On Strong Stability for Linear Integral Equations

by

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SUMMARY

The notion of strong or adjoint stability for linear ordinary differential equations is generalized to the theory of Volterra integral equations. It is found that this generalization is not unique in that equivalent definitions for differential equations lead to different stabilities for integral equations in general. Three types of stabilities arising naturally are introduced: strong stability, adjoint stability, and uniform adjoint stability. Necessary and sufficient conditions relative to the fundamental matrix for these stabilities are proved. Some lemmas dealing with non-oscillation of solutions and a semi-group property of the fundamental matrix are also given.

1. Introduction. We wish here to extend the notion of strong stability for ordinary differential equations, as considered for example by Coppel [1], to systems of Volterra integral equations

$$u(x) = \varphi(x)+\int_a^x K(x, t)u(t)\, dt. \tag{I}$$

Here $K(x, t)$ is a continuous $k\times k$ matrix defined for real x, $t \geq x_0$ and $\varphi(x)$, $u(x)$ are k vectors; a is a fixed but arbitrary constant greater than or equal to x_0. (The basic theory for equation (I) guarantees that there exists a unique solution defined for $x \geq x_0$.) The system (I) contains the initial-value problem for first-order systems of differential equations ($\varphi \equiv$ const. and K independent of x) and we desire to generalize the concept of strong stability so as to contain the essential results of this theory for such equations, as was done in [2] for other basic stabilities. Two features will be seen to stand out in this respect. First, the space of allowable "initial functions" φ plays a significant role, as was found in [2] (see also [3]–[7] and the references cited therein, where this feature is important in other approaches to stability for (I)). Indeed, strong stability on too "large" a space of initial functions is so restrictive that it rules out the dependence of the kernel K on the variable x (see Theorem 1). As a result, for

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integral equations strong stability on "smaller" spaces has more interest and content. We will emphasize $\varphi \equiv$ const. below, although some of the results stated are valid on the space of $C^1[x_0, +\infty)$, where $|\varphi'(x)|$ has a finite integral on $[x_0, +\infty)$. This will be pointed out in context below.

Second, the notion of strong stability for differential equations does not seem to have a unique, "natural" extension to (I) in that equivalent characterizations for differential equations lead to different, although related, notions of stong stability for (I). For example, the definition of strong stability given by Coppel in [1, p. 51] (who credits the concept to Ascoli [8]) can be used, with the necessary modifications for the space of initial functions, for (I); this is done in Section 2. This definition can in fact be used for nonlinear Volterra equations. However, the equivalent definition for linear equations in terms of the adjoint equation (see Cesari [9, p. 45], who calls this notion "restrictive stability") or in terms of the fundamental matrix [1, p. 54] generalizes to a "weaker" stability for (I) (adjoint stability). Another stability (uniform adjoint stability) more or less intermediate to these is possible; both are discussed in Section 3.

We give precise definitions of these three stabilities in Sections 2 and 3, each of which generalizes the concept of strong stability for ordinary differential equations. Also, we state and prove characterizations of each in terms of the fundamental matrix for (I) on an appropriate space of initial functions. These results, among other things, are useful in treating perturbed or linearizable nonlinear Volterra equations (as in [2]) which we hope to do in future work. Although we do not pursue it here, criteria for each of these stabilities relating explicitly to the kernel K can be stated by using the representation theorem and techniques developed by the authors in [10]–[12]. Finally, we point out that some interesting results which we find convenient to state and prove here, concerning the oscillation of solutions of (I) (which, of course, is not possible for homogeneous differential equations), and a composition or semi-group property of the fundamental matrix may be found in Section 2.

2. Strong stability. As shown in [2] the solution to (I) for a given $\varphi \in C^1[x_0, +\infty)$ has the representation

$$u(x) = U(x, a)\varphi(a) + \int_a^x U(x, s)\varphi'(s)\, ds, \tag{R}$$

where the fundamental matrix $U(x, s)$ is defined as the unique solution to the matrix equation

$$U(x, s) = I + \int_s^x K(x, t)U(t, s)\, dt; \qquad x, s \geq x_0.$$

(Integration by parts in (R) leads to a formula for $u(x)$ valid for the larger class of $\varphi \in C^0[x_0, +\infty)$; we will not use this representation here, however.) If (I) is an initial-value problem for a system of ordinary differential equations which has a fundamental solution $Y(x)$, then $U(x, s)$ is nothing more than $Y(x)Y^{-1}(s)$. The representation (R) may be related to similar representations in the literature as in [2].

