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Open Problems and Conjectures

Edited by Gerry Ladas

In this section we present some open problems and conjectures about some interesting types of difference equations. Please submit your problems and conjectures with all relevant information to G. Ladas.

Global Stability of Some Matrix Equations in Population Dynamics

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The system of nonlinear recursion equations

$$\begin{aligned} L_{t+1} &= bA_t \exp(-c_{\mathrm{el}}L_t - c_{\mathrm{ea}}A_t), \\ P_{t+1} &= (1-\mu_{\mathrm{l}})A_t, \\ A_{t+1} &= P_t \exp(-c_{\mathrm{ea}}A_t) + (1-\mu_{\mathrm{a}})A_t \end{aligned} \tag{1}$$

$(t = 0, 1, 2, \ldots)$ arises in population dynamics. This so-called "LPA model" describes the dynamics of an insect population whose individuals pass through larval, pupal and adult life cycle stages. All coefficients are positive and $\mu_{\mathrm{l}} < 1$, $\mu_{\mathrm{a}} < 1$. For example, this system has been extensively used in studies involving flour beetles (sp. *Tribolium*). If $n = (1-\mu_{\mathrm{l}})b/\mu_{\mathrm{a}} < 1$, then the sequence (L_t, P_t, A_t) tends to the origin $(0, 0, 0)$ as $t \to +\infty$. If $n > 1$ there is a unique positive equilibrium

$(L_e, P_e, A_e) > 0$ and this equilibrium is locally asymptotically stable if $n \approx 1$. See [1].

CONJECTURE x.y.1 *If $n > 1$ is sufficiently close to 1, then the positive equilibrium (L_e, P_e, A_e) is globally attracting, i.e., all sequences defined by* (1) *with* $0 \leq (L_0, P_0, A_0) \neq (0,0,0)$ *satisfy* $\lim_{t\to+\infty}(L_t, P_t, A_t) = (L_e, P_e, A_e)$.

This conjecture is a special case of a conjecture for a general class of recursion systems. Recursion formulas of the form

$$x_{t+1} = P(x_t)x_t, \quad t = 0, 1, 2, \ldots \tag{2}$$

frequently arise in population dynamics [1]. Here x_t is an m-vector (of real numbers) and $P = [p_{ij}] \in C^1$, $p_{ij} \geq 0$ is a nonnegative $m \times m$ "projection" matrix. The nonnegative cone $\{x \geq 0\}$ is forward invariant and $x = 0$ is an equilibrium. In most applications $P(x)$ is irreducible and primitive (for each $x \geq 0$) and hence has a positive, strictly dominant, simple eigenvalue with positive right and left eigenvectors. In many applications P has the additive decomposition

$$\begin{aligned} P(x) &= F(x) + T(x), \\ F(x) &= [f_{ij}(x)], \quad f_{ij} \in C^3(R^m \to R^m_+), \\ T(x) &= [\tau_{ij}(x)], \quad \tau_{ij} \in C^3(R^m \to [0,1]). \end{aligned} \tag{3}$$

Suppose $I - T(x)$ is invertible and $F(x)(I - T(x))^{-1}$ has a positive, strictly dominant, simple eigenvalue $n(x)$ with a nonnegative eigenvector $v(x)$ such that $(I - T(x))^{-1}v(x) > 0$. The eigenvalue $n(x)$ is called the *net reproductive number* and $n(0)$ is the *inherent net reproductive number*. The LPA model (ref: LPA) is a three dimensional example in which

$$x_t = \begin{pmatrix} L_t \\ P_t \\ A_t \end{pmatrix},$$

$$T(x) = \begin{pmatrix} 0 & 0 & 0 \\ 1 - \mu_l & 0 & 0 \\ 0 & \exp(-c_{pa}A) & 1 - \mu_a \end{pmatrix},$$

$$F(x) = \begin{pmatrix} 0 & 0 & b\exp(-c_{\mathrm{el}}L - c_{\mathrm{ea}}A) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$n(x) = b\left(\frac{1-\mu_{\mathrm{l}}}{\mu_{\mathrm{a}}}\right)\exp\big(c_{\mathrm{el}}L - (c_{\mathrm{ea}} + c_{\mathrm{pa}})A\big).$$

It can be shown for (2) that $n(0) < 1$ implies $x = 0$ is locally asymptotically stable. If, in addition, $p_{ij}(x) \leq p_{ij}(0)$ for all $x \geq 0$ then all sequences defined by (ref: matrix model) with $x_0 \geq 0$ satisfy $\lim_{t\to\infty} x = 0$. In most applications to population dynamics these inequalities hold; they are the result of "negative" feedback assumptions (referred to by biologists as "density dependence"). If $n(0) > 1$ then $x = 0$ is unstable. In fact if $P(x)x = 0$, $x \geq 0$ implies $x = 0$, then the equation is uniformally persistent (on the nonnegative cone with respect to $x = 0$). A stronger negative feedback condition is

$$\frac{\partial f_{ij}}{\partial x_k} \leq 0, \quad \frac{\partial \tau_{ij}}{\partial x_k} \leq 0 \text{ (and not all equal to 0)} \tag{4}$$

for $k = 1, 2, \ldots, m$. Under this condition, if $n(0) > 1$ is sufficiently close to 1 there exists a positive, locally asymptotically stable equilibrium $x > 0$. See [1].

CONJECTURE x.y.2 *For $n(0) > 1$ sufficiently close to 1 there exists a unique positive equilibrium $x_e > 0$ and all sequences x_t defined by* (2)–(4) *with $0 \leq x_0 \neq 0$ satisfy* $\lim_{t\to\infty} x_t = x_e$.

Remark Suppose $r(x)$ is the dominant eigenvalue of the projection matrix $P(x)$. Then $n(0) < 1$ if and only if $r(0) < 1$ and $n(0) > 1$ if and only if $r(0) > 1$. See [1].

References

[1] J.M. Cushing, *An Introduction to Structured Population Dynamics*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 71, SIAM, Philadelphia, 1998.