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Nonlinear Steklov Problems on the Unit Circle. II (and a Hydrodynamical Application)

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1. INTRODUCTION

This paper is concerned with extending the results of [3] to nonlinear Steklov problems involving harmonic conjugates [cf. (2.1)]. For a short discussion of the literature on Steklov problems see [3]. Unlike the problems treated in [3] the local existence of solution branches bifurcating from the zero solution for the problems treated here has not been previously established (except in the special case of Levi-Civita's water-wave theory [6]). Thus, in Section 1 we state and in Section 6 we prove the existence of such branches using the expansion techniques of Liapunov and Schmidt [10]. Beyond this we establish two main results: (1) that these local branches can be extended to "∞" and (2) that these branches can be characterized by the nodal structure of the solutions on them (see Corollary 4.1). In Section 5 we apply our results to the water-wave theory of Levi-Civita, where amongst other things a long standing conjecture of Levi-Civita is proved globally for solutions on these branches. Our main tool is Leray–Schauder degree theory and we use the notation and properties of this theory outlined in the Appendix of [3].

As noted in [3], the results obtained here in the Banach spaces B_k could also be obtained with essentially no modification of proof within spaces constructed in the same manner but using the nodal structure of the Steklov eigensolutions $r^k \cos k\theta$ (instead of $r^k \sin k\theta$). This remark together with the results below (Corollary 4.1) give rise to *four* branches of solutions to our problem bifurcating from $(0, k)$, $k =$ positive integer.

2. PRELIMINARIES

The problem we consider is the following:

$$\begin{aligned} \Delta u &= 0, \qquad r < 1 \\ \partial u/\partial r &= \lambda(u + f(u, v, \theta)), \qquad r = 1, \qquad -\pi \leqslant \theta \leqslant \pi, \end{aligned} \tag{2.1}$$

where (r, θ) are polar coordinates in the plane, Δ is the Laplace operator, f is a prescribed function of u, v, θ where v is the harmonic conjugate of u vanishing at $r = 0$, and λ is a real constant to be determined as part of the solution. By a solution to this problem (hereafter referred to as problem N) we mean an ordered pair (u, λ); the smoothness of u will be explicitly brought out below. We make the following assumptions concerning the nonlinear term f:

(H1) $f(\xi, \eta, \theta) = O(|\xi|^2 + |\eta|^2), f(0, \eta, \theta) \equiv 0$ for $-\infty < \xi, \eta < +\infty, \theta \in [-\pi, \pi]$;

(H2) f is analytic in its arguments and $f(\xi, \eta, \pi) = f(\xi, \eta, -\pi)$ for all $-\infty < \xi, \eta < +\infty, \theta \in [-\pi, \pi]$;

(H3) $f(-\xi, \eta, \theta) = -f(\xi, \eta, \theta); -\infty < \xi, \eta < +\infty, -\pi \leqslant \theta \leqslant \pi$.

The linearized version of problem N (the Steklov problem [8]) has the boundary condition

$$\partial u / \partial r = \lambda u, \qquad r = 1, \qquad -\pi \leqslant \theta \leqslant \pi, \tag{2.2}$$

and has nontrivial solutions only for $\lambda \in Z^+ \cup \{0\}$ (Z^+ is the set of positive integers) given by $(Ar^k \sin k\theta, k)$, $(Br^k \cos k\theta, k)$, $(C, 0)$ for arbitrary constants A, B, C and $k \in Z^+$(cf. [6]).

