}$ continuously into Y in the compact open topology of C in such a way that $| h(t)(x)|_Y \leqslant \theta | x |_0$ for some constant $\theta > 0$ and all $x \in BC$ satisfying $| x |_0 \leqslant r$.

The following theorem contains our main result concerning the existence of bounded solutions of (P).

Theorem 1. *Let Y be one of the spaces L^p, $1 < p \leqslant \infty$, or M^p, $1 < p < \infty$. Assume that the perturbation term $p(t)(x) = h(t)(x) + g(t)$, $h(t)(0) = 0$, satisfies* H1 *and that* (L) *is (Y, BC)-admissible. Then there exist constants $\theta_0 > 0$ and $r_0 > 0$ such that, for each $g \in Y$ satisfying $| g |_Y \leqslant r_0$, the set of bounded solutions of* (P) *has a locally nonempty partition with respect to the set of bounded solutions of* (L) *provided $\theta \leqslant \theta_0$.*

In the following remarks we discuss the two hypotheses on p and (L) contained in this theorem.

Remark 1. The perturbation term $p(t)(x)$ is frequently of higher order in x: $| p(t)(x)|_Y = o(| x |_0)$ uniformly in $t \geqslant 0$. This is the case, of course, when the approach to a given system is that of "linearization" or "first approximation." We point out that in this case the condition that θ be sufficiently small is always met, provided only that the constant r is taken sufficiently small in H1. Specifically, $| h(t)(x)|_Y/| x |_0 \to 0$ as $| x |_0 \to 0$ uniformly in $t \geqslant 0$ implies that for some $r > 0$ we have $| h(t)(x)|_Y \leqslant \theta_0 | x |_0$ for $| x |_0 \leqslant r$ uniformly in $t \geqslant 0$. Thus, Theorem 1 is always locally valid for higher order perturbations in (P). Of course, H1 does not, however, rule out linear perturbations.

A typical, but broad class of perturbations is included in the example

$$p(t)(x) = f(t, x(t)) + \int_0^t h(t, s, x(s), x(t))\, ds + g(t),$$

where

$$| f(t, z)| \leqslant \alpha(t)| z |^n, \qquad | h(t, x, z, w)| \leqslant \beta(t, s)| z |^m | w |^q$$

for $t \geqslant 0$, $| z | \leqslant r$ and $t \geqslant s \geqslant 0$, $| z | \leqslant r$, $| w | \leqslant r$, respectively. Here $g(t)$ and $\theta(t) = \alpha(t) + \int_0^t \beta(t, s)\, ds$ are assumed to be in the space $Y = L^p$

or M^p. Then $\theta = |\theta|_Y$ in H1. Note that this example includes the case $h(t, s, x(s), x(t)) = \text{col}(0, x_1(t)\, b(t-s)\, x_1(s)), x(t) = \text{col}(x_1(t), x_2(t))$, which is found in frequent applications for $p = +\infty$ (e.g., Volterra's well-known predator–prey equations with hereditary effects [11]).

Remark 2. In the case of differential systems ($B(t, s) \equiv 0$), admissibility has been extensively studied [5]. For autonomous differential systems ($A(t) \equiv A$), the question of admissibility reduces to that of the location of the eigenvalues of A in the complex plane. For example, in this case (L) is (L^p, BC)-admissible for $1 < p \leqslant \infty$ if no eigenvalue has zero real part and is (L^1, BC)-admissible if those eigenvalues with zero real part are simple. For nonautonomous systems, sufficient (and sometimes necessary) conditions for (L^p, BC)-admissibility can be formulated on the basis of appropriate bounds on $Y(t)\, P_1 Y^{-1}(s)$ and $Y(t)\, P_2 Y^{-1}(s)$ where $Y(t)$ is a fundamental matrix for the linear homogeneous system and P_1, P_2 are supplementary projections into R^n, i.e., $P_1 + P_2 = I$. The projection P_1 usually projects onto that subspace of R^n that serves as initial conditions giving rise to bounded solutions of the linear homogeneous system. The existence of such projections and bounds is referred to as the existence of dichotomies of solutions for (L) [5].

