Nontrivial Periodic Solutions of Integrodifferential Equations

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By means of an abstract bifurcation theorem, a multiparameter bifurcation theorem for systems of Volterra integrodifferential equations is proved. The result is applied in detail to general first-order equations with (possibly) two delays, to general second-order equations with delays and to some general 2×2 first-order systems. These applications illustrate not only the main theorem, but the main features of the multiparameter bifurcation approach taken here: namely, that by using several parameters which appear explicitly in the system, one not only gains a knowledge of the nature of the multiparameter bifurcation phenomenon, but also obtains very general results by means of a simple proof and has inherently fewer nondegeneracy or transversality conditions to fulfill in applications. (In fact, for the very general first-order and second-order equations considered as applications here, the nondegeneracy or transversality conditions are always satisfied.) Moreover, one-parameter Hopf-type bifurcation can be viewed as a special case embedded in the multiparameter bifurcation branch. Or, alternatively, one can also prove one-parameter Hopf-type bifurcation results (further illustrating the flexibility of the main theorem here) by rescaling the independent variable and applying the results given here.

1. Introduction

The purpose of this paper is to prove a general (multiparameter) bifurcation theorem for systems of Volterra integrodifferential equations and to study some special cases in detail. The main result (Theorem 2) is a generalization of that in [6] and is proved by means of the implicit function theorem and the method of Lyapunov and Schmidt (which is sometimes called the method of "alternative problems"). This method, which is applicable in a very general setting, is given in an abstract form suitable for our analysis here in Theorem 1.

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Journal of Integral Equations 1, 165–181 (1979)

0163-5549/79/02165-17$01.75

The main feature of the approach taken here is the utilization of several parameters which appear explicitly in the equation, their number being determined (as is to be expected) by the codimension m of the range of the linear operator L obtained from the linearization of the problem at a critical point. In many interesting applications, such as in those involving the question of the existence of nontrivial periodic solutions of nonlinear autonomous differential and integral equations, m is greater than one. This is the case for example in Hopf-type bifurcation problems. In this paper the study of such cases is carried out by using $m \geqslant 2$ parameters which appear explicitly in the equation rather than by using only one such parameter (as is usually done in such problems), which in general must then be further supplemented by other implicit parameters, such as the unknown period or coordinates in the nullspace of L (for example, see [13,18] for differential equations, [2,17,20] for functional differential equations and [14,19] for partial differential equations). As will be seen in the applications in Sec. 4 below, there can be advantages to this approach beyond gaining a knowledge of bifurcation phenomena as a function of several parameters. In a wide variety of applications the technical calculations are fewer and simpler. Moreover, one-parameter (Hopf-type) bifurcation can be viewed as a "special case" embedded in multiparameter bifurcation which corresponds to "transversally slicing" the multiparameter bifurcation surface.

2. A Multiparameter Bifurcation Theorem

Let X, Y denote normed linear spaces, and suppose $L: X \to Y$ is a bounded linear operator with nullspace $N(L)$ and range $R(L)$. We consider the problem of solving

$$Lx = T(x,\lambda) \qquad \text{for} \quad (x,\lambda) \in X \times R^m, \quad x \neq 0, \tag{2.1}$$

where R^m is m-dimensional real Euclidean space and T is an operator such that $T(0,\lambda) = 0$ for all $\lambda \in R^m$ about which more is assumed in H3 below. Let $B(Z,r;z)$ denote the open ball of radius r centered at $z \in Z$ in a normed space Z. The following hypotheses are made.

H1. $R(L)$ and $N(L)$ are closed and admit (continuous) projections, and $\operatorname{codim} R(L) = m < +\infty$.

H2. $N(L) \neq \{0\}$.

H3. The operator $T: D(r) := B(X,r;0) \times B(R^m,r;0) \to Y$ for some $r > 0$ in such a way that $T(\epsilon x,\lambda) = \epsilon \overline{T}(x,\lambda,\epsilon)$ for all $(x,\lambda,\epsilon) \in D(r) \times B(R^1, r;0)$, where $\overline{T}$: $D(R) \times B(R^1,r;0) \to Y$ is $q \geqslant 1$ times continuously Fréchet differentiable in (x,λ,ϵ) with $\overline{T}(y,0,0) = 0$, $\overline{T}_x(y,0,0) = 0$ for some $y \in N(L)$, $0 < |y|_X < r$.

Let M be a closed subspace of X complementary to $N(L)$, and let $A : R(L) \to M$ be bounded right inverse of L. (Such M and A are guaranteed by H1.) Let $P : Y \to R(L)$ be a (continuous) projection. Then $I - P$ is a (continuous) projection of Y onto an m-dimensional subspace complementary to $R(L)$, namely $N(P)$. As a result, for y as in H3, for $z \in M$ with $|z|_X$ small, for $\lambda \in B(R^m, r; 0)$ and for $\epsilon \in B(R^1, r; 0)$, the element $(I - P)\overline{T}(y + z, \lambda, \epsilon)$ has m real components $c = c(z, \lambda, \epsilon) : D(r) \times B(R^1, r; 0) \to R^m$ (with r smaller, if necessary) with respect to a fixed basis of $N(P)$. Note that c has the smoothness properties of T in (z, λ, ϵ) as given in H3. Also note that $c_z(0,0,0) = 0$ by H3. Finally it is assumed that

H4. $d := \det c_\lambda(0,0,0) \neq 0$.