In order to formulate problem N as an operator equation as is done in [3] we first consider the operator Tu which maps the harmonic function u onto its harmonic conjugate v vanishing at $r = 0$. Let

$$\| u \|_D = \max_{\bar{D}} | u | + \sum_{i=1,2} \max_{\bar{D}} | D_i u |$$

where $\bar{D}$ is the closure of the unit circle D and D_i is the differentiation operator in the direction of the i-th rectangular coordinate, and let B be the Banach space of harmonic functions u for which $\| u \|_D < +\infty, u = 0$ at $r = 0$. If we consider the analytic functions $g_n(z) = u_n + iTu_n$ of a complex variable z, $|z| < 1$, $u_n \in B$, then from the Cauchy–Riemann equations $\| u_n \| \to 0$ implies $g_n'(z) \to 0$ uniformly on $\bar{D}$; i.e., $g_n(z) \to 0$ [since $g(0) = 0$] and, hence $\| Tu_n \|_D \to 0$. Thus, T is a continuous linear operator mapping B into itself.

By (H1) the nonlinear operator $\bar{f}u \equiv f(u, Tu, \theta)|_{r=1}$ is continuous as a mapping from B into $C^1[-\pi, \pi]$ endowed with the norm $\| \cdot \|_{\partial D}$. (Note $\| u \|_D = \| u \|_{\partial D}$ for $u \in B$.) Finally, using the Neumann function $N(r, \theta; \sigma)$ and its properties (cf. [2, 3]) we have that if

$$\int_{-\pi}^{\pi} \bar{f}u \, d\theta = 0 \tag{2.3}$$

holds, then

$$u(r, \theta) = \lambda \int_{-\pi}^{\pi} N(r, \theta; \alpha) \bar{f}u \, d\alpha$$

solves problem N. This integral equation for u may be written $\Phi(u, \lambda) = 0$ where $\Phi(u, \lambda) \equiv u - \lambda Au$, $Au \equiv L\bar{f}u$, and $Lu = \int_{-\pi}^{\pi} Nu \, d\alpha$. In order to guarantee (2.3) and that A maps an appropriate Banach space into itself, we introduce the Banach spaces B_k, $k \geqslant 1$, of harmonic functions u, $\| u \|_{\bar{D}} < +\infty$, for which $u(1, \theta)$ is an odd function of period $2\pi/k$ for $\theta \in [-\pi, \pi]$ and $u(r, n\pi/k) \equiv 0$, $r < 1$, $n = 0, \pm 1,..., \pm k$; clearly $B_k \subset B$, $k \geqslant 1$. We need now the hypothesis

(H4)$_k$ $f(\xi, \eta, \theta)$ is an even, periodic function of $\theta \in [-\pi, \pi]$ of period $2\pi/k$ for each $-\infty < \xi, \eta < +\infty$.

If f is independent of θ, then (H4)$_k$ is satisfied for all $k \geqslant 1$. If $u \in B_k$ then an easy Fourier analysis shows Tu is an even function of period $2\pi/k$, $\theta \in [-\pi, \pi]$. Thus, under (H4)$_k$ and (H3) the function $\bar{f}u$ is an odd function of period $2\pi/k$, $\theta \in [-\pi, \pi]$ and consequently, as shown in [3], $L\bar{f}u \in B_k$; i.e., $A: B_k \to B_k$. It was also shown in [3] that L is compact and thus, since $\bar{f}$ has been shown above to be continuous, A is a completely continuous operator of B_k into itself. Whereas (2.3) is fulfilled for $u \in B_k$ (since $\bar{f}u$ is odd) we may reformulate problem N as the operator equation $\Phi(u, \lambda) = 0$, $(u, \lambda) \in B_k \times \mathscr{R}$, $\mathscr{R}$ = reals.

3. Local Theory

It is easy to show that the Fréchet derivative of Au is the operator Lu whose characteristic solutions are the nontrivial solutions to the Steklov problem (2.2) given above. As an operator on B_k, L has only *simple* integer characteristic values (with the solutions given above involving the sine) the smallest being k; $\lambda = 0$ is not a characteristic value since the only constant function in any B_k is the zero function. Consequently, by a result of Krasnosel'skii [4], we can assert the existence [locally near $(0, k)$] of a continuous branch of solutions to problem N in B_k. More constructively we offer the following theorem which is proved in Section 6 below.