Let N be a set of initial functions contained in $C^0[x_0, +\infty)$. Following

Coppel [1, p. 51], we say that (I) (or equivalently, that the null solution corresponding to $\varphi \equiv 0$) is *strongly stable on N* if for each $\epsilon > 0$ and all $a \geq x_0$ there exists a constant $\delta = \delta(\epsilon) > 0$ such that any solution $u(x)$ of (I) for $\varphi \in N$ satisfying $|u(x^*)| \leq \delta$ for some $x^* \geq x_0$ necessarily satisfies $|u(x)| \leq \epsilon$ for all $x \geq x_0$. Here $|\cdot|$ is any k-vector norm. It is obvious that strong stability implies uniform stability (see [2] or Section 3 below for definition), a concept which plays a central role in proving the stability of a small, nonlinear perturbation of (I); see [2].

The first result on strong stability of (I) which we will establish is, roughly speaking, that strong stability on large enough spaces N implies that the kernel K is independent of the variable x. Thus, for integral equations (I), only strong stability on relatively restricted spaces N is of interest.

THEOREM 1. *Suppose $C^m[x_0, +\infty) \subseteq N$ for some $0 \leq m \leq +\infty$ and that $K(x, t)$ is m times continuously differentiable with respect to x. Then the strong stability of N of (I) implies K is independent of x.*

In order to prove this theorem we first obtain some preliminary results. The solution to a first-order homogeneous system of ordinary differential equations with nonzero initial value is nonzero for all $x \geq x_0$ (by nonzero we mean $u(x) \neq \bar{0}$, where $\bar{0}$ is the zero k-vector). This is not true, however, for the more general system (I) even if φ is constant. For example, if $K(x, t) \equiv t-x$ in (I), then $U(x, s) = \cos(s-x)$ and thus $u(x) = \varphi \cos(a-x)$ is the solution to (I) for any $\varphi \equiv$ const. We say then that (I) is *non-oscillatory on N* if $u(x) \neq \bar{0}$, $x \geq x_0$, for all $\varphi \in N$, $\varphi \neq \bar{0}$. The first lemma is relatively easy to establish.

LEMMA 1. *If equation (I) is strongly stable on N, then it is non-oscillatory on N for all $a \geq x_0$.*

Proof. Suppose that (I) is not non-oscillatory on N for all $a \geq x_0$ and that, for some a and $\varphi \in N$, $\varphi \neq \bar{0}$, there exists $x^* \geq x_0$ for which the solution to (I) satisfies $u(x^*) = \bar{0}$. For each positive integer n, the strong stability of (I) on N implies that there exists a $\delta_n > 0$ such that $0 = |u(x^*)| \leq \delta_n$ implies $|u(x)| \leq 1/n$ for all $x \geq x_0$. Thus, since n is arbitrary, $|u(x)| = 0$ for all $x \geq x_0$, a contradiction to $\varphi \neq \bar{0}$.

Our next two lemmas concern a semi-group or translation property of the fundamental matrix $U(x, s)$:

$$\text{(P)} \qquad U(x, s) = U(x, y)U(y, s), \qquad x, y, s \geq x_0.$$

LEMMA 2. *(a) If (P) holds, then (I) is non-oscillatory on E^k (k-dimensional Euclidean space) for all $a \geq x_0$. (b) Suppose that $C^m[x_0, +\infty) \subseteq N$ for some $0 \leq m \leq +\infty$ and that $K(x, t)$ is m times continuously differentiable with respect to x. If (I) is non-oscillatory on N for all $a \geq x_0$, then (P) holds.*

Proof. (a) Suppose that (P) holds and that for some $a \geq x_0$, $\bar{\varphi} \in E^k$, $\bar{\varphi} \neq \bar{0}$, the solution to (I) satisfies $u(x^*) = U(x^*, a)\,\bar{\varphi} = \bar{0}$ for some $x^* \geq x_0$. Then using (P) we have $u(x) = U(x, a)\,\bar{\varphi} = U(x, x^*)U(x^*, a)\,\bar{\varphi} = \bar{0}$ for all $x \geq x_0$. This contradicts $\bar{\varphi} \neq \bar{0}$.