Theorem 1. *If H1 through H4 hold, then there exists an $\epsilon_0 > 0$ such that Eq. (2.1) has a branch of nontrivial solutions $(x, \lambda) \in X \times R^m$ of the form $x = \epsilon[y + z(\epsilon)]$, $\lambda = \lambda(\epsilon)$ for $\epsilon \in B(R^1, \epsilon_0; 0)$ where $z : B(R^1, \epsilon_0; 0) \to M$ and $\lambda : B(R^1, \epsilon_0; 0) \to R^m$ are $q \geqslant 1$ times continuously Fréchet differentiable operators such that $z(0) = 0$ and $\lambda(0) = 0$.*

PROOF. Let $y \in N(L)$ be as in H3 and $z \in M$, and substitute $x = \epsilon(y + z)$ into (2.1). The resulting equation is easily seen to be equivalent to the alternative equations

$$z - AP\overline{T}(y + z, \lambda, \epsilon) = 0 \tag{2.2}$$

$$c(z, \lambda, \epsilon) = 0 \tag{2.3}$$

for $(z, \lambda, \epsilon) \in M \times R^m \times R^1$. Clearly $(z, \lambda, \epsilon) = (0,0,0)$ solves both equations (see H3). Equation (2.2) can be (uniquely) solved for $z = z(\lambda, \epsilon)$, $z(0,0) = 0$ by means of the implicit-function theorem [11], since the Fréchet derivative of the left-hand side with respect to z at $(0,0,0)$ is the identity operator on M. This solution is defined and is $q \geqslant 1$ times continuously Fréchet differentiable in (λ, ϵ) near $(0,0)$. If this solution is substituted into (2.3), one obtains an equation (the so-called bifurcation equation) to solve for $\lambda = \lambda(\epsilon)$. That this bifurcation equation is solvable follows from the implicit-function theorem and H4, since the Jacobian of these m real equations in m real unknowns with respect to λ at $(\lambda, \epsilon) = (0,0)$ is the $m \times m$ matrix $c_\lambda(0,0,0)$. □

In application H1 is usually fulfilled by means of a Fredholm alternative for the linear operator L. Hypothesis H2 is the familiar necessary condition that bifurcation can only occur at $\lambda = 0$ if the linear operator L does not have a bounded inverse, and H4 is a sufficiency or nondegeneracy condition which guarantees that bifurcation (which doesn't always occur) occurs. Assumption H3 requires, roughly speaking, that T be higher order in (x, λ) and $T(0, \lambda) = 0$. Note that different choices of $y \in N(L)$, $y \neq 0$(if $m \geqslant 2$) yield different solution branches.

Suppose now that $w_i \in Y$, $1 \leqslant i \leqslant m$, form a basis for $N(P)$, and P_i is the (continuous) projection of Y onto the span of w_i. Then $c(z, \lambda, \epsilon) = \operatorname{col}(c_i(z, \lambda, \epsilon)) \in R^m$ in H4, where $P_i \overline{T}(y + z, \lambda, \epsilon) = c_i(z, \lambda, \epsilon) w_i$ and $c_\lambda(0,0,0) = ((\partial/\partial\lambda_j) c_i(0,0,0))$. If further

$$T(x, \lambda) = \sum_{j=1}^{m} \lambda_j A_j x + R(x, \lambda), \tag{2.4}$$

where A_j: X → Y is bounded and linear and

H5. $R(x, \lambda)$ satisfies H3 and $(\partial/\partial\lambda_j) R(0,0) = 0$, $1 \leqslant j \leqslant m$,

then $c_\lambda(0,0,0) = (a_{ij})$ in H4, where $P_i A_j y = a_{ij} w_i$, $a_{ij} \in R^1$. Note that terms of order $o(|x|_X)$ or $o(|\lambda|)$ contribute nothing to d in H4.

3. Integrodifferential Systems.

Let $H(s)$ be an $n \times n$ matrix of measurable functions of finite total variation on $s \geqslant 0$, and define

$$Lx := x'(t) + \int_0^\infty dH(s)\, x(t - s), \tag{3.1}$$

$$L_a x := x'(t) - \int_0^\infty dH^t(s)\, x(t + s). \tag{3.2}$$

Both of these operators are linear and bounded as operators from $X = X_n(p)$ to $Y = Y_n(p)$, where $Y_n(p)$ is the Banach space of continuous, p-periodic real n-vector-valued functions defined for all t under the usual supremum norm $|x|_0 = \sup_{-\infty < t < +\infty} |x(t)|$, and $X_n(p)$ is the Banach space of those functions in $Y_n(p)$ which are continuously differentiable under the norm $|x|_1 = |x|_0 + |x'|_0$. Define $(x, y)_p := p^{-1} \int_0^p x(t) y(t)\, dt$ for $x, y \in Y_n(p)$. If $S \subseteq Y_n(p)$ is a subspace, let $S^\perp$ denote the subspace $\{x \in Y_n(p): (x, y)_p = 0 \text{ for all } y \in S\}$. The following lemma was essentially proved in [6].

Lemma. *For the bounded linear operators from $X_n(p)$ to $Y_n(p)$ defined by (3.1) and (3.2) it follows that* $0 \leqslant \dim N(L) = \dim N(L_a) = m < +\infty$ *and* $R(L) = N^\perp(L_a)$ *for any* $p > 0$.

(The solutions found in the Fredholm alternative in [6, Theorem 2] were shown only to be absolutely continuous, but are easily seen to lie in $X_n(p)$.

This is because an absolutely continuous function whose derivative equals almost everywhere a continuous function is in fact continuously differentiable.)

The Lemma implies that H1 holds for L defined by (3.1) on $X = X_n(p)$, $Y = Y_n(p)$ for any period $p > 0$. Assume now that

A1. the linear system $y'(t) + \int_0^\infty dH(s)y(t-s) = 0$ has $m \geqslant 1$ independent, p-periodic solutions $y_i(\cdot,p) \in X_n(p)$ for some period $p > 0$.

