Global Branches of Solutions to Nonlinear Elliptic Eigenvalue Problems

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1. Introduction. This paper was motivated by many recent papers [1–11] concerned with the existence and uniqueness of positive solutions of nonlinear elliptic problems. (For applications of such problems see [2, 9, 12] and the references there cited.) Most of these studies were concerned with problems which do not possess a trivial (*i.e.*, an identically zero) solution, although some results for problems having a trivial solution can be found in [5, 6, 12]. This latter problem leads to a bifurcation phenomenon and is the one with which this paper deals. Specifically, we consider the problem

$$(1.1)\qquad \begin{aligned} Au &= \lambda f(u, x), \qquad x \in D,\\ Bu &= 0, \qquad x \in \partial D, \end{aligned}$$

where $x = (x_1, \cdots, x_m)$ and

$$Au \equiv \sum_{i,j=1}^{m} D_i(a_{ij}(x)\, D_j u) + a_0(x)u, \; D_i = \partial/\partial x_i$$

$$Bu \equiv \alpha(x) \frac{\partial u(x)}{\partial \nu(x)} + \beta(x)u, \qquad x \in D,$$

$$\frac{\partial u(x)}{\partial \nu(x)} \equiv \sum_{i,j=1}^{m} a_{ij}(x) n_i(x)\, D_j u, \qquad x \in \partial D.$$

Here D is a bounded region of type $C^{1+\mu}$ [13] in m dimensional Euclidean space, $n_i(x)$ is the i^{th} component of the outwardly directed unit normal to the boundary ∂D of D at $x \in \partial D$, and λ is a real parameter to be determined as part of the solution. We assume that $a_{ij}(x) = a_{ji}(x)$, $a_0(x) \leqq 0$, $\alpha(x) > 0$ or $\equiv 0$, $\beta(x) > 0$, and $f(u, x)$ are given functions of their arguments for which $\alpha, \beta \in C^0(\partial D)$ and $a_{ij}, a_0 \in C^{(0,\mu)}(D^*)$, $\bar{D} \subset D^*$, $\bar{D} = D + \partial D$. Furthermore it is assumed that L is elliptic; *i.e.*,

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j > 0, \qquad \sum_{i=1}^{m} \xi_i^2 \neq 0, \qquad x \in \bar{D}.$$

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The following assumptions concerning the nonlinear term $f(u, x)$ will be used below. Let R denote the set of real numbers.

$H1$: $f(\xi, x) = r(x)\xi + \xi g(\xi, x)$ where $r(x) \in C^0(\bar{D})$, $r(x) < 0$, $g(\xi, x) \in C^0(R \times \bar{D})$, and $g(\xi, x) = O(\xi)$ as $\xi \to 0$;

$H2$: $\xi f(\xi, x) \leqq 0$, $(\xi, x) \in R \times \bar{D}$;

$H3$: $g(\xi, x) = g_0(x)\xi^{2p} + o(\xi^{2p})$ as $\xi \to 0$ where $g_0(x) \in C^0(\bar{D})$ satisfies $g_0(x) \geqq 0$ or $\leqq 0$ on $\bar{D}$ (but $\not\equiv 0$) and p is a positive integer.

By a solution of (1.1) we mean an ordered pair (u, λ); the smoothness of u will be brought out below. Clearly $(0, \lambda)$ is a solution for all real λ.

The local existence of solutions to (1.1) bifurcating from the trivial solution at characteristic values of the associated linearized problem (cf. §2) of odd multiplicity is known and follows from general theorems on nonlinear completely continuous operators on Banach spaces [14]; in particular, this follows for the first characteristic value λ_1 of the linearized problem which, under the above assumptions, is always simple. Moreover, the associated characteristic solution is of one sign in D while all other characteristic solutions corresponding to any other characteristic value vanish somewhere in D [15].

In this paper we use the following terminology: in a given Banach space, a continuum C (*i.e.*, a compact, connected set) is said to *join* two closed subsets A, B if $A \cap C \neq \emptyset$, $B \cap C \neq \emptyset$. A set D is said to be a *continuum joining* a given element b of the Banach space to "∞" if for every bounded open set Ω containing b there exists a subset of D which is a continuum joining b to the boundary of Ω. We propose to show below that under the above hypotheses and in an appropriate Banach space (i) the nodal properties of the linear solutions described above carry over locally to the solutions of the nonlinear problem (1.1), (ii) there exist branches of positive and negative solutions each of which is a continuum joining $(0, \lambda_1)$ to "∞", and (iii) for each characteristic value $\lambda_n > \lambda_1$ of the linear problem of odd multiplicity there exists a branch of solutions which is a continuum joining $(0, \lambda_n)$ to either "∞" or to some $(0, \lambda_k)$, $k \neq 1, n$. Our main results are contained in Theorem 2.3, Corollary 2.5, and Theorem 3.2 where the exact statements concerning (i)–(iii) are given. In §4 we give some results concerning the asymptotic nature of the global branch of solutions bifurcating from $(0, \lambda_1)$.

The techniques used here were motivated by and are very similar to those used in [16, 17] to study the case $m = 1$. Similar techniques are used in [18, 19].