(b) Let $\varphi \in E^k$ be an arbitrary k-vector. Then for arbitrary but fixed $s, y \geq x_0$, the functions $u(x) = U(x, s)\varphi$, $v(x) = U(x, y)U(y, s)\varphi$ are the solutions to $u(x) = \varphi + \int_s^x K(x, t)u(t)\, dt$ and $v(x) = U(y, s)\varphi + \int_y^x K(x, t)v(t)\, dt$, respectively.

Subtracting these two equations and setting $w(x) = u(x) - v(x)$ we obtain $w(x) = \bar{\varphi}(x) + \int_s^x K(x, t)w(t)\, dt$, where

$$\bar{\varphi}(x) = [I - U(y, s)]\varphi + \int_s^y K(x, t)U(t, y)U(y, s)\varphi\, dt.$$

From our assumptions it follows that $\bar{\varphi} \in C^m[x_0, +\infty)$ and hence $\bar{\varphi} \in N$. But it is assumed that (I) is non-oscillatory on N for $a = s$ and consequently the fact that $u(y) = v(y)$ or $w(y) = \bar{0}$ implies $w(x) = \bar{0}$, $x \geq x_0$. But $\varphi \in E^k$ was arbitrary and consequently $U(x, s) = U(x, y)U(y, s)$, $x \geq x_0$. Since y and s were also arbitrary, we find that (P) holds. This proves the lemma.

Note that if (P) holds, $U(x, s)$ is invertible for all $x, s \geq x_0$. In fact, letting $s = x$ in (P), we find that $U^{-1}(x, y) = U(y, x)$ for all $x, y \geq x_0$.

In the case of ordinary differential equations, $U(x, s) = Y(x)Y^{-1}(s)$, and (P) obviously holds. That a converse is true is contained in the next lemma.

LEMMA 3. *Suppose that $K(x, t)$ is continuous in $x, t \geq x_0$. If (P) holds, then $K(x, t)$ is independent of x.*

Proof. Let $\varphi \in E^k$ and $a \geq x_0$ be fixed, but arbitrary. Once again $u(x) = U(x, a)\varphi$ solves (I) for $x \geq x_0$. Since (P) holds, we have $u(x) = U(x, y)U(y, a)\varphi$ for any $y \geq x_0$ and hence $u(x)$ also solves the equation $u(x) = U(y, a)\varphi + \int_y^x K(x, t)u(t)\, dt$, which when subtracted from (I) yields $\bar{0} = [I - U(y, a)]\varphi + \int_a^y K(x, t)u(t)\, dt$. This equation is valid for all $x, y \geq x_0$; thus, if $x^* \geq x_0$ is arbitrary, we find that $\int_a^y [K(x, t) - K(x^*, t)]u(t)\, dt = \bar{0}$ for all $x, x^* \geq x_0$ and for all $y \geq x_0$. This implies that $[K(x, t) - K(x^*, t)]u(t) = \bar{0}$ for all $t \geq a$ or in particular for $t = a$ that $[K(x, a) - K(x^*, a)]\varphi = \bar{0}$. But $\varphi \in E^k$ is arbitrary so that $K(x, a) = K(x^*, a)$, and inasmuch as $a \geq x_0$ and $x, x^* \geq x_0$ are arbitrary, the lemma is proved.

Proof of Theorem 1. Under the hypotheses of Theorem 1, the strong stability on N of (I) implies (by Lemmas 1 and 2) that (P) holds. The theorem then is seen to follow from Lemma 3.

We now wish to establish necessary and sufficient conditions for strong stability on certain spaces in terms of the fundamental matrix U. The results of the following theorem generalize to (I) the characterization of strong stability for ordinary differential equations as given in [1, p. 54]. The necessary condition (a) is of the type useful in studying nonlinear perturbed equations (see [1]–[5]). The sufficient condition (b) can be used to establish conditions for strong stability relating directly to the kernel $K(x, t)$ as for example is done in [10] and [12] for other stabilities; as this is straightforward, we will not state any general results of this kind here.