Let $w_i(\cdot,p) \in X_n(p)$ be m independent, mutually orthogonal solutions of the adjoint system $w'(t) - \int_0^\infty dH^t(s)w(t+s) = 0$, whose existence (and maximality) is guaranteed by the Lemma. From Theorem 1 follows

Theorem 2. *Suppose A1 holds and T satisfies H3 with* $X = X_n(p)$ *and* $Y = Y_n(p)$ *for some nontrivial linear combination* $y(\cdot,p) \in X_n(p)$ *of the solutions* $y_i(t,p)$. *If*

A2. $$d := \det\left(\left(\frac{\partial}{\partial\lambda_j}T(y,0,0), w_i\right)_p\right) \neq 0,$$

then the system

$$Lx := x'(t) + \int_0^\infty dH(s)x(t-s) = T(x,\lambda) \tag{3.3}$$

has nontrivial p-periodic solutions in $X_n(p)$ *of the form* $x(t,p) = \epsilon y(t,p) + \epsilon z(t,p,\epsilon)$, $\lambda = \lambda(p,\epsilon) \in R^m$ *for small* $|\epsilon|$ *(say* $|\epsilon| < \epsilon_0(p)$*), where z and* λ *are* $q \geqslant 1$ *times continuously Fréchet differentiable in* ϵ *and where* $z(t,p,0) \equiv 0$, $\lambda(p,0) \equiv 0$, $(z,y_i)_p = 0$ *for* $1 \leqslant i \leqslant m$.

Note that T need not be autonomous in Theorem 2, so that p can be prescribed by the system.

If T has the form (2.4), then $d = \det((A_j y, w_i)_p)$. In particular, if $A_j x := \int_0^\infty dH_j(s)x(t-s)$, then

$$d = \det\left(\left(\int_0^\infty dH_j(s)y(t-s), w_i\right)_p\right). \tag{3.5}$$

Although we will not study the case further here, we point out in passing that a theorem similar to Theorem 2 can also be proved for Volterra integral equations by means of Theorem 1 (see [4]). Let $K(t)$ be piecewise continuous on a finite interval $0 \leqslant t \leqslant b$.

Theorem 3. *If L and L_a are redefined as*

$$Lx: = x(t) + \int_{t-b}^{t} K(t-s)x(s)\,ds,$$

$$L_a x: = x(t) + \int_{t}^{t+b} K(s-t)x(s)\,ds,$$

then the Lemma and Theorem 2 as stated are valid for the equation

$$x(t) + \int_{t-b}^{t} K(t-s)x(s)\,ds = T(x,\lambda),$$

where now X and Y are both taken to be $Y_n(p)$.

Theorem 2 is a generalization of the main result in [6] in that T is more general and the parameter λ does not appear in a restricted way.

Although we have had in mind applying Theorem 2 to equations in which appear $m = 2$ explicit parameters, this theorem (as well as Theorem 3) can also be applied to autonomous equations with one parameter by using the then unknown period as a second parameter. The theorems can then be applied on $X_n(1)$, $Y_n(1)$ after a change of independent variable from t to t/p has been made. Using this procedure, one can prove the existence part of the well-known Hopf bifurcation theorem for nondelay equations [$H(s) = Au_0(s)$ where $u_0(s)$ is the unit step function at $s = 0$], and in fact the proof of Theorem 1 is just an abstract version of the proof in [18] and [19] made precisely in this way. In this case the nondegeneracy condition A2 can be shown to have the geometric interpretation that the eigenvalues of the linearization as a function of λ cross the imaginary axis transversally.

In applications x is often the difference between the dependent variable and some equilibrium state, while λ is the difference between a certain parameter $\beta \in R^m$ and the critical value $\beta^0 = \beta^0(p) \in R^m$ of this parameter at which the linear system has nontrivial p-periodic solutions. When applying Theorem 2 to autonomous systems, one in general gets bifurcation of nontrivial periodic solutions along a two-dimensional surface σ of parameter values in R^m defined by $\beta = \beta(\epsilon,p) = \beta^0(p) + \lambda(\epsilon,p)$, occurring at points on a curve C defined by $\beta = \beta^0(p)$ as parametrized by p. This two-dimensional surface σ is made of curves C_p defined by $\beta = \beta(\epsilon,p)$ for fixed p as parametrized by ϵ, along which bifurcation occurs within the space $X_n(p)$ of fixed period p. Within this geometrical framework one can see what happens if only one degree of freedom is allowed in the parameters which constitute the components of β (as for example in [17]), either by varying only one component of β of more generally by constraining β to lie on a one-dimensional curve Λ in R^m. To get bifurcation along Λ the curve Λ must intersect C and lie in the bifurcation surface σ, conditions which express themselves analytically in terms of further nonde-

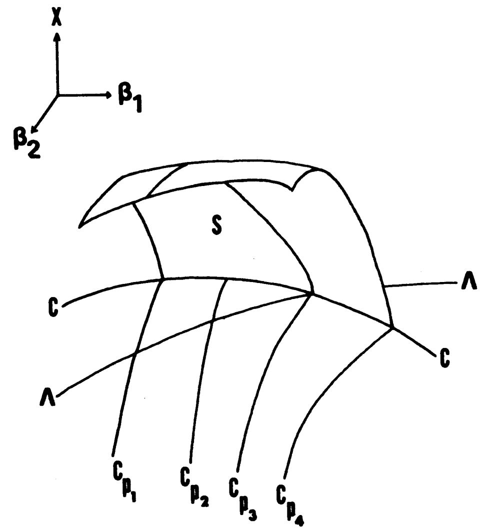

Figure 1. The bifurcation surface S is composed of bifurcation branches whose projections onto the β parameter space are the curves C_{p_i} along which bifurcation occurs within a space of functions of fixed period p_i. One-parameter bifurcation phenomena correspond to constraining β to lie on a one-dimentional curve Λ in the β parameter space and "slicing" the surface S along this curve Λ. Except in the case when Λ coincides with one of the curves C_p, the resulting bifurcation is period-varying.

generacy or solvability conditions. Since even under these added constraints Λ is likely to be (but will not always be) transversal on σ to the curves C_p, the bifurcation along Λ will likely be period-varying as in classical Hopf bifurcation. (See Fig. 1 for a generic picture when $m = 2$.) Note that it is possible when m is large ($m \geqslant 3$) for the curve Λ to intersect σ at a point other than one on the bifurcation curve C. In this case one would have the sudden onset of periodic oscillations of finite (rather than arbitrarily small) amplitude.