2. Preliminaries. The assumptions made above guarantee the existence of a fundamental solution for A on D [13] from which a Green's function $G_i(x, y)$, $i = 1, 2$, can be constructed in the usual manner for both problems (1.1) with $\alpha(x) > 0$, $\alpha(x) \equiv 0$ respectively. The integral operator $L_i\varphi = \int_D G_i(x, y)\varphi(y)dy$ is a linear, compact mapping of $C^{(0,\mu)}(\bar{D})$ into $C^1(\bar{D})$ [13]. Thus, (1.1) with either $\alpha(x) > 0$ or $\alpha(x) \equiv 0$ may be reformulated as $\Phi_i(u, \lambda) = 0$, $i = 1, 2$, respectively, where $\Phi_i(u, \lambda) = u - \lambda L_i \bar{f}u$, $\bar{f}u = f(u(x), x)$. We introduce two Banach spaces $E_1 = C^1(\bar{D})$ and $E_2 = \{u \in C^1(\bar{D}): u(x) = 0, x \in \partial D\}$ under the norm $||u|| = \max |u(x)| + \sum \max |D_i u(x)|$. By H1 the nonlinear operator $\bar{f}$ is continuous and bounded as a mapping from $C^1(\bar{D})$ into $C^0(\bar{D})$ and, hence, $L_i \bar{f}u$ is a completely continuous operator from E_i into itself, $i = 1, 2$, [14]. It is easily shown that $L_i ru$ is the Fréchet derivative of $L_i \bar{f}u$ at $u = 0$. It is well known [14] that bifurcation of nontrivial solutions to $\Phi_i(u, \lambda) = 0$ can only occur at $(0, \lambda_n)$, $n = 1, 2, \cdots$, where λ_n is a characteristic value of $L_i r$; that the spectrum of $L_i r$ is at most countable; and that the spectrum can have only $+\infty$ as an accumulation point. This linear problem is equivalent to the problem $Au = \lambda r(x)u$ (with the corresponding boundary conditions) which has been well studied (*e.g.*, see [15]). It is easily seen from Green's first identity and H2 that for both cases $\alpha(x) > 0$, $\alpha(x) \equiv 0$, we have $\lambda_n > 0$ for all n. Moreover, it has been shown [15] that any characteristic solution u_1 corresponding to λ_1 is of one sign in D and, hence, any characteristic solution u_n corresponding to λ_n, $n \neq 1$, changes sign in D (since $\int_D u_1 u_n r\, dx = 0$, $n \neq 1$). It is now clear that λ_1 is simple, for if two independent characteristic solutions existed for λ_1 then an appropriate linear combination would change sign in D; on the other hand λ_n, $n \neq 1$, may be multiple; *e.g.*, Helmholtz's equation in two dimensions on a square [15].

From a theorem of Krasnosel'skii [14, p. 196] we have the existence of a local continuous branch of solutions to (1.1) in E_i (for problem $i = 1, 2$) bifurcating from $(0, \lambda_n)$ for those λ_n of odd multiplicity (in particular, λ_1). We now wish to show that the solutions on these branches have the nodal structure mentioned above for characteristic solutions of the linear problem. We begin by defining the sets $S_i(\lambda) = \{u \in E_i: \Phi_i(u, \lambda) = 0, ||u|| \neq 0\}$, $N_1^+ = \{u \in E_1: u > 0 \text{ on } \bar{D}\}$, $N_2^+ = \{u \in E_2: u > 0 \text{ on } D, \partial u/\partial \nu < 0 \text{ on } \partial D\}$, $N_i^0 = \{u \in E_i: u \text{ changes sign in } D\}$, and $N_i^- = \{u \in E_i: -u \in N_i^+\}$.

Lemma 2.1. *(a) N_i^j are open subsets of E_i, $j = \pm, 0$ and $i = 1, 2$.*
(b) Under H1, H2, *for all $\lambda \in R$, $j = \pm, 0$ and $i = 1, 2$, we have $S_i(\lambda) \cap \partial N_i^j = \varnothing$.*
(c) N_i^j, for fixed i, are mutually disjoint.

Proof. (a) That N_1^j, N_i^0 are open is obvious from the definition of $||\cdot||$. Suppose $u \in N_2^+$ and $u_n \in E_2 - N_2^+$, $u_n \to u$; then passing to a subsequence if necessary we may assume u_n all change sign in D or all satisfy $\partial u_n/\partial \nu = 0$ somewhere on ∂D. In the first case u_n has a local, negative minimum at some $x_n \in D$ where of course $D_i u(x_n) = 0$ for all $1 \leqq i \leqq m$. Passing to a subsequence if necessary we let $x_n \to x_0 \in \bar{D}$. Since $u_n(x_n) \to u(x_0)$, $u_n(x_n) < 0$, $u(x_0) \geqq 0$ we conclude $x_0 \in \partial D$ where $u(x_0) = 0$. But also $D_i u_n(x_n) \to D_i u(x_0)$ and, hence, $\partial u/\partial \nu = 0$ at $x_0 \in \partial D$ in contradiction to $u \in N_2^+$. In the second case we have $\partial u_n/\partial \nu = 0$ at $x_n \in \partial D$; passing to a subsequence we have $x_n \to x_0 \in \partial D$ and, hence, $\partial u/\partial \nu = 0$ at $x_0 \in \partial D$ once again contrary to $u \in N_2^+$. Thus, N_2^+ is open. Similarly N_2^- is open.