THEOREM 3.1. *Assume* (H1)–(H4)$_k$ *are valid. Problem N has, for* $|\mu|$ *sufficiently small, solutions* $u_k \in B_k$ *of the form*

$$u_k = \sum_{n=1}^{\infty} u_{nk}\mu^n, \qquad \lambda = k + \sum_{n=1}^{\infty} \lambda_{nk}\mu^n, \tag{3.1}$$

where $u_{nk} \in B_k$ *for all* n. *Here* $Tu_k = \sum_1^\infty Tu_{nk}\mu^n$. *The convergence of these series is with respect to* $\| \cdot \|_{\bar{D}}$.

We wish to show that the branches of solutions (3.1) may be characterized by their nodal structure; viz., that they have exactly the nodal structure of $r^k \sin k\theta$. As in [3] the following sets are introduced:

$$\mathscr{B}_k(\rho) = \{u \in B_k : \| u \|_{\bar{D}} < \rho\},$$

$\mathscr{N}_k^+ = \{u \in B_k : u = 0$ only on $\theta = n\pi/k, n = 0, \pm 1, \ldots, \pm k, \ \partial u/\partial\theta \neq 0$ at $r = 1, \ \theta = n\pi/k$, and $\partial u/\partial\theta > 0$ at $r = 1, \ \theta = 0\}$, $\mathscr{N}_k^- = -\mathscr{N}_k^+$, and $\mathscr{S}_k(\lambda) = \{u \in B_k : \Phi(u, \lambda) = 0, \| u \| \neq 0\}$. The following lemma is proved in [3, Lemmas 3.1 and 3.2] and remains valid for Φ as defined here:

LEMMA 3.1. *Let* (H1)–(H4)$_k$ *hold. Then*

(a) $\mathscr{N}_k^\nu$ *is open in* B_k *for* $\nu = +$ *or* $-$ *and all* $k \geq 1$;

(b) $\partial\mathscr{N}_k^\nu \cap \mathscr{S}_k(\lambda) = \varnothing$, $\nu = +$ *or* $-$ *and all* λ;

(c) *there exist functions* $\epsilon(\lambda') > 0$, $\delta(\lambda') > 0$ *such that*

$$\mathscr{S}_k(\lambda) \cap \mathscr{B}_k(\epsilon(k)) \subset \mathscr{N}_k, \qquad \mathscr{S}_k(\lambda) \cap \partial\mathscr{B}_k(\epsilon(k)) = \varnothing, \qquad \lambda' = k,$$

$$\mathscr{S}_k(\lambda) \cap \overline{\mathscr{B}_k(\epsilon(\lambda'))} = \varnothing, \qquad \lambda' \notin Z^+,$$

$$\mathscr{S}_k(\lambda) \cap \mathscr{N}_k \cap \overline{\mathscr{B}_k(\epsilon(\lambda'))} = \varnothing, \qquad \lambda' \neq k,$$

for all $\lambda \in [\lambda' - \delta(\lambda'), \lambda' + \delta(\lambda')]$. *Here* $\mathscr{N}_k = \mathscr{N}_k^+ \cup \mathscr{N}_k^-$.

Part (c) describes exactly the nodal structure of the local branches of solutions (3.1)(and any local solution, for that matter).

We are now in a position to make a degree calculation needed for the global theory in Section 4. For the notation used, see the Appendix of [3].

THEOREM 3.2. *Let* $\epsilon(\lambda)$, $\delta(\lambda)$ *be as in Lemma* 3.1 *and set* $\epsilon'(\lambda) = \frac{1}{2}\min(\epsilon(k), \epsilon(\lambda))$. *Then for* $\lambda \in (k, k + \delta(k))$,

$$d(\Phi(\lambda), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\lambda))}] \cap \mathscr{N}_k) - d(\Phi(\bar{\lambda}), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\bar{\lambda}))}] \cap \mathscr{N}_k) = 2, \tag{3.2}$$

where $\bar{\lambda} = 2k - \lambda \in (k - \delta(k), k)$.