The important and interesting questions of the stability of the bifurcating periodic solutions and the direction of bifurcation are not considered here. The direction of bifurcation is of course determined by the signs of the lowest-order ϵ terms in the components of λ, terms which depend on the nonlinearities and which can be found in principle by the classical method of substituting ϵ-series for x and λ into (3.3) and eliminating secular terms. (See [3] for an example.)

For autonomous equations (3.3) in which $dH(s) = 0$ when $s \geqslant s_0$ for some $s_0 > 0$, the stability analysis of a nontrivial periodic solution by means of an investigation of Floquet multipliers could be carried out in the same manner as is done in the classical one-parameter, Hopf-type bifurcation approach. This is because the existence question (which is all we are considering here) is really distinct from the stability question, and moreover formulas which determine the stability (i.e., the location of the Floquet multipliers) on the basis of the system and its nonlinearities evaluated only at the critical parameter values (i.e. on the bifurcation curve C) are clearly still applicable here. See for example [2,18,20].

4. Applications

Several applications of Theorem 2 will be given in this section. Applications of Theorem 3 can be found in [4]. In these applications R will denote for simplicity a remainder term which has the following properties for a critical value $\beta^0 \in R^m$.

A3. $S(x,\lambda) := R(x,\lambda + \beta^0)$ satisfies the conditions on $T(x,\lambda)$ in H3 with $X = X_n(p)$, $Y = Y_n(p)$ for some $\beta^0 \in R^m$, and in addition $|S|_0 = o(|x|_1)$ near $x = 0$ uniformly for $\lambda \in B(R^m, r; 0)$.

Note that A3 implies that S satisfies H5.

4.1 *First-Order Scalar Equations with Two Delays*

Consider the general scalar ($n = 1$) equation

$$x'(t) = -\beta_1 \int_0^\infty x(t-s)\,dh_1(s) - \beta_2 \int_0^\infty x(t-s)\,dh_2(s) + R(x,\beta), \quad (4.1)$$

where R satisfies A3 with $n = 1$, $m = 2$. Let $\lambda_i = \beta_i - \beta_i^0$(where β_i^0 is to be determined below), and rearrange (4.1) as follows:

$$x'(t) + \beta_1^0 \int_0^\infty x(t-s)\,dh_1(s) + \beta_2^0 \int_0^\infty x(t-s)\,dh_2(s) = T(x,\lambda),$$

$$T(x,\lambda) := -\lambda_1 \int_0^\infty x(t-s)\,dh_1(s) - \lambda_2 \int_0^\infty x(t-s)\,dh_2(s) + S(x,\lambda).$$

Since this T satisfies H3, it is only necessary to consider the hypotheses A1 and A2 in Theorem 2.

First of all, A1 holds, that is to say the linear equation

$$y'(t) + \beta_1^0 \int_0^\infty y(t-s)\,dh_1(s) + \beta_2^0 \int_0^\infty y(t-s)\,dh_2(s) = 0 \quad (4.2)$$

has exactly $m = 2$ independent p-periodic solutions for an isolated point $\beta^0 = \mathrm{col}(\beta_1^0, \beta_2^0)$, if and only if

$$C_1(1) \neq C_2(1) \quad \text{and} \quad \Sigma_1 := C_1(1)S_2(1) - S_1(1)C_2(1) \neq 0; \tag{4.3}$$

$$\begin{aligned} &\text{either} \quad C_1(1)C_2(n) \neq C_1(n)C_2(1) \\ &\text{or} \quad n\Sigma_1 \neq C_1(1)S_2(n) - C_2(1)S_1(n) \end{aligned} \tag{4.4}$$

for all integers $n \geqslant 2$;

$$\beta_1^0 = -\omega C_2(1)/\Sigma_1 \text{ and } \beta_2^0 = \omega C_1(1)/\Sigma_1, \qquad \omega = 2\pi/p, \tag{4.5}$$

where

$$C_i(n) := \int_0^\infty \cos n\omega s \, dh_i(s), \qquad S_i(n) := \int_0^\infty \sin n\omega s \, dh_i(s).$$

This follows from a straightforward Fourier analysis. Under these conditions, two independent solutions are $\sin \omega t$, $\cos \omega t$, which also turn out to be solutions of the adjoint equation.

Secondly, consider the nondegeneracy condition A2. Since T has the form (2.4), the formula (3.5) can be used to calculate d with $y = k_1 \sin \omega t + k_2 \cos \omega t$, $k_1^2 + k_2^2 \neq 0$, and $w_1 = \sin \omega t$, $w_2 = \cos \omega t$. A simple calculation shows that $d = -(k_1^2 + k_2^2)\Sigma_1/4 \neq 0$ and hence A2 holds.

Thus, in this rather general application, the nondegeneracy (or sufficiency) condition A2 is implied by the (necessary) condition A1. The lack of any further nondegeneracy or transversality conditions (in addition to the generality of the delay integrators and nonlinearities) is one of the nice features of this multiparameter approach. This also occurs for the case of general second-order delay equations, as will be seen below.