(b) From Green's first identity and H2 it is easily shown that nontrivial solutions to (1.1) (with either $\alpha \equiv 0$ or $\alpha > 0$) exist only for $\lambda > 0$. Suppose

$u \in S_1(\lambda) \cap \partial N_1^+$; then $u \geqq 0$ on $\bar{D}$ and $u(x_0) = 0$ for some $x_0 \in \bar{D}$. By H2 it follows that $Au \leqq 0$ and by a well known max-min principle for A we conclude $x_0 \in \partial D$. But the boundary condition $Bu = 0$ with $\alpha(x) > 0$ implies also that $\partial u/\partial \nu = 0$ at x_0, in contradiction to a known theorem [20, p. 65] unless $u \equiv 0$. But $u \equiv 0$ contradicts $u \in S_i(\lambda)$ and, hence, $S_i(\lambda) \cap \partial N_1^+ = \emptyset$. Similar proofs may be constructed for the other intersections $S_i(\lambda) \cap \partial N_i^j$.

Part (c) is obvious.

Let $\tilde{g}(u, x)$ be a function for which $\tilde{f}(u, x) = r(x)u + u\tilde{g}(u, x)$ satisfies H1, H2 and define $\Phi_i(u, \lambda, t) = u - \lambda \int G_i(x, y)[tf + (1 - t)\tilde{f}]dy$, $0 \leqq t \leqq 1$. The following lemma is proved, with only slight modifications, exactly as a similar lemma in [17]; the proof is accordingly omitted here. Let $S_i(\lambda, t) = \{u \in E_i : \Phi_i(u, \lambda, t) = 0, ||u|| \neq 0\}$ and $\sigma = \{\lambda_n\}$ denote the spectrum of $L_i r$.

Lemma 2.2. *Let* $\inf \emptyset = +\infty$, *assume* H1, H2 *hold, and set* $N_i = N_i^+ \cup N_i^-$. *Define the functions*

$$r_i(\lambda) = \inf_{E_i} \{||u||: \ u \in S_i(\lambda, t) \cap (E_i - \bar{N}_i) \quad \text{for some} \quad t \in [0, 1]\},$$

$$r_i^0(\lambda) = \inf_{E_i} \{||u||: \ u \in S_i(\lambda, t) \cap (E_i - \bar{N}_i^0) \quad \text{for some} \quad t \in [0, 1]\},$$

$$\rho_i^j(\lambda) = \inf_{E_i} \{||u||: \ u \in S_i(\lambda, t) \cap N_i^j \quad \text{for some} \quad t \in [0, 1]\}, \qquad j = \pm, 0.$$

Then $r_i(\lambda)$, $\rho_i^0(\lambda)$ *and* $r_i^0(\lambda)$, $\rho_i^j(\lambda)$, $j = \pm$, *are positive and lower semi-continuous on* $(R - \sigma) \cup \{\lambda_1\}$ *and* $R - \{\lambda_1\}$ *respectively.*

We may now prove the following theorem which describes the nodal structure of local solution branches to (1.1). We set $B_i(\epsilon) = \{u \in E_i : ||u|| < \epsilon\}$.

Theorem 2.3. *Assume* H1, H2. *To each* $\lambda \in R$ *there exist two constants* $\epsilon(\lambda) > 0$, $\delta(\lambda) > 0$ *such that for* $\mu \in [\lambda - \delta(\lambda), \lambda + \delta(\lambda)]$, *all* $t \in [0, 1]$, *and* $i = 1, 2$,

(a) $\lambda = \lambda_1 \Rightarrow S_i(\lambda, t) \cap \overline{B_i(\epsilon(\lambda_1))} \subset N_i$;

(b) $\lambda \in \sigma - \{\lambda_1\} \Rightarrow S_i(\lambda, t) \cap \overline{B_i(\epsilon(\lambda))} \subset N_i^0$;

(c) $\lambda \notin \sigma \Rightarrow S_i(\lambda, t) \cap \overline{B_i(\epsilon(\lambda))} \cap N_i^j = \emptyset$.

Furthermore, if H3 *holds for both* g *and* $\tilde{g}$ *with the same integer* p, *then* $\epsilon(\lambda)$, $\delta(\lambda)$ *for* $\lambda \in \sigma$ *may be chosen such that for all* $t \in [0, 1]$

(d) $S_i(\lambda, t) \cap \partial B_i(\epsilon(\lambda)) = \emptyset$, $i = 1, 2$.

Proof. (a) For any $0 < \delta(\lambda_1) < \frac{1}{2} \operatorname{dist}(\lambda_1, \sigma - \{\lambda_1\})$, set $\epsilon(\lambda_1) = \frac{1}{2} \inf \{r_1(\mu), r_2(\mu)\}$ for $\mu \in [\lambda_1 - \delta(\lambda_1), \lambda_1 + \delta(\lambda_1)]$. By Lemma 2.2 $\epsilon(\lambda_1) > 0$ and part (a) follows from the definition of $r_i(\lambda)$ and Lemma 2.1(b).

(b) For any $0 < \delta(\lambda) < \frac{1}{2} \operatorname{dist}(\lambda, \sigma - \{\lambda\})$ set $\epsilon(\lambda) = \frac{1}{2} \inf \{r_1^0(\mu), r_2^0(\mu)\}$ for $\mu \in [\lambda - \delta(\lambda), \lambda + \delta(\lambda)]$. Once again part (b) follows from Lemma 2.2 and the definition of $r_i^0(\lambda)$.