Proof. That the degrees in (3.2) are defined follows from the definition of $\epsilon'(\lambda)$ and Lemma 3.1(c); i.e., no solutions exist on the boundaries of the open sets involved. The homotopy invariance of degree (cf. P3, Appendix [3]) and Lemma 3.1(b) imply

$$d(\Phi(\lambda), \mathscr{B}_k(\epsilon(k))) = d(\Phi(\bar{\lambda}), \mathscr{B}_k(\epsilon(k)))$$

while the additivity of degree in turn implies

$$\begin{aligned} &d(\Phi(\lambda), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\lambda))}] \cap \mathscr{N}_k) + d(\Phi(\lambda), \mathscr{B}_k(\epsilon'(\lambda))) \\ &\quad = d(\Phi(\bar{\lambda}), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\bar{\lambda}))}] \cap \mathscr{N}_k) + d(\Phi(\bar{\lambda}), \mathscr{B}_k(\epsilon'(\bar{\lambda}))). \quad (3.3) \end{aligned}$$

But at $(0, \lambda)$ and $(0, \bar{\lambda})$, L is invertible and, hence (P4[3])

$$\begin{aligned} d(\Phi(\lambda), \mathscr{B}_k(\epsilon'(\lambda))) &= i(\Phi(\lambda), 0, 0) = -1, \\ d(\Phi(\bar{\lambda}), \mathscr{B}_k(\epsilon'(\bar{\lambda}))) &= i(\Phi(\bar{\lambda}), 0, 0) = +1 \end{aligned}$$

which, together with (3.3), implies (3.2).

This proof also yields a proof of the existence of local solutions near $(0, k)$ in $\mathscr{N}_k$ since (3.2) implies at least one of the two degrees is nonzero.

4. The Global Theory

In this section we show that the existence of solution branches bifurcating from $(0, k)$ and lying in $\mathscr{N}_k$ is a global phenomenon. Our aim is to establish Theorem 4.1 of [3] for problem N.

Let $B_k \times \mathscr{R}$ have the product topology and $\mathscr{O}_k$ be an arbitrary bounded open set in $B_k \times \mathscr{R}$ such that $(0, k) \in \mathscr{O}_k$. Set $\mathscr{S}_k^\nu = \{(u, \lambda): u \in \mathscr{S}_k(\lambda) \cap \mathscr{N}_k^\nu$ for some $\lambda \in \mathscr{R}\}$. Just as in [3] we have

Lemma 4.1. (a) *$C_k^\nu = \overline{\mathscr{S}_k^\nu \cap \mathscr{O}_k}$ is compact in $B_k \times \mathscr{R}$, $\nu = +$ or $-$;*

(b) *$\mathscr{S}_k^\nu \cap \partial\mathscr{O}_k = \varnothing \Rightarrow C_k^\nu \subset \mathscr{O}_k$.*

Theorem 4.1. *If* (H1)–(H4)$_k$ *are satisfied, then $\mathscr{S}_k^\nu \cap \partial\mathscr{O}_k \neq \varnothing$ for all $k \geqslant 1$, $\nu = +$ or $-$.*

Proof. Let $U_\lambda^k = \{u \in B_k : (u, \lambda) \in \mathscr{O}_k\}$ and $\epsilon'(\lambda)$, $\delta(\lambda)$ be as in Lemma 3.1 and Theorem 3.2. Just as in [3] we can assert

$$d(\Phi(\mu), [U_\mu^k - \overline{\mathscr{B}_k(\epsilon'(\mu))}] \cap \mathscr{N}_k^\nu) = 0 \quad (4.1)$$

for all $\mu \neq k$, $\nu = +$ or $-$, under the assumption $\mathscr{S}_k^\nu \cap \partial\mathscr{O}_k = \varnothing$. We will derive a contradiction using (4.1).