Theorem 4. *If the linear equation (4.2) has exactly $m = 2$ independent p-periodic solutions for an isolated point $\beta^0 = \text{col}(\beta_1^0, \beta_2^0)$ (i.e. (4.3)–(4.5) hold), and if the remainder term R satisfies A3, then (4.1) has a bifurcating branch of nontrivial p-periodic solutions as described in Theorem 2 with $\beta_i(\epsilon, p) = \beta_i^0(p) + \lambda_i(\epsilon, p)$, β_i^0 being given by (4.5).*

As an example, consider the following equation with two constant lags:

$$\begin{aligned} &x'(t) = \beta_1 f_1(x(t - \tau_1)) + \beta_2 f_2(x(t - \tau_2)), \qquad \tau_2 \geqslant \tau_1 \geqslant 0, \quad \beta_i \in R^1, \\ &f_i \in C^3(R^1, R^1), \qquad f_i(0) = 0, \quad f_i'(0) = -1. \end{aligned} \tag{4.6}$$

In this case $C_i(n) = \cos n\omega\tau_i$, $S_i(n) = \sin n\omega\tau_i$, $\omega = 2\pi/p$. By Theorem 4 one gets p-periodic bifurcation if p can be chosen so that (4.3) and (4.4) hold and if

$$\beta_1^0 = -\frac{\omega \cos \omega\tau_2}{\sin \omega(\tau_2 - \tau_1)}, \qquad \beta_2^0 = \frac{\omega \cos \omega\tau_1}{\sin \omega(\tau_2 - \tau_1)}. \tag{4.7}$$

The trigonometric inequalities (4.3) and (4.4) will not be studied in depth here, but two illustrative examples will be considered. Clearly (4.3) fails if both $\tau_1 = \tau_2 = 0$, so that is necessary that at least one $\tau_i > 0$. Since the case when one τ_i is zero is considered in detail in [3], we will assume that $\tau_2 \geqslant \tau_1 > 0$. By rescaling t we lose no generality in assuming $\tau_1 = 1, \tau_2 = \tau \geqslant 1$.

(i) Take $\tau_1 = 1, \tau_2 = \tau > 1$ and $p = 2\tau/k$ for some integer $k \geqslant 1$. Then (4.3) reduces to $\sin k\pi/\tau \neq 0$. Secondly (4.4) becomes

$$\text{either} \quad \cos nk\pi/\tau \neq (-1)^{(n+1)k} \cos \pi k/\tau$$

$$\text{or} \quad \sin nk\pi/\tau \neq (-1)^{(n+1)k} n \sin \pi k/\tau$$

for all integers $n \geqslant 2$, whose truth can be seen as follows. If both inequalities are assumed to be equalities for some integer $n \geqslant 2$, then the contradiction $(n^2 - 1)\sin^2 k\pi/\tau = 0$ can be reached by squaring both equalities and adding. *Thus, Eq. (4.6) with $\tau_1 = 1, \tau_2 = \tau > 1$ has a bifurcating branch of nontrivial $2\tau/k$-periodic soltutions for an integer $k \geqslant 1$ and for β_i near the critical values*

$$\beta_1^0 = \frac{k\pi}{\tau} \csc \frac{k\pi}{\tau}, \qquad \beta_2^0 = (-1)^{k+1} \frac{k\pi}{\tau} \cot \frac{k\pi}{\tau}$$

provided $\sin k\pi/\tau \neq 0$. This follows immediately from Theorem 4.

(ii) Let $\tau_1 = 1, \tau_2 = 2$ and $p = 6/k$, where $k =$ integer $\geqslant 1$. It is straightforward to show that (4.3) holds if and only if $k = 6m + 1$ or $k = 6m + 5$ for some integer $m \geqslant 0$ and that the second inequality in (4.4) always holds for $n \geqslant 2$. (Note that $|n\Sigma_1| > 2$ for $n > 2$.) Theorem 4 yields *a bifurcating branch of nontrivial $6/k$-periodic solutions of Eq. (4.6) for $k = 6m + 1$ or $6m + 5$, $m \geqslant 0$, with $\tau_1 = 1, \tau_2 = 2$, and β_i near the critical values $\beta_1^0 = \beta_2^0 = (-1)^\nu k\pi/3\sqrt{3}$, where $\nu = 0$ if $k = 6m + 1$ and $\nu = 1$ if $k = 6m + 5$.*

Because (4.3) and (4.4) are inequalities, if they hold for some $p = p^0$ then they will hold for p near p^0. (In particular, this applies to examples (i) and (ii).) It is customary to draw bifurcation diagrams in which the parameters β_i are plotted against the norm of the solutions in $X_n(p)$. If this is done, then one gets for a fixed period p a curve of points $(|x|_1, \beta_1, \beta_2) \in R^3$ which intersects the β_1, β_2-plane at $(0, \beta_1^0, \beta_2^0)$ and projects onto the curve C_p given by $\beta_i(\epsilon, p)$ as parametrized by ϵ. As p varies one constructs from such curves C_p a surface S which intersects the β_1, β_2-plane in a bifurcation curve C given by $\beta^0(p)$ as parametrized by p (see Fig. 1). If one varies the β_i through only one degree of freedom along a curve Λ which intersects the bifurcation curve C as drawn in Fig. 1, then one obtains bifurcation along Λ by "slicing" S along Λ, which, unless Λ coincides with some C_p, produces period varying bifurcation. (In Fig. 1 only one-sided bifurcation is drawn.)