(c) For any $0 < \delta(\lambda) < \frac{1}{2}$ dist (λ, σ), set $\epsilon(\lambda) = \frac{1}{2} \inf \{\rho_i^j(\mu)\} > 0$ for $\mu \in [\lambda - \delta(\lambda), \lambda + \delta(\lambda)]$ and $i = 1, 2, j = \pm, 0$.

(d) From a known result in bifurcation theory [10, p. 154], the local bifurcation of nontrivial solutions of (1.1) near $(0, \lambda_n)$ is one-sided since $g_0(x) \geqq 0$ ($\leqq 0$), but $\not\equiv 0$ (and the direction of bifurcation is determined by the sign of $g_0(x)$); in particular then there exists a constant $\rho_n > 0$ such that no nontrivial solution of (1.1) exists for $\lambda = \lambda_n$ satisfying $||u|| = \rho_n$. Clearly, if g, $\tilde{g}$ satisfy H3 with the same integer p then $tg + (1 - t)\tilde{g}$ satisfies H3 with the integer p for all $t \in [0, 1]$. It is clear from the theory presented in [10, p. 154] that ρ_n can be taken independent of t. Thus,

$$S_i(\lambda_n, t) \cap \partial B_i(\rho_n) = \varnothing \quad \text{for all} \quad t \in [0, 1]. \tag{2.1}$$

Now suppose that part (d) is false with $\epsilon(\lambda_n) = \rho_n$ and, hence, there exist functions $w_k \in S_i(\mu_k, t_k) \cap \partial B_i(\rho_n)$ for $\mu_k \to \lambda_n$, $t_k \to t_0 \in [0, 1]$. Since $\Phi_i(w_k, \mu_k, t_k) = 0$ implies $\{w_k\}$, a bounded sequence, is the image of a bounded set under a completely continuous operator, $\{w_k\}$ has a convergent subsequence. Assume $w_k \to w_0 \in E_i$; then by the continuity of the operator Φ_i it follows that $\Phi_i(w_0, \lambda_n, t_0) = 0$. Whereas $||w_0|| = \rho_n$ we have a contradiction to (2.1) and, hence, part (d) is true.

We conclude this section with a Leray–Schauder degree calculation. The degree of a mapping Φ on a bounded open set Ω with respect to 0 will be denoted by $d(\Phi, \Omega)$. (A summary of the properties of the Leray–Schauder degree needed here may be found in [16, 17, 18]. For a complete treatment see [14].)

Theorem 2.4. *Assume* H1, H2, H3 *and let* $\epsilon(\lambda)$, $\delta(\lambda)$ *be as in Theorem* 2.3. *Set* $\epsilon_n(\lambda) = \frac{1}{2} \min \{\epsilon(\lambda_n), \epsilon(\lambda)\}$. *Then for* $\lambda \in (\lambda_n, \lambda_n + \delta(\lambda_n))$, $\bar{\lambda} \in (\lambda_n - \delta(\lambda_n), \lambda_n)$, $\lambda_n \in \sigma$, $n \neq 1$, *of odd multiplicity we have*

$$\begin{aligned} &d(\Phi_i(\lambda), [B_i(\epsilon(\lambda_n)) - \overline{B_i(\epsilon_n(\lambda))}] \cap N_i^0) \\ &-d(\Phi_i(\bar{\lambda}), [B_i(\epsilon(\lambda_n)) - \overline{B_i(\epsilon_n(\bar{\lambda}))}] \cap N_i^0) = \pm 2. \end{aligned} \tag{2.2}$$

Also for $n = 1$ *we have for both* $j = \pm$

$$d(\Phi_i(\mu), B_i(\epsilon(\lambda_1)) - \overline{B_i(\epsilon_1(\mu))}] \cap N_i^j) = \operatorname{sgn}(\mu - \lambda_1) \tag{2.3}$$

for either all $\mu \in (\lambda_1, \lambda_1 + \delta(\lambda_1))$ *or all* $\mu \in (\lambda_1 - \delta(\lambda_1), \lambda_1)$ *where* $\operatorname{sgn} a = a/|a|$ *for* a *real and nonzero. Here we have set* $\Phi_i(u, \lambda) \equiv \Phi_i(\lambda)$.

Corollary 2.5. (*a*) *There exist two local branches of solutions to* (1.1), *one consisting of positive and one of negative solutions, bifurcating from* $(0, \lambda_1)$.

(*b*) *If* $\lambda_n \in \sigma$, $n \neq 1$, *is of odd multiplicity then there exists a local branch of solutions which change sign in* D *bifurcating from* $(0, \lambda_n)$.