Let $\lambda \in (k, k+\delta(k))$; then (3.2) holds. From (4.1) and the additivity property of degree we have

$$d(\Phi(\mu), [U_\mu{}^\nu - \overline{\mathscr{B}_k(\epsilon(k))}] \cap \mathscr{N}_k{}^\nu) \\ + d(\Phi(\mu), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\mu))}] \cap \mathscr{N}_k{}^\nu) = 0. \qquad (4.2)$$

Homotopic invariance of degree implies

$$d(\Phi(\lambda), [U_\lambda{}^\nu - \overline{\mathscr{B}_k(\epsilon(k))}] \cap \mathscr{N}_k{}^\nu) = d(\Phi(\bar{\lambda}), [U_{\bar{\lambda}}{}^\nu - \overline{\mathscr{B}_k(\epsilon(k))}] \cap \mathscr{N}_k{}^\nu)$$

and, hence, by letting $\mu = \lambda, \bar{\lambda}$ in (4.2) and subtracting the resulting equations we find

$$d(\Phi(\lambda), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\lambda))}] \cap \mathscr{N}_k{}^\nu) \\ - d(\Phi(\bar{\lambda}), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\bar{\lambda}))}] \cap \mathscr{N}_k{}^\nu) = 0$$

in contradiction to (3.2). The assumption that $\mathscr{S}_k{}^\nu \cap \partial\mathscr{O}_k = \varnothing$ is false and the theorem is proved.

With this theorem established the following corollary follows as in [3]:

COROLLARY 4.1. *If* (H1)–(H4)$_k$ *are satisfied then there exists a continuum of solutions* (u, λ), $u \in \mathscr{N}_k{}^\nu$, $\nu = +$ *and* $-$, *connecting* $(0, k)$ *to* ∞ *in* $B_k \times \mathscr{R}$.

Concerning the spectrum of problem N it is easily shown that the theorems of Section 5 in [3] are valid. Of these we will only state the following which will apply to the Levi-Civita problem studied in the next section:

THEOREM 4.2. *If* $g \equiv u + f$ *has a nonzero u-zero (i.e.,* $u_0 + f(u_0, \eta, \theta) \equiv 0$, $-\infty < \eta < +\infty$, $\theta \in [-\pi, \pi]$ *for some* $u_0 = \text{const.} \neq 0$), *then the spectrum of problem N corresponding to solutions lying on the branch bifurcating from* $(0, k)$ *is an unbounded interval in* $\mathscr{R}^+$. *Moreover, for these solutions the inequality* $\max_{\bar{D}} |u| < u_0$ *holds. Furthermore, if* $\xi g(\xi, \eta, \theta) \geqslant 0$ *for all* $-\infty < \xi$, $\eta < +\infty$, $\theta \in [-\pi, \pi]$, *then the entire spectrum of problem N is positive.*

5. LEVI-CIVITA'S PROBLEM

The exact mathematical theory of steady permanent progressing water waves on an infinitely deep ocean was studied by Levi-Civita in 1925 [6], who reduced the existence of such waves to problem N with $f(u, v, \theta) \equiv -u + e^{-3v} \sin u$. In this theory u is the angle of the velocity vector of fluid flow as a function of position as measured from the horizontal (and a coordinate system riding with the wave profile at the crest), v is $\ln q/c$ where q is the length of the velocity vector and c is the speed of the wave profile (measured from some rest position), and $\lambda = gl/2\pi c^2$ where l is a wavelength of the wave. It is assumed without loss of generality that u is to vanish for $\theta = 0, \pi$. Levi-Civita showed that any existent wave gives rise to infinitely many solutions to this problem, one for each $\lambda = ng\bar{l}/2\pi c^2$, $n = 1, 2, \ldots$, where $\bar{l}$ is the smallest wavelength. Conversely, any solution (u, λ) gives rise to a wave as soon as l is chosen, c being determined by $\lambda = gl/2\pi c^2$. Levi-Civita then proved that this problem has a solution for $\lambda \in (1, 1 - \delta)$ for $\delta > 0$ sufficiently small. (Nekrassov [7] also independently proved this result; a more modern approach may be found in [9].) As Levi-Civita showed, solutions for λ close to $k = 2, 3, \ldots$ may be constructed from those for $\lambda \in (1 - \delta, 1)$; such solutions arise from considering $n\bar{l}$ as the wavelength instead of $\bar{l}$ and correspond to $\lambda = ng\bar{l}/2\pi c^2$, but of course do not give rise to physically different waves and, hence, may be ignored.

