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Existence, Uniqueness, and Extendibility of Solutions of Volterra Integral Systems with Multiple, Variable Lags

By

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1. Introduction.

Our purpose is to give some conditions sufficient to guarantee the local existence, uniqueness, extendibility, and continuity of solutions of a general Volterra integral equation with multiple, variable lags. Specifically we consider, for $t \in [a, b)$, $a < b \leq +\infty$, the system

$$x_i(t) = f_i(t) + \int_a^t F_i(t, s, x_1(g_{i1}(s)), \cdots, x_n(g_{in}(s)))ds, \qquad i=1, 2, \cdots, n,$$

where x_i, f_i, F_i, and g_{ij} are scalar valued functions of their arguments. Using the simplifying notation $x(t)=\mathrm{col}\,(x_i(t))$, $f(t)=\mathrm{col}\,(f_i(t))$, $F(t, s, x(g(s)))=\mathrm{col}\,(F_i(t, s, x_1(g_{i1}(s)), \cdots, x_n(g_{in}(s))))$ we may write the system more concisely as

$$\text{(E)} \qquad x(t)=f(t)+\int_a^t F(t, s, x(g(s)))ds.$$

Throughout it is assumed that $g_{ij}(s) \leq s$ for all $a \leq s < b$. The solution of (E) is to be prescribed on a suitable initial interval which depends on the g_{ij}; namely, if $A=\{s: a \leq s < b,\ g_{ij}(s) < a \text{ for some } i, j\}$ and $I=\{t: t < a,\ t=g_{ij}(s) \text{ for some } s \in A\}$ then it is required that

$$\text{(IC)} \qquad x(t)=h(t), \qquad t \in I$$

for a given function $h(t)$ defined on I. We seek conditions on F, f, and g_{ij} under which the "initial value problem" (E) and (IC) has a local solution $x(t)$ on $[a, c]$ for some $c \in (a, b)$; we also consider the problem of the uniqueness of this solution, its extendibility in t as a solution and the dependence of x on f and F.

We point out that the problem (E)–(IC) includes the initial value problem for ordinary differential and integrodifferential systems with or without lags as well as the familiar Volterra equation without lags. Problems of the type (E)–(IC), however, have arisen in certain applications to impulse theory [4, 5] and have been considered in a few recent papers [2, 7] where, in these references, comparison theorems and resulting uniqueness theorems for (E) are presented without proof, and some properties of a resolvent for a linearized version of (E) are asserted, again without proof. Our main motivation here is simply to establish the basic theory for (E)–(IC) for future reference. Although there are no particular surprises in the results, the complications introduced by variable time-lags require care, and it seems advisable to set down the explicit results here, separate from other more detailed considerations regarding the system (E)–(IC). Our results serve as generalizations of such results for ordinary differential systems [1] as well as Volterra integral equations (without lags) [3].

We will make the following assumptions throughout the paper: $F(t, s, x)$ is measurable in $(s, x) \in [a, t]\times\Omega$, Ω=open$\subseteq R^n$ and bounded in (s, x) on compact subsets of $[a, t]\times\Omega$ for every $t \in [a, b)$; $F(t, s, x)\equiv 0$ for $s>t$ and all $x \in \Omega$; $f(t)$ is continuous on $[a, b)$; the sets A and I are measurable; and $h(t)$ and $g_{ij}(t)$ are bounded and measurable on I and A respectively. It is also assumed that $h(t) \in \Omega$ for $t \in I$ and $f(a) \in \Omega$. *A solution* of (E)–(IC) on an interval $[a, b)$, $a<b\leq+\infty$ (or $[a, c]$, $c<+\infty$) is a function continuous on $[a, b)$ (or $[a, c]$), satisfying (IC) on I and (E) on $[a, b)$ (or $[a, c]$).

2. Local existence and uniqueness.

We will prove two existence theorems, one by using the Schauder-Tychonoff fixed point theorem and the other by using the contraction principle on the operator T defined by

$$T\phi=\begin{cases} f(t)+\int_a^t F(t, s, \phi(g(s)))ds, & t \in [a, a+\delta] \\ h(t), & t \in I \end{cases}$$

for a $\delta>0$ sufficiently small.