Proof. All degrees in the theorem and its proof below are well defined, for by Lemma 2.1 and Theorem 2.3 no solutions exist on the boundaries of the sets involved. Using λ as a parameter, the homotopic invariance of degree implies for any $t \in [0, 1]$

$$d(\Phi_i(\lambda, t), B(\epsilon(\lambda_n))) = d(\Phi_i(\bar{\lambda}, t), B(\epsilon(\lambda_n)))$$

while the additivity property in turn yields

$$\begin{aligned} d(\Phi_i(\lambda, t), [B_i(\epsilon(\lambda_n)) - \overline{B_i(\epsilon_n(\lambda))}] \cap N_i^0) \\ + d(\Phi_i(\lambda, t), B_i(\epsilon_n(\lambda))) \\ = d(\Phi_i(\bar{\lambda}, t), [B_i(\epsilon(\lambda_n)) - \overline{B_i(\epsilon_n(\bar{\lambda}))}] \cap N_i^0) \\ + d(\Phi_i(\bar{\lambda}, t), B_i(\epsilon_n(\bar{\lambda}))). \end{aligned} \tag{2.4}$$

However, since $u - \lambda L_i r u$ is invertible for $\lambda \notin \sigma$, λ_n is of odd multiplicity, and Theorem 2.3(c) holds we have (see [14, p. 136])

$$\begin{aligned} d(\Phi_i(\mu, t), B_i(\epsilon_n(\mu)) \cap N_i^0) &= d(\Phi_i(\mu, t), B_i(\epsilon_n(\mu))) \\ &= i(\Phi(\mu, t), 0, 0) = (-1)^m, \qquad \mu \neq \lambda_n , \end{aligned} \tag{2.5}$$

where m is the sum of the multiplicities of all $\lambda_k < \mu$, $\lambda_k \in \sigma$. Using (2.5) at $\mu = \lambda, \bar{\lambda}$ and the fact that λ_n has odd multiplicity together with (2.4) at $t = 1$, we obtain (2.2).

To prove (2.3) we first note that the above argument and, hence, (2.2) with $+2$ and all $t \in [0, 1]$ are valid with $n = 1$ provided N_i^0 is replaced by $N_i = N_i^+ \cup N_i^-$. As remarked above in the proof of Theorem 2.3(d), because of H3, the bifurcation at λ_n is one sided and, hence, by taking $\epsilon(\lambda_n)$, $\delta(\lambda_n)$ possibly smaller we know that one of the degrees in (2.2) must be zero. For definiteness we assume bifurcation occurs to the right and consequently

$$d(\Phi_i(\lambda, t), [B_i(\epsilon(\lambda_1)) - \overline{B_i(\epsilon_1(\lambda))}] \cap N_i) = 2.$$

By Lemma 2.1 and the additivity of degree, this yields

$$\sum_{j=\pm} d(\Phi_i(\lambda, t), [B_i(\epsilon(\lambda_1)) - \overline{B_i(\epsilon_1(\lambda))}] \cap N_i^j) = 2. \tag{2.6}$$

We now set $\tilde{g}(u, x) = g(|u|, x)$ and notice that for $t = 0$, u is a solution if and only if $-u$ is also a solution; *i.e.*, $\Phi_i(\lambda, 0)$ is an odd operator in u. Hence, from the definition of degree and N_i^j we conclude

$$\begin{aligned} d(\Phi_i(\lambda, 0), [B_i(\epsilon(\lambda_1)) - \overline{B_i(\epsilon_1(\lambda))}] \cap N_i^+) \\ = d(\Phi_i(\lambda, 0), [B_i(\epsilon(\lambda_1)) - \overline{B_i(\epsilon_1(\lambda))}] \cap N_i^-). \end{aligned} \tag{2.7}$$

Using t as a homotopy parameter we also have

$$d(\Phi_i(\lambda, t), [B_i(\epsilon(\lambda_1)) - \overline{B_i(\epsilon_1(\lambda))}] \cap N_i^j) = C_j , \qquad j = \pm,$$

for $t \in [0, 1]$, $C_j =$ const., and finally we find (since $C_+ = C_-$ by (2.7))

$$\begin{aligned} d(\Phi_i(\lambda, 1), [B_i(\epsilon(\lambda_1)) - \overline{B_i(\epsilon_1(\lambda))}] \cap N_i^+) \\ = d(\Phi_i(\lambda, 1), [B_i(\epsilon(\lambda_1)) - \overline{B_i(\epsilon_1(\lambda))}] \cap N_i^-) \end{aligned}$$

which together with (2.6) at $t = 1$ and $\Phi_i(\lambda, 1) \equiv \Phi_i(\lambda)$ yields (2.3) for $\mu = \lambda > \lambda_1$. But λ in this argument was arbitrary, $\lambda_n < \lambda < \lambda_n + \delta(\lambda_n)$, since the case of bifurcation to the right was considered. The case of bifurcation to the left (*i.e.*, $\mu < \lambda_1$ in (2.2)) is proved similarly. This completes the proof of the theorem.

The direction of local bifurcation (an important consideration in applications, particularly at λ_1 [5–7]) is determined by the sign of $g_0(x)$: if $g_0(x)$ is positive, then the bifurcation is to the right while the bifurcation is to the left if $g_0(x)$ is negative [10].

3. Global theorems. In this section we wish to show that the local branches described in Corollary 2.5 exist as continua out of the local neighborhood of $(0, \lambda_n)$. We let $E_i \times R$ have the product topology and define $S_i^j = \{(u, \lambda) : u \in S_i(\lambda) \cap N_i^j\}$, $j = \pm, 0$. By definition $(0, \lambda_n) \notin S_i^j$; but by Theorem 2.5, $(0, \lambda_n) \in \partial S_i^0$, $n \neq 1$, and $(0, \lambda_1) \in \partial(S_i^+ \cup S_i^-)$. Also by Theorem 2.3, we see that $(0, \lambda) \notin \bar{S}_i^j$, $j = \pm, 0$, for all $\lambda \notin \sigma$.