Since $f = -u + e^{-3v} \sin u$ satisfies all of the hypotheses above (for all k), our results above contribute to this theory by providing some answers to the following questions: (1) how "far" do the local branches of solutions constructed by Levi-Civita extend and (2) are there solutions not on the branch bifurcating from (0, 1) which give rise to waves not found from solutions on this branch? Levi-Civita conjectured in [6] that the answer to the second question is "no", but up to now this has been proved only locally (cf. [1, 5]). The characterization of the branches according to their nodal structure given in Corollary 4.1 provide a global answer to Levi-Civita's conjecture at least insofar as one considers only solutions on the branches from $(0, k)$ (we have not ruled out the existence of solutions off these branches). This is because solutions (u, λ), $u(r, \theta) \in \mathcal{N}_k^{\nu}$, give rise to waves which are physically no different from those given by

$$\bar{u}(r, \theta) = u(r, \theta/k), \quad 0 \leqslant \theta \leqslant \pi, \quad \bar{u}(r, -\theta) = -\bar{u}(r, \theta) \qquad \text{for} \quad \bar{\lambda} = \lambda/k$$

which is a solution in $\mathcal{N}_1^{\nu}$ lying on the branch from (0, 1) (cf. [1, 6]).

Concerning the first question above we see from Corollary 4.1 that the branches extend to "∞". Moreover, since $u_0 = \pi$ is a u-zero of $g = e^{-3v} \sin u$ and $ue^{-3v} \sin u \geqslant 0$ for all u, v, Theorem 4.2 implies that the spectrum of Levi-Civita's problem is positive and that that part of the spectrum corresponding to solutions in $\mathcal{N}_1^{\nu}$ is an interval $[a, +\infty)$, $a > 0$; the solutions on $\mathcal{N}_1^{\nu}$ satisfying $\max_{\bar{D}} | u | < \pi$. There is no conflict here with the result of Krasovskii [5] which states that necessarily $\lambda \in [a, b]$, $b < +\infty$, since he restricts his attention to solutions for which $\max_{\bar{D}} | u | \leqslant \pi/6$; apparently $\pi/6 \leqslant \max_{\bar{D}} | u | < \pi$ for $\lambda > b$. Waves of the type considered here probably do not exist for which $\max_{\bar{D}} | u | > \pi/2$ or if they do are probably unstable; in fact, it appears [9] that $\max_{\bar{D}} | u | \leqslant \pi/6$ for waves appearing in nature.

We do not study here any of these questions, in particular the behavior of $\max_{\bar{D}} | u |$ for large λ.

Finally we point out that only solutions on $\mathcal{N}_1^-$ are of interest, the solutions in $\mathcal{N}_1^+$ corresponding to the same waves with a coordinate system riding on the trough of the wave instead of the crest (cf. [6]).

6. Proof of Theorem 4.1

In [2] an *a priori* estimate for nonhomogeneous Steklov problems is proved. The proof of that estimate can be carried through using the $\| \cdot \|_{\bar{D}}$ norm as defined here by virtue of the properties of the Neumann function mentioned above (and in [3]). The estimate is stated below as we will need it to prove Theorem 3.1.

Lemma 6.1. *Consider the problem*

$$\partial u / \partial r = ku + \psi,$$

where $k \in Z^+$, $u \in B_k$, *and* ψ *is the boundary values of a function in* B_k. *If* $u \in B_k$ *is the solution to this problem satisfying*

$$\int_{-\pi}^{\pi} u(1, \theta) \sin k\theta \, d\theta = 0 \tag{6.1}$$

then

$$\| u \|_{\bar{D}} \leqslant c \| \psi \|_{\partial D} \tag{6.2}$$

for some constant $c > 0$ *where* $\| \psi \|_{\partial D} = \max_{[-\pi, \pi]} | \psi(\theta) | + \max_{[-\pi, \pi]} | \psi'(\theta) |$.