In the case of integrodifferential systems, sufficient conditions for admissibility of the same type can be given, although the problem is more complex. For "autonomous" systems $A(t) \equiv A$ and $B(t, s) \equiv B(t-s)$, recent work of Miller [6] and Miller and Nohel [1] can be interpreted in this way. Their approach uses the location of the complex roots of the equation $\det(sI - A - B^*(s)) = 0$ where $B^*(s)$ is the Laplace transform of the kernel $B(z)$. This approach is of course a generalized version of the eigenvalue approach for differential systems; it is complicated mainly by the fact that the equation for the "eigenvalues" is not in general algebraic. (For example, it may have more then n roots and, in fact, infinitely many roots that, furthermore, may have very large multiplicities.) In [2] it is established that (L) is (L^p, BC)-admissible for $1 \leqslant p < \infty$ if and only if there exists a matrix function $P(s) \in L^q$, $1/p + 1/q = 1$ (with $q = \infty$ for $p = 1$) such that for some constant $K > 0$,

$$\begin{aligned} &\left(\int_0^t |V(t,s)|^q\, ds\right)^{1/q} \leqslant K, \qquad \left(\int_t^\infty |W(t,s)|^q\right)^{1/q} ds \leqslant K \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \geqslant 0 \text{ when } p \neq 1, \\ &|V(t,s)| \leqslant K, \qquad t \geqslant s \geqslant 0, \qquad \text{and} \qquad |W(t,s)| \leqslant K \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } s \geqslant t \geqslant 0 \text{ when } p = 1, \end{aligned} \tag{2.1}$$

where $V(t, s) = U(t, s) - U(t, 0)\, P(s)$ and $W(t, s) = U(t, 0)\, P(s)$. Here,

$U(t, s)$ is the fundamental matrix of (L) which is defined to be the unique $n \times n$ matrix solution of

$$U_t(t, s) = A(t)\, U(t, s) + \int_s^t B(t, u)\, U(u, s)\, du, \qquad t \geqslant s \geqslant 0,$$

satisfying $U(s, s) = I$. Then the solution of (NH) is given by the variation of constants formula

$$z(t) = U(t, 0)\, z(0) + \int_0^t U(t, s)\, g(s)\, ds.$$

In addition, condition (2.1) with $q = 1$ is sufficient for (L^∞, BC)-admissibility. In the special case of autonomous differential systems mentioned above, it turns out that $P(s) = P_2 Y(s)$ so that $V(t, s) = Y(t)\, P_1 Y^{-1}(s)$, $W(t, s) = Y(t)\, P_2 Y^{-1}(s)$ and (2.1) are familiar bounds characterizing admissibility for these systems. This approach has been used by the author, through the construction of $P(s)$, to study Volterra's predator–prey model [2].

For the spaces M^p, sufficiency conditions can be found in a manner similar to that used for L^p. Suppose $P(s)$ is a matrix such that

$$\int_0^t \left(\int_u^{u+1} | V(t, s)|^q\, ds\right)^{1/q} du \leqslant K, \qquad \int_t^\infty \left(\int_u^{u+1} | W(t, s)|^q\, ds\right)^{1/q} du \leqslant K$$

for $t \geqslant 0$, $p \neq 1$, or such that

$$\int_0^t \sup_{u \leqslant s \leqslant u+1} | V(t, s)|\, du \leqslant K, \qquad \int_t^\infty \sup_{u \leqslant s \leqslant u+1} | W(t, s)|\, du \leqslant K$$

for $t \geqslant 0$, $p = 1$, where $K > 0$ is some constant and V, W are defined as above; then (L) is (M^p, BC)-admissible. This can be seen as follows. For $g \in M^p$ consider

$$z(t) = \int_0^t V(t, s)\, g(s)\, ds - \int_t^\infty W(t, s)\, g(s)\, ds, \qquad t \geqslant 0.$$

From the estimates (recall that all functions vanish for negative arguments),

$$\begin{aligned}
\int_0^t | V(t, s)|\, | g(s)|\, ds &= \int_0^t | V(t, s)|\, | g(s)| \int_{s-1}^s du\, ds \\
&\leqslant \int_{-1}^t \int_u^{u+1} | V(t, s)|\, | g(s)|\, du\, ds \\
&\leqslant \int_0^t \left(\int_u^{u+1} | V(t, s)|^q\, ds\right)^{1/q} \left(\int_u^{u+1} | g(s)|^p\, ds\right)^{1/p} du \\
&\leqslant K\, | g |_{m,p}
\end{aligned}$$

and a similar estimate for $\int_t^\infty Wg\,ds$ we conclude that z is well defined and bounded for $t \geqslant 0$. By rewriting

$$z(t) = \int_0^t U(t, s)\, g(s)\, ds - U(t, 0) \int_0^\infty P(s)\, g(s)\, ds, \qquad t \geqslant 0,$$

we see that this $z(t)$ is a solution of (NH), for this is just the variation of constants formula for the unique solution of (NH) with initial condition $-\int_0^\infty P(s)\, g(s)\, ds$. Thus, under the above conditions, (NH) has for each $g \in M^p$ at least one bounded solution, which is the same as to say (L) is (M^p, BC)-admissible. See [3] for other perturbation results for integral equations involving the spaces M^p.