Let $\|U\| = \sup_{|\xi|=1} |U\xi|$.

THEOREM 2. *(a) If (I) is strongly stable on N, where $E^k \subseteq N$, then $U(x, s)$ is invertible for all $x, s \geq x_0$ and there exists a constant $L > 0$, independent of s and x, for which*

$$\|U(x, s)\| \leq L, \qquad \|U^{-1}(x, s)\| \leq L, \qquad x, s \geq x_0. \tag{2.1}$$

(b) Conversely, if $U^{-1}(x, s)$ exists for all $x, s \geq x_0$ and (2.1) holds for some $L > 0$ independently of $x, s \geq x_0$, then (I) is strongly stable on E^k.

Proof. (a) If (I) is strongly stable on E^k, then (by Lemma 1) (I) is non-oscillatory on E^k for all $a \geq x_0$. Thus, $U(x, a)$ is invertible for all x, $a \geq x_0$, for if $U(x^*, a^*)\varphi^* = \bar{0}$ for some x^*, $a^* \geq x_0$ and $\varphi^* \in E^k$, $\varphi^* \neq \bar{0}$, then $u(x) = U(x, a)\varphi^*$ is a solution to (I) with $\varphi = \varphi^*$, $a = a^*$, for which $u(x^*) = \bar{0}$, contrary to (I) being non-oscillatory. Fixing $\epsilon_0 > 0$, let $\delta_0 > 0$ be as in the definition of strong stability. For arbitrary but fixed x^*, $a \geq x_0$, consider $u(x) = U(x, a)U^{-1}(x^*, a)\xi$, where ξ is any vector in E^k for which $|\xi| \leq \delta_0$. Since this $u(x)$ solves (I) with $\varphi = U^{-1}(x^*, a)\xi$ and $u(x^*) = \xi$, we conclude from the strong stability of (I) that $|U(x, a)U^{-1}(x^*, a)\xi| \leq \epsilon_0$ for all $x \geq x_0$. Since ξ, $|\xi| \leq \delta_0$, is arbitrary this means $\| U(x, a)U^{-1}(x^*, a)\| \leq L$, $L = \epsilon_0/\delta_0$, where, from the definition of strong stability, δ_0, and hence L, is independent of a and x^* as well as x. Putting respectively $x = a$ and $x^* = a$ in this estimate, we obtain the bounds of (2.1).

(b) Conversely, the solution to (I) for $\varphi \in E^k$ is $u(x) = U(x, a)\varphi$, $x \geq x_0$, or $u(x) = U(x, a)U^{-1}(x^*, a)u(x^*)$ for arbitrary $x^* \geq x_0$. For given $\epsilon > 0$, choose $\delta = \epsilon L^{-2}$. Then, if $|u(x^*)| \leq \delta$ for some $x^* \geq x_0$, we have $|u(x)| \leq \| U(x, a)\| \cdot \| U^{-1}(x^*, a)\| \cdot |u(x^*)| \leq L^2\delta = \epsilon$ for $x \geq x_0$ and (I) is strongly stable on E^k.

As an example of a strongly stable equation (I) and of a utilization of Theorem 2 we consider the scalar ($k = 1$) equation with $\varphi \in E^1$ and $K(x, t) = a(x)b(t)$ for continuous scalar-valued functions $a(x)$ and $b(x)$, $x \geq x_0$. Using the formula $U(x, s) = 1 + a(x) \int_s^x b(t) \exp\left(\int_t^x a(z)b(z)\, dz\right) dt$, which follows from the techniques used by the authors in [10] and [12], we easily see that some simple criteria which imply the strong stability of (I) on E^1 are: $0 \leq b(x)$ and $0 \leq a(x) \leq A$ for $x \geq x_0$ and some constant $A > 0$; and $B = \int_{x_0}^{+\infty} b(x)\, dx < +\infty$. Indeed, under these conditions, $1 \leq U(x, s) \leq AB \exp(AB) < +\infty$ for all x, $s \geq x_0$. For more general kernels one can use the representation formula for $U(x, s)$ in [11].