For example, a piece of one branch of the bifurcation curve C for equation (4.6) as given by

$$\beta_1^0 = -\frac{\omega \cos 2\omega}{\sin \omega}, \qquad \beta_2^0 = \omega \cot \omega, \qquad \omega = \frac{2\pi}{p} \tag{4.8}$$

(this is from (4.7) with $\tau_1 = 1$, $\tau_2 = 2$) is drawn in Fig. 2. Above, in (i) with $k = 1$, we justified bifurcation for p near $2\tau = 4$. Note that C in Fig. 2 crosses the β_1-axis transversally at the point $\beta = (\pi/2, 0)$, corresponding to $p = 4$. C also crosses the β_1-axis transversally at the points $\beta = ((-1)^m \cdot (2m+1)\pi/2, 0)$, $m \geqslant 0$, corresponding to $p = 4/(2m+1)$, because $(\partial/\partial p)\beta_2^0 \neq 0$ at these points (these points are not shown in Fig. 2). Bifurcation occurs near these points also because (i) above applies with $\tau = 2$, $k = 2m + 1$.

Taking Λ to be the β_1-axis in the above discussion corresponds to studying the equation

$$\begin{aligned} &x'(t) = \beta_1 f(x(t-1)), \\ &f \in C^3(R^1, R^1), \qquad f(0) = 0, \quad f'(0) = -1, \end{aligned} \tag{4.9}$$

for which we obtain the bifurcation values $\beta_1^0 = (-1)^m[(2m+1)\pi/2)]$, $m = \text{integer} \geqslant 0$, with $p = 4/(2m+1)$ where the β_1-axis crosses the bifurcation curve C given by (4.8). Whether the β_1-axis crosses the curves C_p, these bifurcations are period-varying, with the consequence that depends on the nonlinearities in f. In general one would expect this, but it

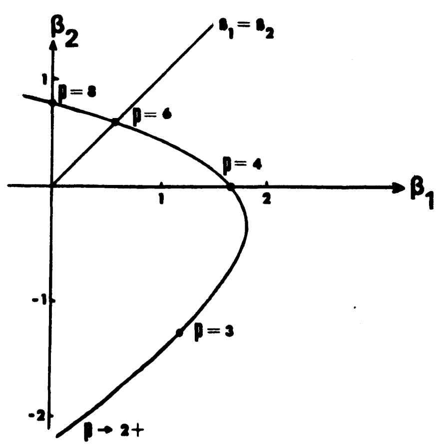

Figure 2. A piece of one branch of the bifurcation curve C is shown for Eq. (4.6) with $\tau_1 = 1$, $\tau_2 = 2$ as given by (4.8) for periods $2 < p < 8$.

would not necessarily occur. The result of Kaplan and Yorke [12] implies that under certain restrictive conditions on f (including oddness) the bifurcation is not period-varying but occurs in $X_1(4)$. In case (ii) bifurcation was found near $\beta_i^0 = \pi/3\sqrt{3}$ for $p = 6$. Clearly in Fig. 2 the curve Λ given by the straight line $\beta_2 = \beta_1$ is transversal to the bifurcation curve C at the intersection point $(\pi/3\sqrt{3}, \pi/3\sqrt{3})$, corresponding to $p = 6$. This straight line corresponds to bifurcation for the equation

$$x'(t) = \beta_1[f_1(x(t-1) + f_2(x(t-2))],$$
$$f_i \in C^3(R^1, R^1), \qquad f_i(0) = 0, \quad f_i'(0) = -1$$

for β_1 near $\pi/3\sqrt{3}$ and period p near 6. The results of Kaplan and Yorke [12] and Nussbaum [15,16] again imply that under certain restrictive conditions on f_i this bifurcation is not period-varying, but in general one expects the line $\beta_2 = \beta_1$ to cross the curves C_p.

More generally, if in Theorem 4 there is a period p for which $C_1(1) = 0$ and $(d/dp)C_1(1) \neq 0$, then the β_1-axis will cross the bifurcation curve C given by (4.5) as parametrized by p, which corresponds to bifurcation for the single-variable problem

$$x'(t) = -\beta_1 \int_0^\infty x(t-s)\,dh_1(s) + R(x, \beta_1).$$

(When applied to Volterra's population equation this would yield a corrected, period-varying version of Theorem 2 in [7]; see [8].)

4.2 *Another Look at Eq. (4.9)*

In order to further illustrate the use of Theorem 2, we show how bifurcation results for Eq. (4.9) can be obtained directly from Theorem 2 by using the unknown period p as a second parameter. If a change of variable from t to t/p is made, Eq. (4.9) reduces to $x'(t) = p\beta_1 f(x(t - 1/p))$, which we rewrite as

$$x'(t) + \alpha_1^0 x(t - \alpha_2^0) = \alpha_1^0 x(t - \alpha_2^0) - \alpha_1 x(t - \alpha_2) + R(x, \alpha_1, \alpha_2),$$

where $\alpha_1 = p\beta_1$, $\alpha_2 = 1/p$, and where α_i^0 are critical values chosen so that the linear homogeneous equation

$$y'(t) + \alpha_1^0 y(t - \alpha_2^0) = 0$$

has two independent 1-periodic solutions $\sin 2\pi t$, $\cos 2\pi t$. A simple Fourier analysis shows that $\alpha_1^0 = (-1)^m 2\pi$, $\alpha_2^0 = (2m+1)/4$ for some integer $m \geqslant 0$ [i.e. $\beta_1^0 = (-1)^m(2m+1)\pi/2$, $p^0 = 4/(2m+1)$]. Theorem 2 applies with $\lambda_i = \alpha_i - \alpha_i^0$ and $q = 1$ on x , $_1(1)$, $Y_1(1)$ provided $d \neq 0$. The adjoint solutions are also $\sin 2\pi t$, $\cos 2\pi t$ and

$$\cdot\ T(x,\lambda) = \alpha_1^0 x(t - \alpha_2^0) - (\lambda_1 + \alpha_1^0)x(t - \lambda_2 - \alpha_2^0) + S,$$

from which, together with (3.5), it follows that

$$d = \det \begin{vmatrix} -(y(t - \alpha_2^0), \sin 2\pi t)_1 & -(y(t - \alpha_2^0), \cos 2\pi t)_1 \\ \alpha_1^0(y'(t - \alpha_2^0), \sin 2\pi t)_1 & \alpha_1^0(y'(t - \alpha_2^0), \cos 2\pi t)_1 \end{vmatrix},$$

where $y = k_1 \sin 2\pi t + k_2 \cos 2\pi t$, $k_1^2 + k_1^2 \neq 0$. Hence it follows that $d = (-1)^{m+1}\pi^2(k_1^2 + k_2^2) \neq 0$.