3. Proofs

We begin by establishing some preliminary lemmas. Consider the abstract operator equation $Lx = p(x)$ where L is a linear operator. Let F be a convex Fréchet space and let X be a Banach subspace of F where the norm $|\cdot|_X$ yields a topology stronger than that determined by the metric on F. Assume that the domain D of L is a linear subspace of X and that the range of L lies in a Banach space Y. We further need the hypothesis

H2. There exists a subspace S of D on which L_s, the restriction of L to S, is one–one, (X, Y)-closed, and has Y-closed range. Moreover, L_s^{-1} is (Y, F)-compact as an operator from the range R_s of L_s into F.

Let $\bar{D}$ denote the closure of D in F and $\sum_X(r)$ or $\sum_Y(r)$ the balls in X or Y respectively of radius r centered at 0. Regarding the operator $p(x)$, which is written without any loss of generality as $p(x) = h(x) + g$, $h(0) = 0$, $g \in Y$, we assume the hypothesis

H3. There exists a constant r, $0 < r \leqslant \infty$, such that $p(x)$ maps $\bar{D} \cap \sum_X(r)$ into R_s (F, Y)-continuously in such a way that $|h(x)|_Y \leqslant \theta\, |x|_X$ for all $x \in \bar{D} \cap \sum_X(r)$ and some constant $\theta > 0$.

Let $N = \{x \in D : Lx = 0\}$ and $\Omega = \{x \in D \cap \sum_X(r) : Lx = p(x)\}$, where r is as in H3.

Lemma 1. *Suppose* H2 *and* H3 *hold. There exist constants* $\theta_0 > 0$ *and* $r_0 > 0$ *such that* $\theta \leqslant \theta_0$ *and* $g \in R_s \cap \sum_Y(r_0)$ *imply that* Ω *has a locally nonempty partition with respect to* N.

We recall that the Schauder–Tychonoff fixed point theorem is valid on any convex Fréchet space F; i.e., a continuous, compact operator mapping a closed, convex subset of F into itself has at least one fixed point in this subset [10].

Proof of Lemma 1. For $n \in N$ define the operator $T_n x = n + L_s^{-1} p(x)$ which maps $\bar{D} \cap \sum_X (r)$ into D. In order to construct a locally nonempty partition $\{K_n\}$ of Ω, we define K_n to be the set of fixed points of T_n contained in $D \cap \sum_X (r)$. Note that since T_n maps into D the set K_n is the same as the set of fixed points in the closed, convex set $\bar{D} \cap \sum_X (r)$.

First we show that the collection $\{K_n\}$ is a partition of Ω. If $x \in K_n$ for $n \in N$, then $T_n x = x$, and hence, $LT_n x = Lx$ or $p(x) = Lx$ (since $LL_s^{-1} p(x) = p(x)$). Thus, $x \in \Omega$ or $K_n \subseteq \Omega$ and as a result $\bigcup_N K_n \subseteq \Omega$. Conversely, if $x \in \Omega$, then $Lx = p(x)$, and hence, $T_n x = x$ with $n = x - L_s^{-1} p(x)$. That this n lies in N is easily checked. Thus, $\Omega \subseteq \bigcup_N K_n$ and we conclude $\Omega = \bigcup_N K_n$. Finally, if $x \in K_n \cap K_{n'}$ for $n, n' \in N$, then $T_n x = x = T_{n'} x$, for which it follows, by the very definition of T_n, that $n = n'$; i.e., $K_n \cap K_{n'} = \varnothing$ whenever $n \neq n'$.