Lemma 3.1. Let Ω_i^n be an arbitrary bounded open subset of $E_i \times R$ such that $(0, \lambda_n) \in \Omega_i^n$, $(0, \lambda_k) \notin \partial\, \Omega_i^n$, $k \neq n$. Then for $j = \pm, 0$ and all $n \geqq 1$,

(*a*) $\overline{S_i^j \cap \Omega_i^n}$ *is a compact subset of* $E_i \times R$;

(*b*) $S_i^j \cap \partial\Omega_i^n = \varnothing \Rightarrow \overline{S_i^j \cap \Omega_i^n} \subset \Omega_i^n$.

Proof. Part (a) follows from the boundedness of Ω_i^n and the complete continuity of $L_i \bar{f}$.

The proof of part (b) is by contradiction. We first notice that $\overline{S_i^j \cap \Omega_i^n} \subset \overline{\Omega_i^n}$. Now let $(u, \lambda) \in \overline{S_i^j \cap \Omega_i^n} \cap \partial\Omega_i^n$; then we may find $(w_k, \mu_k) \in S_i^j \cap \Omega_i^n$ such that $(w_k, \mu_k) \to (u, \lambda)$. By the continuity of Φ_i it follows that $\Phi_i(u, \lambda) = 0$. But also $(u, \lambda) \in \partial\Omega_i^n$, so since $(0, \lambda_k) \notin \partial\Omega_i^n$ for all k we conclude $u \not\equiv 0$ by the remarks preceding the lemma. Thus, $(u, \lambda) \in S_i^j$. But since also $(u, \lambda) \in \partial\Omega_i^n$ we have a contradiction to the hypothesis in (b).

We now give our main result.

Theorem 3.2. Let H1, H2, H3 *hold and let Ω_i^n be any bounded open set of $E_i \times R$ for which $(0, \lambda_k) \in \Omega_i^n$, $(0, \lambda_n) \notin \bar{\Omega}_i^n$, $k \neq n$, where λ_n has odd multiplicity. Then $S_i^0 \cap \partial\Omega_i^n \neq \varnothing$ for $n \neq 1$ and $S_i^j \cap \partial\Omega_i^1 \neq \varnothing$ for $j = \pm$.*

Proof. The proof follows closely those found in [18, 19], both of these being motivated and modelled after the work of Rabinowitz in [16]. The proof is by contradiction and uses Leray–Schauder degree.

We first consider the case $n \neq 1$ and assume $S_i^0 \cap \partial\Omega_i^n = \varnothing$. The degrees that follow are well defined as this assumption together with Lemma 2.1 and Theorem 2.3 insure that no solutions appear on the boundaries of the open sets involved. Set $\Omega_i^n(\lambda) = \{u \in E_i : (u, \lambda) \in \Omega_i^n\}$, an open subset of E_i. Let $\epsilon(\lambda)$, $\delta(\lambda)$ be as in Theorem 2.3 and $\epsilon_n(\lambda)$ be as in Theorem 2.4. Choose $\lambda > \lambda_n$, $\lambda \notin \sigma$, such that $\Omega_i^n(\lambda) \neq \varnothing$ and $\lambda^* > \lambda$ such that $\Omega_i^n(\lambda^*) \neq \varnothing$, $\Omega_i^n(\lambda^*) \cap \overline{S_i^0} = \varnothing$. This is possible since by the preceding Lemma 3.1 $\overline{S_i^0 \cap \Omega_i^n}$ is a compact subset of Ω_i^n. If $\{\lambda_{n_k}\}$ is the set of those points in σ lying in $[\lambda, \lambda^*]$ we know that $(0, \lambda_{n_k}) \notin \overline{\Omega_i^n}$ and hence $(B_i(\rho) \times [\lambda_{n_k} - \rho, \lambda_{n_k} + \rho]) \cap \overline{\Omega_i^n} = \emptyset$ for some $\rho > 0$ sufficiently small and all k. Clearly $m_1 = \inf_\mu \operatorname{dist} \{(0, \mu); \partial\Omega_i^n(\mu)\}$, $\mu \in \bigcup_k [\lambda_{n_k} - \rho, \lambda_{n_k} + \rho]$, is positive. Denote by Λ the closed set of reals lying in $[\lambda, \lambda^*] - \bigcup_k (\lambda_{n_k} - \rho, \lambda_{n_k} + \rho)$; by Theorem 2.3 $m_2 = \inf_{\Lambda, k} \epsilon_{n_k}(\mu) > 0$. Now choose $\epsilon^* > 0$ so small that $\epsilon^* < \min(m_1, m_2, \epsilon_n(\lambda))$. By construction no solution of $\Phi_i(u, \mu) = 0$ lies on the boundary of $[\Omega_i^n - (\overline{B_i(\epsilon^*)} \times [\lambda, \lambda^*])] \cap S_i^0$ for $\mu \in [\lambda, \lambda^*]$, because if (u, μ) were such a solution on the boundary of this set then either $(u, \mu) \in \partial\Omega_i^n$ or $\|u\| = \epsilon^*$ and $\mu \in \Lambda$ (by the definition of m_1 and the choice of $\epsilon^* < m_1$), both cases being impossible by $S_i^0 \cap \partial\Omega_i^n = \emptyset$ and Theorem 2.3 respectively. The homotopic invariance of degree thus gives