4.3 Second-Order Equations with Delays

The system

$$x_1' = x_2, \qquad x_2' = -\beta_1 \int_0^\infty x_1(t-s)\,dh_1(s) - \beta_2 \int_0^\infty x_2(t-s)\,dh_2(s) + R(x_1, x_2, \beta_1, \beta_2), \tag{4.10}$$

$\beta_i \in R^1$, is equivalent to the second-order equation

$$x''(t) = -\beta_1 \int_0^\infty x(t-s)\,dh_1(s) - \beta_2 \int_0^\infty x'(t-s)\,dh_2(s) + R(x, x', \beta_1, \beta_2). \tag{4.11}$$

Suppose that β_i^0 are chosen so that the linear equation

$$y''(t) = -\beta_1^0 \int_0^\infty y(t-s)\,dh_1(s) - \beta_2^0 \int_0^\infty y'(t-s)\,dh_2(s) \tag{4.12}$$

has exactly two independent p-periodic solutions. This is possible for an isolated point $\beta^0 = \mathrm{col}(\beta_1^0, \beta_2^0)$ if and only if

$$C_2(1) \neq 0 \quad \text{and} \quad \Sigma_2 := S_1(1)S_2(1) + C_1(1)C_2(1) \neq 0, \tag{4.13}$$

$$\begin{aligned} &\text{either} \quad nS_1(1)S_2(n) + C_2(1)C_1(n) \neq n^2\Sigma_2 \\ &\text{or} \quad nS_1(1)C_2(n) - C_2(1)S_1(n) \neq 0 \end{aligned} \tag{4.14}$$

for all integers $n \geqslant 2$, and

$$\beta_1^0 = \omega^2 C_2(1)/\Sigma_2, \qquad \beta_2^0 = \omega S_1(1)/\Sigma_2, \tag{4.15}$$

in which case the solutions are $\sin \omega t$, $\cos \omega t$. Let $\lambda_i = \beta_i - \beta_i^0$ and $x = \mathrm{col}(x, x') = \mathrm{col}(x_1, x_2)$, and write the equivalent system (4.10) in the form (3.3) in Theorem 2 with

$$H(s) = \begin{vmatrix} 0 & -1 \\ \beta_1^0 h_1(s) & \beta_2^0 h_2(s) \end{vmatrix},$$

$$T(x,\lambda) = \sum_{j=1}^{2} \lambda_j \int_0^\infty dH_j(s)x(t-s) + \overline{R}(x,\lambda),$$

$$H_1(s) = \begin{vmatrix} 0 & 0 \\ -h_1(s) & 0 \end{vmatrix}, \qquad H_2(s) = \begin{vmatrix} 0 & 0 \\ 0 & -h_2(s) \end{vmatrix}, \qquad \overline{R} = \text{col}(0, S(x,\lambda)).$$

By choice of β_i^0 hypothesis A1 is fulfilled, and two independent p-periodic solutions of the linear homogeneous system are

$$y^{(1)}(t) = \text{col}(\cos\omega t, -\omega\sin\omega t), \qquad y^{(2)}(t) = \text{col}(\sin\omega t, \omega\cos\omega t).$$

The adjoint system

$$w_1' = \beta_1^0 \int_0^\infty w_2(t+s)\,dh_1(s), \qquad w_2' = -w_1 + \beta_2^0 \int_0^\infty w_2(t+s)\,dh_2(s)$$

has two independent, orthogonal solutions given by

$$\text{col}(\omega C_1(1)C_2(1)\cos\omega t - \omega S_1(1)C_2(1)\sin\omega t, -\Sigma_2 \sin\omega t)$$
$$\text{col}(\omega C_1(1)C_2(1)\sin\omega t + \omega S_1(1)C_2(1)\cos\omega t, \Sigma_2 \cos\omega t).$$

If $y = k_1 y^{(1)} + k_2 y^{(2)} = \text{col}(y_1, y_2)$, $k_1^2 + k_2^2 \neq 0$, then from (3.5)

$$d = \det \begin{vmatrix} \left(-\int_0^\infty y_1(t-s)\,dh_1(s), -\Sigma_2\sin\omega t\right)_p & \left(-\int_0^\infty y_1(t-s)\,dh_1(s), \Sigma_2\cos\omega t\right)_p \\ \left(-\int_0^\infty y_2(t-s)\,dh_2(s), -\Sigma_2\sin\omega t\right)_p & \left(-\int_0^\infty y_2(t-s)\,dh_2(s), \Sigma_2\cos\omega t\right)_p \end{vmatrix},$$

or $d = -(\omega/4)(k_1^2 + k_2^2)\Sigma_2^3 \neq 0$.