We will need the following hypotheses:

H1: $\begin{cases}\text{For every } t \in [a, b) \text{ and almost all } s \in [a, t] \text{ the function } F(t, s, x) \text{ is con-} \\ \text{tinuous for } x \in \Omega;\end{cases}$

H2: There exists a constant $c' \in (a, b)$ and a function $m(t, s)\geq 0$ for which $m(t, \cdot)$ is integrable for every $t \in [a, c']$,

$$|F(t, s, x)|\leq m(t, s)$$

for all $a\leq s\leq t\leq c'$ and $x \in \Omega$.

Let $c \in (a, b)$ and denote by B_c the Banach space of functions $\phi: I\cup[a, c]\to R^n$ which are continuous on $[a, c]$ and equal to $h(t)$ on I with norm $|\phi|_c = \sup_{[a,c]} |\phi(t)|$. Let $B_c(\Omega) = \{\phi \in B_c : \phi(t) \in \Omega, t \in I \cup [a, c]\}$; $B_c(\Omega)$ is an open subset of B_c. It is not difficult to see that under the assumptions made, $F(t, s, \phi(g(s)))$ is bounded and measurable in $s \in [a, t]$ for each $t \in [a, c]$ and $\phi \in B_c(\Omega)$ and hence $T\phi$ is well-defined. In order to guarantee that $T\phi$ is itself continuous in t it is necessary to make a further assumption on $F(t, s, x)$ with respect to its first variable.

H3: There exists a constant $c' \in (a, b)$ such that

$$\sup_\phi \int_a^{c'} |F(t, s, \phi(g(s))) - F(\bar{t}, s, \phi(g(s)))| \, ds \to 0$$

as $t \to \bar{t}$ for all $\bar{t} \in [a, c']$ where the supremum is taken over any bounded subset of $B_{c'}(\Omega)$.

Note that if H2 or H3 hold for some $c' \in (a, b)$ then they hold for all c' closer to a (with the same $m(t, s)$ in H2). This hypothesis is a little clumsy as it stands but it is a natural one for our purposes below. It would not be difficult to put conditions on $F(t, s, x)$ as a function of (t, s, x) sufficient to guarantee H3 (an obvious one would be a Lipschitz type condition in t with suitable Lipschitz "constant" $k(s, x)$); we will refrain from doing this however, as H3 is a more straightforward assumption for our work below. Under H3, $T\phi$ is continuous on $[a, c]$ for $\phi \in B_c(\Omega)$, $c \leq c'$.

Theorem 1. *Assume* H1, H2, *and* H3. *Then* $\exists c \in (a, b) \ni$ *problem* (E)–(IC) *has a solution* $x(t)$ *on* $[a, c]$.

Proof. Let c' be as in H3 and H2. Since it is assumed that $f(t)$ is continuous on $[a, b)$ and $f(a) \in \Omega$ we conclude that for all $c'' \leq c'$ close to a we have $f(t) \in \Omega$ for $t \in [a, c'']$. Thus the function

$$\phi = \begin{cases} f(t), & t \in [a, c''] \\ h(t), & t \in I \end{cases}$$

lies in $B_{c''}(\Omega)$. For $c \leq c''$ define $S_c = \{\phi \in B_c(\Omega) : |\phi(t) - f(t)| \leq 1, t \in [a, c]\}$; S_c is a closed, convex subset of the Banach space B_c. Using H3 with $\bar{t} = a$ we see that there exists $c \in (a, b)$ close enough to a such that

$$\sup_{S_{c''}} \int_a^t |F(t, s, \phi(g(s)))| \, ds \leq 1$$

for $t \in [a, c]$ and hence

$$\sup_{S_c} \int_a^t |F(t, s, \phi(g(s)))| \, ds \leq 1, \qquad t \in [a, c].$$

We will show: (i) $T: S_c \to S_c$ continuously and (ii) TS_c is precompact. It will then follow by the Schauder-Tychonoff fixed point theorem [3, 6] that there exists a least one $x \in S_c$ such that $Tx = x$. This function x is a solution of (E)–(IC) on $I \cup [a, c]$, continuous on $[a, c]$.

(i) For $\phi \in S_c$ we have for $t \in [a, c]$ the estimate

$$|T\phi - f(t)| \leq \int_a^t |F(t, s, \phi(g(s)))| \, ds \leq 1$$

which shows T maps S_c into itself.