$$d(\Phi_i(\mu), {}^*\Omega_i^n(\mu)) = C = \text{constant}, \qquad \mu \in [\lambda, \lambda^*], \tag{3.1}$$

where we have set ${}^*\Omega_i^n(\mu) = [\Omega_i^n(\mu) - \overline{B_i(\epsilon^*)}] \cap N_i^0$. Since by choice $\Omega_i^n(\lambda^*) \cap \overline{S_i^0} = \emptyset$, no solution lies in ${}^*\Omega_i^n(\lambda^*)$ and, hence setting $\mu = \lambda^*$ in (3.1) yields $C = 0$. Since no solution lies in $[B_i(\epsilon_n(\lambda)) - \overline{B_i(\epsilon^*)}] \cap N_i^0$, the additivity of degree together with (3.1) for $\mu = \lambda$ yields

$$d(\Phi_i(\lambda), [\Omega_i^n(\lambda) - \overline{B_i(\epsilon_n(\lambda))}] \cap N_i^0) = 0. \tag{3.2}$$

A similar argument holds for $\lambda < \lambda_n$, $\lambda \notin \sigma$, and hence (3.2) is valid for all $\lambda \notin \sigma$.

Suppose now that $\lambda \in (\lambda_n, \lambda_n + \delta(\lambda_n))$ and $\bar\lambda \in (\lambda_n - \delta(\lambda_n), \lambda_n)$. By homotopic invariance

$$d(\Phi_i(\mu), [\Omega_i^n(\mu) - \overline{B_i(\epsilon(\lambda_n))}] \cap N_i^0) = \text{constant} \tag{3.3}$$

for $\bar\lambda \leqq \mu \leqq \lambda$, and by additivity of degree together with (3.2)

$$d(\Phi_i(\mu), [\Omega_i^n(\mu) - \overline{B_i(\epsilon(\lambda_n))}] \cap N_i^0) + d(\Phi_i(\mu), [B_i(\epsilon(\lambda_n)) - \overline{B_i(\epsilon_n(\mu))}] \cap N_i^0) = 0 \tag{3.4}$$

for $\mu = \lambda, \bar\lambda$. Subtracting the two equations (3.4) for $\mu = \lambda, \bar\lambda$ and using (3.3) we get the contradiction that the difference in (2.2) is zero and not ± 2 as asserted by Theorem 2.4. This contradiction proves that $S_i^0 \cap \partial\Omega_i^n = \emptyset$ is false, $n \neq 1$.

Finally, for $n = 1$ exactly the same argument can be used with S_i^j, $j = +$ or $-$, replacing S_i^0 throughout and a contradiction to (2.3) being reached.

The following corollary is proved exactly as Corollary 1.34 in [16].

Corollary 3.3. *Under the hypotheses of Theorem 3.2 there exists a continuum of solutions (u, λ) to (1.1), $u \in N_i^0$, joining $(0, \lambda_n)$, $n \neq 1$, to $\partial\Omega_i^n$ and a continuum of solutions (u, λ), u in each of N_i^+, N_i^- joining $(0, \lambda_1)$ to $\partial\Omega_i^1$.*

Since Ω_i^n was an arbitrary bounded open set such that $(0, \lambda_n) \in \Omega_i^n$, $(0, \lambda_k) \notin \overline{\Omega}_i^n$ for $k \neq n$ it is clear that a continuum joins $(0, \lambda_n)$ to $(0, \lambda_k)$, $k \neq n$, or to "∞" (or both). However, by Theorem 3.2 it is clear that a continuum from $(0, \lambda_n)$, $n \neq 1$, in N_i^0 can not join $(0, \lambda_1)$ nor can a continuum from $(0, \lambda_1)$ in either N_i^j, $j = \pm$, join $(0, \lambda_n)$, $n \neq 1$. Thus, we have

Corollary 3.4. Under the hypotheses of Theorem 3.2 *there exist continua of solutions to* (1.1) *in each of* N_i^+, N_i^- *joining* $(0, \lambda_1)$ *to* "∞" *and a continuum in* N_i^0 *joining* $(0, \lambda_n)$, $n \neq 1$, *to either* $(0, \lambda_k)$, $k \neq 1$, n *or to* "∞".

4. Asymptotic behavior of the branches from λ_1. In this section we want to consider some aspects of the nature of the sets $G_i^j = \{(||u||_0, \lambda): (u, \lambda) \in K_i^j\}$, $j = \pm$, where $||u||_0 = \max |u(x)|$ and K_i^j is a continuum of solutions (u, λ) to (1.1), $u \in N_i^j$, connecting $(0, \lambda_1)$ to "∞"; the graph of this set is the so called bifurcation diagram for (1.1) at λ_1. We first remark that, because of the inequality $||u|| \leqq K\, ||f(u(x), x)||_0$, $K > 0$, which follows from $u = \lambda \int G(x, y) f(u, y)\, dy$ as applied to solutions of (1.1) and the continuity of $f(u, x)$ in u, convergence with respect to the product topology on $E_i \times R$ using $||\cdot||_0$ implies convergence with respect to the product topology using $||\cdot||$ insofar as solutions (u, λ) are concerned. Since the converse is obviously true, G_i^j is a continuum and its asymptotic behavior reflects that of the set $\{(||u||, \lambda): (u, \lambda) \in K_i^j\}$ as well; thus, G_i^j connects $(0, \lambda_1)$ to "∞". We define the sets $\Sigma_i^j = \{\lambda: (||u||_0, \lambda) \in G_i^j, u \in N_i^j\}$, $j = \pm$. Since G_i^j is a continuum connecting $(0, \lambda_1)$ to "∞", we know that Σ_i^j and $M_i^j = \{||u||_0: (||u||_0, \lambda) \in G_i^j, \lambda \in R\}$ are intervals on the positive real axis at least one of which must be infinite. As far as we know in general Σ_i^j may be open, closed, half open or closed and either finite or infinite, but it contains only positive reals as an easy application of Green's first identity to *any* solution of (1.1) will show (provided H2 holds). (Actually, as mentioned above, this proves that the *entire* spectrum of (1.1), under H2, is positive.) We now offer a theorem concerning M_i^j. We need