3. Adjoint and Uniform Adjoint Stability. For ordinary differential equations, condition (2.1) is equivalent to the boundedness of both $Y(x)$ and $Y^{-1}(x)$; thus, Theorem 2 generalizes the characterization of strong stability in [1, p. 54]. Because of this, strong stability for linear differential equations is equivalent to the simultaneous stability of the equation and its adjoint (see [9, p. 45] in this context). The boundedness of both Y and Y^{-1}, however, is perhaps more appropriately interpreted in the context of integral equations (I) as the boundedness of $U(x, s)$ in x for fixed s and vice versa. This leads one to consider a notion of strong stability for (I) in terms of such a boundedness property for $U(x, s)$, which, it turns out, is different from that defined above in Section 2. Such a definition, of course, would only be applicable to linear equations, and inasmuch as we hope in future work to consider nonlinear Volterra equations, we first give a (δ, ϵ)-definition and then prove the characterization in terms of $U(x, s)$ in the following theorem.

Let $N \subseteq C^0[x_0, +\infty)$ be a normed space of initial functions with norm $\| \cdot \|_N$. In [2] the following definition was made: (I) is *stable on* N if to $\epsilon > 0$ there exists a constant $\delta = \delta(a, \epsilon) > 0$ such that $\|\varphi\|_N \leq \delta$ implies $|u(x)| \leq \epsilon$ for all $x \geq a$. If δ is independent of $a \geq x_0$, then (I) is said to be *uniformly* *stable on N*. We say that (I) is *adjointly stable on N* if it is stable on N and if to any $\epsilon > 0$ and any $x^* \geq x_0$ there corresponds a constant $\delta = \delta(\epsilon, x^*) > 0$ such that $\|\varphi\|_N \leq \delta$, $\varphi \in N$, implies $|u(x^*)| \leq \epsilon$ for all $a \geq x_0$.

THEOREM 3. *(a) If (I) is adjointly stable on N for which $E^k \subseteq N$, then there exist constants $L(s)$, $M(x) > 0$ such that*

$$(3.1) \qquad \|U(x, s)\| \leq L(s) \text{ for } x \geq x_0, \qquad \|U(x, s)\| \leq M(x) \text{ for } s \geq x_0.$$

(b) If (3.1) holds for some constants L, M, then (I) is adjointly stable on E^k.

Proof. (a) That adjoint stability of (I) implies the existence of the required constant $L(s)$ follows from the existence of such a constant for stable equations on E^k [2]. From the definition of adjoint stability, $u(x) = U(x, s)\varphi$, $\varphi \in E^k$, satisfies $|u(x)| \leq \epsilon$ for $|\xi| \leq \delta = \delta(\epsilon, x)$ for all $s \geq x_0$. But $|U(x, s)\xi| \leq \epsilon$ for all $|\xi| \leq \delta$, $s \geq x_0$, implies $\|U(x, s)\| \leq M$, where $M = \epsilon/\delta(\epsilon, x)$.

(b) The existence of M and L implies the bounds $|u(x)| \leq L(a)|\varphi|$, $x \geq x_0$; $|u(x)| \leq M(x)|\varphi|$, $a \geq x_0$ for all $\varphi \in E^k$. These clearly imply adjoint stability.

That adjoint and strong stability are different stabilities will be pointed out, together with other relationships between all these various stabilities, in Section 4.

The boundedness of both Y and Y^{-1} for differential equations may also be stated as the boundedness of $Y(x)Y^{-1}(s)$ for all x, $s \geq x_0$. Thus, we are also naturally led to a third concept of stability, which also generalizes the notion of strong stability for differential equations, in terms of the boundedness of $U(x, s)$ on the quadrant x, $s \geq x_0$. This concept will also turn out to be distinct from strong stability defined in Section 2. Once again we begin with a (δ, ϵ)-definition because of its wider range of applicability. We say (I) is *uniformly adjointly stable on N* provided to each $\epsilon > 0$ there exists a constant $\delta = \delta(\epsilon) > 0$ (independent of $a \geq x_0$) such that $\|\varphi\|_N \leq \delta$, $\varphi \in N$, implies $|u(x)| \leq \epsilon$ for all x, $a \geq x_0$. This definition differs from that of uniform stability of (I) in that in the latter case $|u(x)|$ is to be small (independently of $a \geq x_0$) for all $x \geq a$, whereas for uniform adjoint stability it is to be small for all $x \geq x_0$.