Theorem 5. *If the linear equation (4.12) has exactly $m = 2$ independent p-periodic solutions for an isolated β^0 (i.e. if (4.13)–(4.15) hold) and if R satisfies A3, then the second-order equation (4.11) has a bifurcating branch of nontrivial p-periodic solutions as described in Theorem 2 with $\beta_i(\epsilon,p) = \beta_i^0(p) + \lambda_i(\epsilon,p)$ and β_i^0 given by (4.15).*

Note that if there is no delay in the derivative in (4.11)(as in [1,9,10]) then $h_2(s) = u_0(s)$ and $C_2(n) = 1$, $S_2(n) = 0$ for all n.

As an example, suppose $h_1(s) = u_\tau(s)$, $\tau > 0$, and $h_2(s) = u_0(s)$, so that (4.11) reduces to the equation

$$x''(t) = -\beta_1 x(t-\tau) - \beta_2 x'(t) + R(x, x', \beta_1, \beta_2). \qquad (4.16)$$

In this case (4.13) reduces to $\cos \omega t \neq 0$. The condition (4.14), which reduces to

$$\begin{aligned} &\text{either} \quad \cos n\omega\tau \neq n^2 \cos \omega\tau \\ &\text{or} \quad n \sin \omega\tau - \sin n\omega\tau \neq 0 \end{aligned} \tag{4.17}$$

for all integers $n \geqslant 2$, can be shown to hold as follows. Suppose that the inequalities in (4.17) are both equalities for some integer $n \geqslant 2$. By squaring both sides of these equalities and adding, one gets the contradiction $n^2 + n^2(n^2 - 1)\cos^2 \omega\tau = 1$. Thus, from Theorem 5 it follows that *if* $\cos 2\pi\tau/p \neq 0$, *then the second-order equation* (4.16) *has a bifurcating branch of nontrivial p-periodic solutions for* β_i *near the critical values given by*

$$\beta_1^0 = \frac{2\pi}{p} \sec \frac{2\pi}{p}, \qquad \beta_2^0 = \frac{2\pi}{p} \tan \frac{2\pi}{p}. \tag{4.18}$$

The bifurcation curve C, given by (4.18) as parametrized by p, is drawn in Fig. 3 for Eq. (4.16) when $\tau = 1$. As a specific example, the delay van der Pol equation studied by Grafton [9,10] is obtained from Eq. (4.16) by letting $R = \beta_1 x^2$.

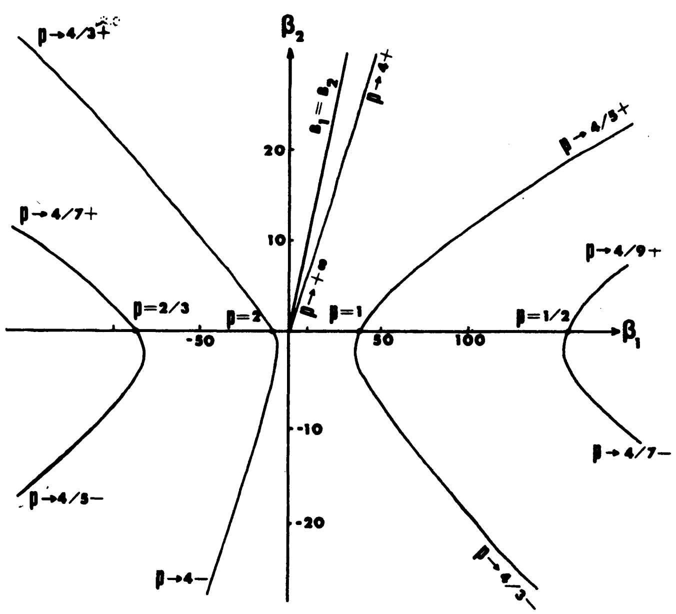

Figure 3: Several branches of the bifurcation curve C, as given by (4.18), for the second-order delay equation (4.16) with $\tau = 1$.

4.4 Some Systems with Delays

Many two-species models with delays found in mathematical ecology are of the form [5,6]

$$x_i' = -\beta_i\Big(\int_0^\infty x_1(t-s)\,dh_{1i}(s) + \int_0^\infty x_2(t-s)\,dh_{2i}(s)\Big) + R(x,\beta_1,\beta_2), \tag{4.19}$$

where $R = \mathrm{col}(R_i)$ satisfies A3. Suppose that the linearized system

$$y_i' = -\beta_i^0\Big(\int_0^\infty y_1(t-s)\,dh_{1i}(s) + \int_0^\infty y_2(t-s)\,dh_{2i}(s)\Big) \tag{4.20}$$

has exactly $m = 2$ independent p-periodic solutions for some $0 < \beta_i^0 \in R^1$. It is easy to write down necessary and sufficient conditions involving the Fourier integrals of the integrators $h_{ij}(s)$ for this assumption to hold, but it is not necessary here to do this. Let these p-periodic solutions and those of the adjoint system be given by the real and imaginary parts of $ae^{i\omega t}$ and $be^{i\omega t}$, $\omega = 2\pi/p$, respectively, where $a = \mathrm{col}(a_1, a_2)$ and $\mathrm{col}(b_1, b_2)$ are complex vectors. One can straightforwardly apply Theorem 2 as in the above applications. After some lengthy calculations it is found that

$$d = \frac{\omega^2}{4\beta_1^0\beta_2^0}(k_1^2 + k_2^2)\,\mathrm{Im}\, a_1\bar{a}_2\bar{b}_1 b_2 \tag{4.21}$$

(where the bars denote complex conjugation).

Thus, *the two-species model* (4.19) *has a bifurcating branch of nontrivial p-periodic solutions for* β_i *near the values* β_i^0 *where the linearization* (4.20) *has exactly* $m = 2$ *independent p-periodic soltutions provided* $d \neq 0$, *where d is given by* (4.21). This result is sufficient to prove the bifurcation theorems for the specific ecological models in [5, ch. 4].

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