H4: $f(\xi, x) = 0$ for $0 \leqq |\xi| < +\infty$, $x \in \bar{D}$, if and only if $\xi = 0$.

Theorem 4.1. If H1, H2, H3, H4 *hold, then* $M_i^j = (0, +\infty)$ *for* $j = \pm$, $i = 1, 2$.

Proof. We begin by remarking that it is easy to show using H4 that for any $\epsilon > 0$

$$u \in B_i(\epsilon) \Rightarrow ||u/f(u, x)||_0 \leqq C < +\infty, \tag{4.1}$$

for some constant $C = C(\epsilon) > 0$. Applying Green's second identity to (u_1, λ_1), $(u, \lambda) \in K_i^j$, we find

$$0 = \int_D [\lambda u_1 f(u, x) - \lambda_1 u_1 u r(x)]\, dx$$

or

$$\lambda = \lambda_1 \frac{\int_D u_1 u r(x)\, dx}{\int_D u_1 f(u, x)\, dx}, \tag{4.2}$$

the denominator being non-zero since the integrand is of one sign for $u \in S_i(\lambda) \cap N_i^j$. Using a Mean Value Theorem for integrals ([21], p. 269) we have

$$\int_D u_1 f(u, x)\, dx = \int_D \frac{f(u, x)}{ur(x)} uu_1 r(x)\, dx$$
$$= \frac{f(u(\xi), \xi)}{u(\xi)r(\xi)} \int_D uu_1 r(x)\, dx, \qquad \xi \in \bar{D},$$

and, hence, from (4.2) we find

$$0 < \lambda \leqq \lambda_1 \left| \frac{u(\xi)r(\xi)}{f(u(\xi), \xi)} \right| \leqq \lambda_1 K \left\| \frac{u}{f(u, x)} \right\|_0 \tag{4.3}$$

where $K = \max |r(x)| < +\infty$. Clearly $M_i^j = (0, a)$ or $(0, a]$ for some $a > 0$ (possibly $a = +\infty$). Suppose $a < +\infty$; then $(u, \lambda) \in K_i^j$ implies $\|u\|_0 \leqq a < +\infty$ and (4.1), (4.3) imply $0 < \lambda \leqq \lambda_1 KC(a) < +\infty$. Thus, the finiteness of M_i^j implies the finiteness of Σ_i^j which contradicts the definition of K_i^j. It follows that $a = +\infty$ and the proof is complete.

Theorem 4.1 gives rather mild conditions under which (1.1) possess solutions of one sign of arbitrarily large norm. An interesting and important question is that of the structure of the corresponding spectrum Σ_i^j. We do not attempt here an extensive study of this question, but only offer two theorems which are easy consequences of the preceding work. The following lemma is an obvious result of Corollary 3.4.

Lemma 4.2. *If any solution* (u, λ) *of* (1.1) *satisfies an a priori estimate of the form* $\|u\|_0 \leqq C(\lambda)$ *where* $C(\lambda)$ *is a non-negative real valued function defined on* $[0, +\infty)$ *which is bounded on finite intervals then* Σ_i^j *is an infinite interval.*

Theorem 4.3. *If, in addition to* H1, H2, H3, $f(u, x)$ *satisfies*

$$|f(\xi, x)| \leqq a(x)|\xi|^q, \qquad x \in \bar{D}, \tag{4.4}$$

for some $0 \leqq q < 1$ *and function* $a(x)$ *bounded on* $\bar{D}$, *then* Σ_i^j *is an infinite subinterval of the positive reals,* $j = \pm$, $i = 1, 2$.

Proof. If $(u, \lambda) \in K_i^j$, then $u(x) = \int_D \lambda G(x, y) f(u(y), y) dy$ from which, together with (4.4), we have $\|u\|_0 \leqq K\lambda \|u\|_0^q$, $K =$ constant > 0. We can thus set $C(\lambda) = (K\lambda)^{1/(1-q)}$ in Lemma 4.2.

Finally we state a theorem which follows immediately from the Weak Positivity Lemma in [6].

Theorem 4.4. *Suppose, in addition to* H1, H2, H3, *that* $g(u, x)$ *satisfies the condition* $g(\xi, x) \leqq 0$ *for all* ξ *and* $x \in \bar{D}$. *Then* $\Sigma_i^j \subseteq (0, \lambda_1)$, $M_i^j = (0, +\infty)$, $j = \pm$, $i = 1, 2$.

More information on the structure of the graph of G_i^j can be obtained by using the results of this paper together with the many uniqueness and nonexistence theorems and a priori bounds found in the literature.

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