THEOREM 4. *(a) If (I) is uniformly adjointly stable on N, $E^k \subseteq N$, then there exists a constant $L > 0$ such that*

$$(3.2) \qquad \|U(x, s)\| \leq L \quad \text{for all } x, s \geq x_0.$$

(b) If a constant $L > 0$ exists such that (3.2) holds, then (I) is uniformly adjointly stable on E^k.

Proof. (a) From the definition of uniform adjoint stability, $|U(x, a)\varphi| \leq \epsilon$ for $|\varphi| \leq \delta(\epsilon)$, $\varphi \in E^k$, and for all x, $a \geq x_0$. This implies $\|U(x, a)\| \leq L = \epsilon/\delta$ for x, $a \geq x_0$.

(b) The existence of L such that (3.2) holds implies $|u(x)| \leq L|\varphi|$, $\varphi \in E^k$ for all x, $a \geq x_0$, which implies uniform adjoint stability because of the independence of L on x and a.

We note that all three stabilities defined here are equivalent to (and, hence, generalize) strong stability for ordinary differential equations and that the three theorems (Theorems 2, 3, 4) generalize the characterization theorem for strong stability in terms of the fundamental solution (see Coppel [1]).

We also point out that the conclusion of part (b) of Theorem 4 can be strengthened somewhat from that of uniform adjoint stability on E^k to the larger space of functions in $C^1[x_0, +\infty)$ with finite norm $\|\varphi\|_N = |\varphi(a)| + \int_{x_0}^{+\infty} |\varphi'(s)|\, ds$. Thus, on this space, the bounds appearing in this theorem constitute necessary and sufficient conditions in terms of the fundamental matrix $U(x, s)$ for uniform adjoint stability. Likewise, in Theorem 3 (b), if the first bound L is independent of s for $x \geq s \geq x_0$, then (I) is adjointly stable on this larger space. The proofs involve minor changes from those given above. The remarks preceding Theorem 2 concerning the utility of these types of results also apply to Theorems 3 and 4.

4. Remarks. We conclude with a few remarks concerning the relationships among the various stabilities discussed on the space E^k. Uniform stability implies stability on E^k, as is obvious from their definitions. That the converse is false is illustrated by examples from the theory of ordinary differential equations [1]. (These two stabilities are equivalent for convolution kernels $K \equiv K(x-t)$ because in this case $U \equiv U(x-s)$; see [2].) Since uniform stability on E^k is characterized by the boundedness of $U(x, s)$ on the "infinite triangle" $x \geq s \geq x_0$, it is immediate from either the definitions or from Theorems 2 and 4 that both strong stability and uniform adjoint stability on E^k imply uniform stability on E^k. However, examples from the theory of ordinary differential equations show that uniform stability does not imply strong stability and, hence, inasmuch as adjoint, uniform adjoint, and strong stability for (I) all "collapse" to strong stability for the special case of differential equations, uniform stability on E^k implies none of these on E^k. The example mentioned above with $K(x, t) \equiv t-x$ and $U(x, s) \equiv \cos(x-s)$ shows that (I) may be either adjointly or uniformly adjointly stable on E^k but not strongly stable on E^k. Thus, we have the following implications for the indicated stabilities on E^k:

$$\text{strong} \Rightarrow \text{uniform adjoint} \Rightarrow \text{uniform} \Rightarrow \text{stable}$$

$$\text{uniform adjoint} \Rightarrow \text{adjoint} \Rightarrow \text{stable}.$$

All indicated implications have false converses. There remain two possible implications left unsettled: does adjoint stability imply uniform stability and does adjoint stability imply uniform adjoint stability? We conjecture that the former is true while the latter is false; however, we have not been able to prove or disprove either.

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