Next we show that TS_c is an equicontinuous family of functions at each $\bar{t} \in [a, c]$. Given $\varepsilon > 0$, by H3 (for c closer to a if necessary) and the continuity of f, there exists a $\delta = \delta(\varepsilon) > 0$ such that $|t - \bar{t}| \leq \delta$ (or $\bar{t} \leq t \leq \bar{t} + \delta$ if $\bar{t} = a$) implies

$$|(T\phi)(t) - (T\phi)(\bar{t})| \leq |f(t) - f(\bar{t})| + \int_a^c |F(t, s, \phi(g(s))) - F(\bar{t}, s, \phi(g(s)))| \, ds \leq \varepsilon$$

for all $\phi \in S_c$; i.e., TS_c is equicontinuous at $\bar{t} \in [a, c]$.

Now we can show T is continuous on S_c. Suppose $\phi_n \in S_c$ and $\phi_n \to \phi \in S_c$ uniformly on $[a, c]$. Then $\phi_n(g(s)) \to \phi(g(s))$ for each $s \in [a, c]$ (since $\phi_n = \phi = h$ on I) and, by H1, $F(t, s, \phi_n(g(s))) \to F(t, s, \phi(g(s)))$ for every $t \in [a, c]$ and almost all $s \in [a, t]$. Then H2 and Lebesgue's Dominated Convergence Theorem imply $T\phi_n \to T\phi$ at each $t \in [a, c]$. Since TS_c is equicontinuous at each $t \in [a, c]$ it follows that $T\phi_n \to T\phi$ uniformly on $[a, c]$; i.e., $T: S_c \to S_c$ is continuous.

(ii) The precompactness of TS_c follows from Ascoli's Lemma since TS_c has been shown equicontinuous at each $t \in [a, c]$ and since, by H2, the family TS_c is bounded at each $t \in [a, c]$ as is seen by the estimate

$$|(T\phi)(t)| \leq |f(t)| + \int_a^t m(t, s) ds < \infty$$

for each $t \in [a, c]$ and all $\phi \in S_{c'}$. ∎

Our next result is obtained by making T a contraction on S_c. In place of H1 and H3 we need

H1′ $\begin{cases} \exists \text{ a function } k(t, s) \geq 0 \ni k(t, s) \text{ is integrable in } s \text{ on } [a, t] \text{ for all } a \leq t < b \\ \text{satisfying } \int_a^t k(t, s) ds \to 0 \text{ as } t \to a+ \text{ and a constant } c' \in (a, b) \ni \\ \qquad |F(t, s, x) - F(t, s, y)| \leq k(t, s) \, |x - y| \\ \text{for each } t \in [a, c'], \text{ almost all } s \in [a, t], \text{ and all } x, y \in \Omega. \end{cases}$

H3′ $\begin{cases} \exists c' \in (a, b) \ni \int_a^{c'} F(t, s, \phi(g(s))) ds \to \int_a^{c'} F(\bar{t}, s, \phi(g(s))) ds \\ \text{as } t \to \bar{t} \text{ for all } t, \bar{t} \in [a, c'] \text{ and for all } \phi \text{ in any bounded subset of } B_{c'}(\Omega). \end{cases}$

If H1$'$ and/or H3$'$ hold for $c' \in (a, b)$ then they hold for c' closer to a. Hypothesis H3$'$ guarantees that $T\phi$ is continuous on $[a, c']$ for $\phi \in S_{c'}$ and H2 guarantees (as in the proof of Theorem 1) that T maps S_c into itself for c close enough to a.

Theorem 2. *Assume* H1$'$, H3$'$, *and* H2 *with* $\lim_{t\to a+} \int_a^t m(t, s)ds = 0$. *Then* $\exists\, c \in (a, b) \ni$ *problem* (E)–(IC) *has a unique solution on* $[a, c]$.

Proof. From H3$'$ follows easily the continuity of $T\phi$ for $\phi \in S_c$ and for $c \leq c'$ (c' as in H1$'$, H3$'$, and H2). Let $c \leq c'$ be chosen so close to a that $\int_a^t m(t, s)ds \leq 1$, $t \in [a, c]$. Then, for $\phi \in S_c$, by H2

$$|T\phi - f| \leq \int_a^t m(t, s)ds \leq 1$$

and T maps S_c into itself.