It remains to prove the main point of the lemma, namely, that this partition is locally nonempty. The inverse L_s^{-1} is (Y, X)-closed (since L_s is (X, Y)-closed) and has a Banach space R_s as domain (R_s is assumed Y-closed). Consequently, the closed graph theorem implies L_s^{-1} is (Y, X)-continuous as a linear operator from R_s into S. Let $| L_s^{-1} |$ denote the norm of L_s^{-1}, and let $\theta_0 = \beta / | L_s^{-1} |$ where β is any fixed constant $\beta \in (0, 1)$. Set $r_0 = r(1 - \beta)/| L_s^{-1} |$. Then, for $g \in R_s \cap \sum_Y (r_0)$ and $n \in N \cap \sum_X (r(1 - \beta)/2)$ we find, using the inequality in H3 with $\theta \leqslant \theta_0$,

$$\begin{aligned} | T_n x |_X &\leqslant | n |_X + | L_s^{-1} h(x) |_X + | L_s^{-1} g | \\ &\leqslant r(1 - \beta)/2 + \beta r + r(1 - \beta)/2 = r \end{aligned}$$

for all $x \in \bar{D} \cap \sum_X (r)$. Thus, T_n maps $\bar{D} \cap \sum_X (r)$ into itself. The set $\bar{D} \cap \sum_X (r)$ is a closed, convex subset of the convex Fréchet space F. Moreover, that L_s^{-1} is (Y, F)-compact (which implies it is (Y, F)-continuous) implies T_n is (F, F)-compact and continuous. The Schauder–Tychonoff theorem then implies $K_n \neq \varnothing$ for $n \in N \cap \sum_X (r(1 - \beta)/2)$, and in turn, that $\{K_n\}$ is locally nonempty. ∎

Theorem 1 will follow as a direct application of Lemma 1 with

$$Lx \equiv x'(t) - A(t)\, x(t) - \int_0^t B(t, s)\, x(s)\, ds \tag{3.1}$$

and $p(x) \equiv p(t)(x)$ and with the necessary spaces taken to be $F = C$, $X = BC$, and $Y = L^p$ or M^p once it is verified that H2 and H3 hold. With respect to the hypothesis H2 on L defined by (3.1) above, we take as its domain $D = D_Y$ the linear space of all functions in BC which are absolutely continuous for

$t \geqslant 0$ and satisfy $Lx \in Y$. Let $R_1 \subset R^n$ be the linear space of initial conditions that produce bounded solutions of the linear homogeneous system (L); i.e., $R_1 = \{y(0) \in R^n$ where $y \in BC$ and $Ly = 0\}$. Let R_2 be any complementary subspace to R_1 in R^n so that $R^n = R_1 \oplus R_2$. In H2 we take $S = S_Y = \{x \in D_Y : x(0) \in R_2\}$.

LEMMA 2. *If* (L) *is* (Y, BC)*-admissible for* $Y = L^p$, $1 < p \leqslant \infty$, *or* M^p, $1 < p < \infty$, *then* L *defined by* (3.1) *satisfies* H2 *with the above choices of spaces.*

Proof. We must consider L_s the restriction of L defined by (3.1) to $S = S_Y$ as defined above.

(i) L_s *is one–one.* Suppose $g = Lx_1 = Lx_2$ for $x_1, x_2 \in S$. This is the same as saying that x_1 and x_2 are both bounded solutions of (NH) with initial states in R_2. Thus, $x = x_1 - x_2$ is a bounded solution of (L) and $x(0) \in R_2$ which implies, by the way R_1 and R_2 were defined, that $x(0) = 0$. From the uniqueness of solutions of (L) we find that $x \equiv 0$ or $x_1 = x_2$ for $t \geqslant 0$.

(ii) L_s *is* (BC, Y)*-closed.* This part of the proof is carried out for $Y = M^p$ only. The proof for $Y = L^p$ is similar and simpler in that the estimates used are more standard; this case, in fact, is proved in [2].

Suppose $x_n \in S_Y$ and $g_n = L_s x_n \in M^p$ are such that $x_n \to x_0$ and $g_n \to g_0$ in BC and Y, respectively. We must show $x_0 \in S_Y$ and $L_s x_0 = g_0$. Integrating $g_n = L_s x_n$, we find from (3.1),

$$x_n(t) = x_n(0) + \int_0^t \left(A(s) + \int_s^t B(u, s)\, du\right) x_n(s)\, ds + \int_0^t g_n(s)\, ds. \tag{3.2}$$