To prove Theorem 3.1 we substitute (3.1) into the boundary condition (2.1) and equate like powers of μ. This will generate a sequence of linear problems for u_{nk} which will be solvable for an appropriate choice of λ_{nk}. We then must prove that the resulting sequences in (3.1) converge. If the series for u converges in the $\| \cdot \|_{\bar{D}}$ norm then

$$\begin{aligned} \left\| Tu - T\left(\sum_{n=1}^{N} u_{nk}\mu^n\right)\right\|_{\bar{D}} &= \left\| T\left(\sum_{n=N+1}^{\infty} u_{nk}\mu^n\right)\right\|_{\bar{D}} \\ &\leqslant B \left\| \sum_{n=N+1}^{\infty} u_{nk}\mu^n \right\|_{\bar{D}} \leqslant \epsilon \end{aligned}$$

for N sufficiently large and, hence,

$$Tu = \sum_{n=1}^{\infty} Tu_{nk}\mu^n. \tag{6.3}$$

Ignoring the question of convergence for the moment we substitute (3.1) and (6.3) into

$$f(u, v, \theta) = \sum_{i+j=2}^{\infty} c_{ij}(\theta)\, u^i v^j$$

[cf. (H1), (H2)] to obtain

$$f(u, Tu, \theta) = \sum_{n=2}^{\infty} G_n \mu^n.$$

Here for $n \geqslant 3$

$$G_n \equiv G_n(u_{ik}, Tu_{ik}, c_{ij}(\theta)),$$
$$0 \leqslant i \leqslant n-1, \qquad 2 \leqslant i+j \leqslant n,$$

is a polynomial expression in its variables with positive integer coefficients. Thus

$$u + f(u, Tu, \theta) = \sum_{n=1}^{\infty} (u_{nk} + G_n)\, \mu^n$$

where we define $G_1 \equiv G_0 \equiv 0$. Finally

$$\lambda(u + f(u, Tu, \theta) = \sum_{n=1}^{\infty} \left[\sum_{l=0}^{n-1} \lambda_{lk}(u_{n-l,k} + G_{n-l}) \right] \mu^n,$$

where we have let $G_0 \equiv 0$ and $\lambda_{0k} = k$. The boundary condition (2.1) then becomes, upon equating like powers of μ, the sequence of conditions for $n \geqslant 1$

$$\frac{\partial u_{nk}}{\partial r} = \sum_{l=0}^{n-1} \lambda_{lk}(u_{n-l,k} + G_{n-l}), \qquad r = 1$$

or

$$\partial u_{1k}/\partial r = k u_{1k},$$
$$\partial u_{nk}/\partial r = k u_{nk} + \lambda_{n-1,k} u_{1k} + H_n, \qquad r = 1, \quad n \geqslant 2, \tag{6.4}$$

where

$$H_n = kG_n + \sum_{l=1}^{n-2} \lambda_{lk}(u_{n-l,k} + G_{n-l}), \qquad n \geqslant 3,$$
$$H_1 = 0, \qquad H_2 = 0. \tag{6.5}$$

Thus, $u_{1k} = Ar^k \sin k\theta$ which we normalize by taking $A = 1$. From [2] we know that in B_k, (6.4) has a unique solution satisfying (6.1) provided the orthogonality condition

$$\int_{-\pi}^{\pi} [\lambda_{n-1,k} u_{1k} + H_n]\, u_{1k}\, d\theta = 0$$

is satisfied; this is done by choosing

$$\lambda_{n-1,k} = -\frac{1}{\pi} \int_{-\pi}^{\pi} H_n \sin k\theta\, d\theta, \qquad n \geqslant 2, \tag{6.6}$$

which determines the coefficients in the series expansion (3.1) of λ uniquely. In this manner the coefficients of (3.1), $u_{nk} \in B_k$ and λ_{nk}, are recursively and uniquely defined (since H_n depends only on u_i for $0 \leqslant i \leqslant n - 1$).