We need only show that T is a contraction on S_c for c close to a. On $[a, c]$, $c \leq c'$, we have from H3$'$ the estimate, for ϕ and $\psi \in S_c$,

$$(1) \qquad |T\phi - T\psi| \leq \int_a^t k(t, s)\, |\phi(g(s)) - \psi(g(s))|\, ds.$$

Now since $\phi = \psi = h$ on I and $g_{ij}(s) \leq s$ on $[a, b)$ we see that $\sup_{[a,c]} |\phi(g(s)) - \psi(g(s))| \leq \sup_{[a,c]} |\phi(s) - \psi(s)|$. Using this in (1), with τ replacing t, and then taking the supremum of both sides of the resulting inequality over the interval $\tau \in [a, t]$ we obtain

$$|T\phi - T\psi|_c \leq \sup_{[a,c]} \int_a^t k(t, s)ds\, |\phi - \psi|_c.$$

From H1$'$, there exists a c (closer yet to a if necessary) so that $\int_a^t k(t, s)ds \leq 1/2$ for $t \in [a, c]$. For such a $c \in (a, b)$, $|T\phi - T\psi|_c \leq \frac{1}{2} |\phi - \psi|_c$. ■

The final result of this section is a uniqueness result independent of question of existence.

Theorem 3. (a) *Assume* H1$'$. *Then* $\exists c \in (a, b)$ *such that problem* (E)–(IC) *can have at most one solution on* $[a, c]$.

(b) *Assume* H1$'$ *holds for every* $c' \in (a, b)$ *for a function* $k(t, s)$ *satisfying the stronger condition* $\int_t^{t+\eta} h(t, s)ds \to 0$ *as* $\eta \to 0+$ *uniformly in* t *on compact subintervals of* $[a, b)$. *Then problem* (E)–(IC) *can have at most one solution on* $[a, b)$.

Proof. (a) Suppose $x(t), y(t)$ are two solutions of (E)–(IC) continuous on a common interval $[a, c]$ where c is so close to a that $\int_a^t k(t, s)ds \leq \frac{1}{2}$ for $t \in [a, c]$ (see H1′). Then $x = y = h$ on I and

$$x(t) - y(t) = \int_a^t [F(t, s, x(g(s))) - F(t, s, y(g(s)))]ds$$

on $[a, c]$. Thus, by H1′, we find

$$|x(t) - y(t)| \leq \int_a^t k(t, s)ds\,|x - y|_c \leq \tfrac{1}{2}|x - y|_c$$

on $[a, c]$. Since the continuous function $x - y$ assumes its supremum $|x - y|_c$ somewhere in $[a, c]$ we have a contradiction, unless $x \equiv y$ on $[a, c]$.

(b) Suppose x and y are two solutions continuous on $[a, b)$. Let $d \in (a, b)$ be arbitrary. We wish to show $x \equiv y$ on $[a, d]$. By (a) we know $x \equiv y$ on $[a, c]$ for some $c \in (a, b)$. If $c \geq d$ then the assertion is proved. Thus we assume $d > c$.

The solutions x and y both solve problem (E)–(IC) on the interval $[c, d]$ with the lag interval I and h modified in the obvious manner, f replaced by $\bar{f}(t) = f(t) + \int_a^c F(t, s, x(g(s)))ds$ and a replaced by c. Noting that the assumptions made on k in (b) insures

$$\int_c^t k(t, s)ds \leq \tfrac{1}{2}$$

on $[c, 2c]$ (considering how c was chosen in the proof of (a)), we find that

$$|x(t) - y(t)| \leq \tfrac{1}{2} \sup_{[c, 2c]} |x - y|$$

which again leads to a contradiction unless $x \equiv y$ on $[c, 2c]$. Thus, $x \equiv y$ on $[a, 2c]$. If $2c \geq d$ we have the desired result that $x \equiv y$ on $[a, d]$; if not the argument may be repeated m times where m is a positive integer such that $mc > d$ in order to conclude $x \equiv y$ on $[a, d]$. ∎

3. Extendibility.

We treat the problem of extending a local solution on $[a, c]$ until $(t, x(t))$ reaches the boundary of $[a, b) \times \Omega$ in the standard way from the theory of differential and integral equations. The problem (E)–(IC) is translated so that $(c, x(c))$ becomes the new "initial point" and Theorem 1 or 2 is reapplied. In order to do this it is necessary to assume that H2, H3 or H1′, H3′ hold *for every* $c' \in [a, b)$ instead of for some $c' \in [a, b)$ as stated above. These corresponding hypotheses will be denoted by $\overline{\text{H2}}$, $\overline{\text{H3}}$, or $\overline{\text{H1}'}$, $\overline{\text{H3}'}$ respectively.