Let $t \geqslant 0$ be fixed, but arbitrary. Since $x_n(s) \to x_0(s)$ uniformly on $[0, t]$, we may pass $n \to \infty$ under the first integral on the right hand side of (3.2). Now $g_n \to g_0$ in M^p means $| g_n - g_0 |_{m,p} = \sup_{t \geqslant 0} (\int_t^{t+1} | g_n - g_0 |^p\, ds)^{1/p} \to 0$ as $n \to \infty$. From the estimates

$$\begin{aligned}\left| \int_0^t g_n(s)\, ds - \int_0^t g_0(s)\, ds \right| &\leqslant \int_0^t | g_n(s) - g_0(s)| \int_{s-1}^s du\, ds \\ &\leqslant \int_{-1}^t \int_u^{u+1} | g_n(s) - g_0(s)|\, ds\, du \\ &\leqslant \int_{-1}^t \left(\int_u^{u+1} | g_n(s) - g_0(s)|^p\right)^{1/p} ds\, du \\ &\leqslant (t+1)\, | g_n - g_0 |_{m,p}\end{aligned}$$

we conclude that $\int_0^t g_n\, ds \to \int_0^t g_0\, ds$ as $n \to \infty$ for each fixed $t \geqslant 0$. Thus, from (3.2), as $n \to \infty$, we find that x_0 also satisfies (3.2) for every $t \geqslant 0$ (i.e., $Lx_0 = g_0$). That the x_n are uniformly bounded for $t \geqslant 0$ implies that x_0 is also bounded in $t \geqslant 0$, and, in addition, $x_n(0) \in R_2$ also implies $x_0(0) \in R_2$; this means $x_0 \in S$. Thus, $L_s x_0 = g_0$, and we conclude that L_s is closed.

(iii) R_s *is* Y-*closed.* The assumption that (L) is (Y, BC)-admissible implies that the range R_s of L_s is all of Y, and hence, closed. To see this, choose any $g \in Y$. The admissibility assumption implies the existence of an $z_1 \in BC$ such that $Lz_1 = g$. If P_1 is the projection of R^n onto R_1, let y be the solution of (L) satisfying $y(0) = P_1 z_1(0)$. Since $y(0) \in R_1$, it follows that $y \in BC$. Set $z = y - z_1$. By superposition, $Lz = g$. Also $z \in BC$ and $z(0) = y(0) - z_1(0) = (I - P_1)\, z_1(0) \in R_2$. Thus, $z \in S$ and $Lz = L_s z = g$. This shows that $Y \subseteq R_s$. Since the converse set inclusion is obvious from the definition of the domain D_Y of L, we have proved $Y = R_s$.

(iv) L_s^{-1} *is* (Y, C)-*compact.* Let $Y = M^p$, the proof for $Y = L^p$ being similar. Suppose g_n is a bounded sequence in M^p: $| g_n |_{m,p} \leqslant \alpha$ for all n. We must show $L_s^{-1} g_n$ has a subsequence convergent in C.

First we recall that it was shown above that L_s^{-1} is continuous as a linear operator from Y into BC; consequently, $x_n = L_s^{-1} g_n$ is a bounded set of functions on $t \geqslant 0$. The compactness of L_s^{-1} as an operator into C will follow from the Ascoli theorem by our showing that these functions x_n are equicontinuous on every finite interval $[0, i]$, $i = 2, \ldots$. To this end let t, t' be arbitrary points in $[0, i]$ and assume without loss of generality that $t < t'$ and $| t' - t | \leqslant 1$. Then we have

$$\int_t^{t'} | g_n(s) |\, ds = \left(\int_t^{t'} | g_n(s) |^p\, ds \right)^{1/p} | t - t' |^{1/q} \leqslant | g_n |_{m,p}\, | t - t' |^{1/q}. \quad (3.3)$$

Since A and B are continuous and the x_n are uniformly bounded on $[0, i]$, the expression $A(t)\, x_n(t) + \int_0^t (B(t, s)\, x_n(s)\, ds$ is uniformly bounded in n and $t \in [0, i]$; let $\beta > 0$ be a bound. From this fact and the estimate (3.3), we have

$$| x_n(t') - x_n(t) | \leqslant \int_t^{t'} | x_n'(s) |\, ds \leqslant \beta(t' - t) + \alpha(t' - t)^{1/q},$$

and hence, the equicontinuity of x_n on $[0, i]$. ∎

Proof of Theorem 1. With L defined by (3.1), $F = C$, $X = BC$, $Y = L^p$ or M^p, and $S = S_Y$ defined as above, Theorem 1 follows immediately from Lemma 1, since Lemma 2 shows H2 holds, and since hypothesis H1 on the perturbation term $p(x) = p(t)(x)$ implies, in this abstract setting, the other necessary hypothesis H3. ∎

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