We now turn to the important task of proving the convergence of the series (3.1) and thus justifying the above construction of solutions to problem N. Applying the *a priori* estimate (6.2) to u_{nk} as a solution to (6.4) we have

$$\| u_{nk} \|_{\bar{D}} \leqslant c(| \lambda_{n-1,k} | + \| H_n \|_{\partial D}), \qquad n \geqslant 2;$$

but from (6.6)

$$| \lambda_{n-1,k} | \leqslant 2 \| H_n \|_{\partial D}, \qquad n \geqslant 2$$

so that

$$\| u_{nk} \|_{\bar{D}} \leqslant 3c \| H_n \|_{\partial D}, \qquad n \geqslant 2.$$

Letting $p_{nk} = \| u_{nk} \|_{\bar{D}} + | \lambda_{n-1,k} |$, we may combine the last two estimates into the estimate

$$0 \leqslant p_{nk} \leqslant K \| H_n \|_{\partial D}, \qquad n \geqslant 2,$$

where $K = 2 + 3c > 0$ is independent of n. Using

$$\begin{aligned} \| G_n \|_{\partial D} &\leqslant G_n(\| u_{ik} \|_{\bar{D}}, \| Tu_{ik} \|_{\bar{D}}, \| c_{ij}(\theta) \|_{\partial D}) \\ &\leqslant G_n(p_{ik}, Bp_{ik}, q_{ij}), \end{aligned}$$

where $q_{ij} = \| c_{ij}(\theta) \|_{\partial D} < +\infty$ and using (6.5) we have for p_{nk} the estimate

$$0 \leqslant p_{nk} \leqslant K \sum_{l=1}^{n-2} p_{l+1,k} p_{n-l,k} + K \sum_{l=0}^{n-2} p_{l+1,k} G_{n-l}(p_{ik}, Bp_{ik}, q_{ij}) \tag{6.7}$$

for $n \geqslant 2$. We wish to show $\sum_{n=1}^{\infty} p_{nk} \mu^n$ converges for μ sufficiently small; this will guarantee the convergence of (3.1) in the $\| \ \|_{\bar{D}}$ norm by the definition of p_{nk}.

Consider the function $h(z, \mu)$ of two real variables defined by

$$h(z, \mu) = Kz^2 + Kz\mu^{-1}g(\mu z, \mu Bz) - z + p_{2k}\mu,$$
$$g(\xi, \eta) = \sum_{i+j=2}^{\infty} q_{ij}\xi^i\eta^j.$$

Since $h(0, 0) = 0$, $h_z(0, 0) = -1$, we know from the Implicit Function Theorem that the equation $h(z, \mu) = 0$ defines an analytic function

$$z = \sum_{n=0}^{\infty} z_{n+1}\mu^n \qquad (6.8)$$

for $|\mu|$ sufficiently small. Clearly $z_1 = 0$ and by implicit differentiation $z_2 = p_{2k}$. To find a recursion formula for z_n we substitute (6.8) into $h(z, \mu) = 0$; this leads to

$$z_{n+1} = K\sum_{l=1}^{n-1} z_{l+1}z_{n-l+1} + K\sum_{l=1}^{n-1} z_{l+1}G_{n-l+1}$$

for $n \geqslant 2$; an easy induction shows, together with (6.7), that

$$z_{n+1} \geqslant K\sum_{l=1}^{n-1} p_{l+1,k}p_{n-l+1,k} + K\sum_{l=0}^{n-1} p_{l+1,k}G_{n-l+1}$$
$$\geqslant p_{n+1,k} > 0, \qquad n \geqslant 2,$$

so that $\sum p_{nk}^n$ converges for $|\mu|$ sufficiently small by the comparison test; this is true for each $k \in Z^+$.

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