Suppose $x(t)$ is a solution on $[a, c]$ and that H1, $\overline{\text{H2}}$, $\overline{\text{H3}}$ or $\overline{\text{H1}'}$, $\overline{\text{H2}}$, $\overline{\text{H3}'}$ hold. Consider the translated system

$$\text{(E)}' \qquad z(t)=\bar{f}(t)+\int_c^t F(t, s, z(g(s)))ds$$

for $t \geq c$ where

$$\bar{f}(t)=f(t)+\int_a^c F(t, s, x(g(s)))ds.$$

Define $A'=\{s: g_{ij}(s)<c$ for some $s \in [c, b)$ and some $i, j\}$ and $I'=\{g_{ij}(s): s \in A'\}$ and impose the initial condition

$$\text{(IC)}' \qquad z(t)=\begin{cases} x(t), & t \in I' \cap [a, c] \\ h(t), & t \in I' \cap I. \end{cases}$$

It is not difficult to see that if $x(t)$ solves (E)–(IC) on $[a, d]$ for $d>c$ then $z(t)=x(t)$ for $t \in [c, d]$ solves (E)′–(IC)′ on $[c, d]$. Conversely, if $z(t)$ solves (E)′–(IC)′ on $[c, d]$ for some $d>c$, then $x(t)$ extended to $[a, d]$ by defining $x(t)=z(t)$ for $t \in [c, d]$ solves (E)–(IC) on $[a, d]$. Theorem 1 or 2 can be applied to (E)′–(IC)′ provided $\bar{f}(c)=x(c) \in \Omega$ and consequently either $(c, x(c))$ is on the boundary of $[a, b)\times\Omega$ or $x(t)$ can be extended as a solution of (E)–(IC) to a larger interval $[a, d]$, $d>c$. Continuing in this manner we see that one of the following possibilities arises: (i) $(c, x(c)) \in \partial([a, b)\times\Omega)$ or (ii) $x(t)$ can be extended as a solution to an interval $[a, c) \subseteq [a, b)$.

We now examine the latter case more closely when $c<b$. Let $t_n \in [a, c)$ be any sequence for which $t_n \to c$; we wish to show that the sequence $x(t_n) \in \Omega$ converges. Suppose $m<n$ (and hence $t_m<t_n$); then

$$\begin{aligned} |x(t_n)-x(t_m)| &\leq |f(t_n)-f(t_m)| + \left|\int_a^{t_n} F(t_n, s, \phi(g(s)))ds - \int_a^{t_m} F(t_m, s, \phi(g(s)))ds\right| \\ &= |f(t_n)-f(t_m)| + \left|\int_a^c F(t_n, s, \phi(g(s))) - F(t_m, s, \phi(g(s)))ds\right|. \end{aligned}$$

By either $\overline{\text{H3}}$ or $\overline{\text{H3}'}$ the integral tends to zero as $n, m\to\infty$. Thus, since in addition f is assumed continuous on $[a, b)$ (and, hence, at c) we see that $x(t_n)$ is a Cauchy sequence and consequently under the stated conditions the solution $x(t)$ can be extended to $[a, c]$ as a solution (by defining $x(c^-)$ to be the limit of $x(t_n)$). But then either $x(c^-) \in \partial\Omega$ or x can be extended to a larger closed interval as described above. Thus, even in case (ii) we see that either $x(c^-) \in \partial\Omega$ or x can be extended beyond c, provided $c<b$. In this way we conclude in case (ii) that x can be extended as a solution to an interval $[a, c)$ such that either $x(c^-) \in \partial\Omega$ or $c=b$.

Theorem 4. *Assume either* H1, $\overline{\text{H2}}$, $\overline{\text{H3}}$ *or* $\overline{\text{H1}'}$, $\overline{\text{H2}}$, $\overline{\text{H3}'}$. *Then the local solution whose existence is guaranteed by Theorems* 1 *or* 2 *respectively can be extended as a solution of* (E)–(IC) *until* $(t, x(t))$ *reaches the boundary of* $[a, b)\times\Omega$; *that is,* $x(t)$ *can be extended to an interval* $[a, c)$ *such that either* $x(c^-)\in\partial\Omega$ *or* $c=b$.

Note that b can be $+\infty$ in everything done above and if $[a, b)\times\Omega$ is unbounded then ∞ is said to be on its boundary.

4. Local dependence of solutions on *f*, *h*, and *F*.

Suppose $x(t)$ is a solution of (E)–(IC) with kernel F, forcing function f, and initial function h as described in § 1 and assume F satisfies H1′. Without loss of generality we suppose $x(t)$ is a solution on an interval $[a, c]$ where c is such that $\left|\int_a^t k(t, s)ds\right|_c = k_c < 1$ (cf. H1′ and recall that $|\phi|_c = \sup_{[a,c]}|\phi(t)|$). Let $x_1(t)$ be a solution of (E)–(IC) on the same interval $[a, c]$ with kernel F_1, forcing function f_1 and initial function h_1 as described in § 1 where $F_1(t, s, x)$ and $F(t, s, x)$ satisfy the condition

H4: $|F(t, s, x) - F_1(t, s, x)| \le \varepsilon$ for all $(t, s, x)\in[a, c]\times[a, t]\times\Omega$.

Then we can prove the following theorem:

Theorem 5. *If* $x(t)$, $x_1(t)$ *are solutions of* (E)–(IC) *as described above then*

$$|x - x_1|_c \le \max\{\sup_I |h - h_1|, [1 - k_c]^{-1}[|f - f_1|_c + c\varepsilon]\}.$$

Proof. Observe that

$$|x(g) - x_1(g)|_c \le \max\{\sup_I |h - h_1|, |x - y|_c\} \le |x - y|_c$$

unless $\sup_I |h - h_1| > |x - y|_c$. Thus, from (E) for $t\in[a, c]$

$$\begin{aligned} |x(t) - y(t)| &\le |f(t) - f_1(t)| + \int_a^t |F(t, s, x(g(s))) - F_1(t, s, x_1(g(s)))|\, ds \\ &\le |f - f_1|_c + \int_a^t |F(t, s, x(g(s))) - F(t, s, x_1(g(s)))|\, ds \\ &\quad + \int_a^t |F(t, s, x_1(g(s))) - F_1(t, s, x_1(g(s)))|\, ds \\ &\le |f - f_1|_c + \int_a^t k(t, s)ds\, |x(g) - x_1(g)|_c + \varepsilon c. \end{aligned}$$

Taking the supremum of both sides of this inequality for $t\in[a, c]$ we get

$$|x - y|_c \le |f - f_1|_c + k_c |x(g) - x_1(g)|_c + \varepsilon c.$$

Thus, either $\sup_I |h-h_1|>|x-y|_c$ or

$$|x-y|_c \leq |f-f_1|_c + k_c |x-y|_c + \varepsilon c$$

from which the theorem follows. ∎

5. An application.

The many uses of "variation of constants" formulas for solutions of linear systems of differential, integro-differential, and integral equations are of course well known. For example, many stability, boundedness, and other qualitative results for linear and for nonlinear perturbations of linear systems rely on such representation formulas. It is possible to derive a "variation of constants" formula for the solution of linear systems of the form (E) as we will show below. This representation formula is of course dependent on the concept of a "fundamental solution" of linear systems (E). As an application of our results above we will prove the global existence and uniqueness of such a fundamental solution. (The authors plan to use the variation of constants formula in future work on the stability of solutions of problem (E)–(IC).)

Consider system (E) with a linear kernel

$$\text{(L)} \qquad x(t)=f(t)+\int_a^t C(t,s)x(g(s))ds$$

where $C(t, s)$ is an $n\times n$ matrix with properties sufficient to guarantee the hypotheses of Theorem 2 on F used above. Let $S(t)=A\cap[a,t]$ for $t\geq a$,

$$\bar{f}(t)=f(t)+\int_{S(t)} C(t,s)h(g(s))ds$$

and

$$\bar{C}(t,s)=\begin{cases} 0 & \text{for } s\in A,\ a\leq s\leq t \\ C(t,s) & \text{for } s\notin A,\ a\leq s\leq t. \end{cases}$$

Note that $S(a)=\emptyset$ so that $\bar{f}(a)=f(a)$. Then system (L)–(IC) is equivalent to (L′)–(IC) where

$$\text{(L}'\text{)} \qquad x(t)=\bar{f}(t)+\int_a^t \bar{C}(t,s)x(g(s))ds, \qquad t\geq a.$$

We define the *fundamental solution* of (L) to be the $n\times n$ matrix solution $X(t, s)$ of the matrix problem

(X) $$X(t,s)=\begin{cases} I+\int_s^t \bar{C}(t,u)X(g(u),s)du, & a\leq s\leq t, \\ 0, \quad a\leq t\leq s. \end{cases}$$

Here I is the $n\times n$ identity matrix. Formally, one can show by straightforward substitution and verification that, if $\bar{f}(t)$ is of bounded variation on finite subintervals of $[a, b)$, then the function defined by the "variation of constants" formula

$$x(t)=\begin{cases} X(t,a)\bar{f}(a)+\int_a^t X(t,s)d\bar{f}(s), & t\geq a \\ h(t), \quad t\in I \end{cases}$$

solves (L′)–(IC) and hence (L)–(IC).

To justify the meaningfullness of this formula we apply our theorems above to assert that the fundamental solution $X(t, s)$ exists for all $a\leq s\leq t$ and is continuous in $t\geq s$ for each fixed $s\geq a$. (One can also prove the $X(t, s)$ is continuous in $s\in[a, t]$ for fixed $t\geq a$, a fact which is used in the manipulations to prove that the above formula for $x(t)$ actually solves (L′)–(IC). This fact, however, is not an application of our theorems above and consequently will not be proved here. A rather straightforward proof can be made by use of the standard Gronwall inequality.)

If we let e_i be the i^{th} column of I and if $x_i(t)$ is a solution of

(X_i) $$x_i(t,s)=e_i+\int_s^t \bar{C}(t,u)x_i(g(u),s)du, \qquad a\leq s\leq t$$

and

(IC) $$x_i(t,s)=0 \quad \text{on} \quad I_s,$$

then $X(t, s)$ is the $n\times n$ matrix constructed from the n column vectors x_i (with $X(t, s)$ formally defined to be zero for all $a\leq t<s$, $t\notin I_s$). The linear problem (X_i)–(IC) is of the form (E)–(IC) with s replacing a to which our theorems apply. Hence under any suitable assumptions on $C(t, s)$ which will guarantee H1′, H2, and H3′ (for example, $C(t, s)$ continuous in t, s is sufficient) we can conclude from Theorem 2 the local existence and uniqueness of $X(t, s)$ for $t\in[s, c_s)$ for each fixed $s\geq a$ where $c_s>s$ is some constant.

Finally we use Theorem 4 to show that $X(t, s)$ exists as a fundamental solution for all $a\leq s\leq t$. We do this by ruling out the alternative $x(c_s^-)\in\partial R^n$ in Theorem 4 and concluding as a result that $c_s=+\infty$ (in this application of Theorem 4 we use $\Omega=R^n$ and $b=+\infty$). In order to accomplish this we suppose this $C(t, s)$ is bounded on compact subsets of R^2. Then from equation (X_i) we find that for fixed $s\geq a$

$$|x_i(t,s)|\leq 1+M_s\int_s^t |x_i(g(u),s)|\,du, \qquad s\leq t<c_s$$

where M_s is a bounded for $|C(t, s)|$ on $s \leq t \leq c_s$. Inasmuch as $x_i(t, s)=0$ for $t<s$ and $g(u) \leq u$ for all $u \geq a$ we have that

$$\max_{s\leq\tau\leq u} |x_i(g(\tau), s)| \leq \max_{s\leq\tau\leq u} |x_i(\tau, s)| = m_i(u, s)$$

and hence

$$|x_i(t, s)| \leq 1 + M_s \int_s^t m_i(u, s) du, \qquad s \leq t < c_s.$$

Replacing t by the symbol τ and taking the maximum on $s \leq \tau \leq t$ of both sides of this inequality we find that

$$m_i(t, s) \leq 1 + M_s \int_s^t m_i(u, s) du, \qquad s \leq t < c_s$$

from which follows by Gronwall's inequality the estimate

$$|x_i(t, s)| \leq m_i(t, s) \leq \exp\,(M_s(t-s)), \qquad s \leq t < c_s.$$

This clearly shows that $x_i(c_s^-, s) \notin \partial R^n$ and Theorem 4 now yields the global existence of the fundamental solution $X(t, s)$ for $a \leq s \leq t$.

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