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Periodic Cycles of Nonlinear Discrete Renewal Equations

J. M. CUSHING*
Interdisciplinary Program on Applied Mathematics, Department of Mathematics, Building 89, University of Arizona, Tucson, AZ 85721
cushing@math.arizona.edu

A general class of discrete, nonlinear renewal equations containing a real parameter is studied. Bifurcation theory methods are used to prove the existence of nontrivial periodic solutions and asymptotically periodic solutions. Fundamental to the approach is the "limit equation" whose periodic solutions are shown to be asymptotic limits of solutions of the renewal equation. An application is made to a model of age-structured population dynamics in which the bifurcation of nontrivial equilibria and 2-cycles is shown to occur with increasing inherent net reproductive value.

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1. INTRODUCTION

The linear difference equation

$$x(t+1) = g(t) + \Sigma_{j=0}^{t} k(j)x(t-j), \quad t = 0, 1, 2, \ldots$$
$$x(0) = x_0$$

is a discrete analog of the famous renewal integral equation [6], [7], [8]. It arises in, for example, the study of population dynamics where the asymptotic behavior of solutions are of interest (see the application in Sec. 5 below). For linear equations the asymptotic dynamics can be studied by the z-transform [9]. However, in many applications the equation is nonlinear and the asymptotic dynamics become more difficult to ascertain. In this paper we study the existence of periodic solutions (including equilibria) of a general class of nonlinear discrete renewal equations. Our approach will utilize bifurcation theory and will consider the existence of periodic solutions as a function of a real parameter appearing in the equation. Specifically, we consider the equation

$$x(t+1) = g(\lambda)(t) + \lambda \sum_{j=0}^{t} k(j)x(t-j) + r(\lambda, x)(t), \quad t = 0, 1, 2\ldots$$
$$x(0) = x_0 \tag{1}$$

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Here $r(\lambda, x)$ is nonlinear and of order higher than x near $x = 0$. The "kernel sequence" $\lambda k(j)$, the "forcing function" $g(\lambda)(t)$ and the nonlinear term $r(\lambda, x)$ depend on a real parameter λ. For initial condition $x_0 = 0$ and forcing function $g = 0$ this equation has a trivial periodic solution, namely $x = 0$. We will use bifurcation theory techniques to obtain nontrivial periodic and asymptotically periodic solutions for certain values of λ.

A nonlinear equation with a known solution can be transformed by a simple change of dependent variable into one of the form (1) in which the known solution corresponds to the trivial solution $x = 0$. In this way our results can be applied to obtain the bifurcation of periodic solutions from other known periodic solutions (e.g. equilibria). An example occurs in the application given in Sec. 5.

In Sec. 2 it is established that the problem of asymptotically periodic solutions for renewal equations can be decomposed, by direct sum means, into two problems: that of periodic solutions of a "limit equation" and of asymptotically 0 solutions of a decoupled equation (Theorem 3). In Sec. 3 both linear and nonlinear limit equations are studied and a bifurcation theorems for 1-periodic and 2-periodic solutions are obtained (Theorems 8 and 6). Bifurcation theorems for 1-periodic and 2-periodic solutions of (1) are given in Sec. 4 (Theorems 11 and 10). The bifurcation of asymptotic 1-periodic and 2-periodic solutions is also considered in Sec. 4. In Sec. 5 an application to age-structured population dynamics is given in which a primary bifurcation of nontrivial equilibria and a secondary bifurcation of nontrivial 2-cycles is proved as the inherent net reproductive value of the population is increased through critical values.

2. A DIRECT SUM DECOMPOSITION

In this section we prove a decomposition theorem for asymptotically periodic solutions of nonlinear renewal equations of the form

$$\begin{aligned} x(t+1) &= g(t) + \textstyle\sum_{j=0}^{t} k(j)x(t-j) + r(x)(t), \qquad t = 0,1,2,\ldots \\ x(0) &= x_0 \end{aligned} \tag{2}$$

We have temporarily suppressed the presence of the parameter λ. We begin with some preliminary definitions and lemmas.

Let $P(n)$ denote the linear space of real sequences $p = \{p(t)\}_{t=-\infty}^{+\infty}$ of period $n \geq 1$ (n-cycles) and let Z denote the linear space of real sequences $z = \{z(t)\}_{t=0}^{+\infty}$ that converge to 0 as $t \to +\infty$. These spaces are Banach spaces under the supremum norm $\|\cdot\|_0$. The direct sum $A(n) = P(n) \oplus Z$ will be referred to as the space of *asymptotic n-cycles*. Each sequence $x \in A(n)$ can be written as $x = p + z$ where $p \in P(n)$ is an n-cycle and $z \in Z$ is a sequence converging to 0. $A(n)$ is a Banach space under the norm $\|x\| \doteq \|p\|_0 + \|z\|_0$.

We will look for solutions $x = p + z$ of (2) in $A(n)$ under the assumption that

A1: $g \in Z$.

We will do this by decomposing (2) into equations for p and z. To obtain equations for $p \in P(n)$ and $z \in Z$ we substitute $x = p + z$ into (2) and equate from both sides the periodic terms and the terms asymptotic to 0. In order to carry out this procedure we need to determine the $P(n)$ and Z projections of the terms on the right hand side of equation (2). The following lemma does this for the linear term.

Lemma 1 *Assume the kernel sequence* $k = \{k(j)\}_{j=0}^{+\infty}$ *satisfies*

$$\mathbf{A2}: 0 < \|k\|_1 \doteq \sum_{j=0}^{+\infty} |k(j)| < +\infty.$$

Then for $p \in P(n)$ *and* $z \in Z$

$$(a)\left\{\sum_{j=0}^{+\infty} k(j)p(t-j)\right\}_{t=-\infty}^{+\infty} \in P(n)$$

$$(b)\left\{\sum_{j=t+1}^{+\infty} k(j)p(t-j)\right\}_{t=0}^{+\infty} \in Z$$

$$(c)\left\{\sum_{j=0}^{t} k(j)z(t-j)\right\}_{t=0}^{+\infty} \in Z$$

Proof. The two series defining the sequences in (a) and (b) are absolutely convergent, both being bounded by the product $\|k\|_1\,\|p\|_0$. That the sequence in (a) is n-periodic is obvious. The inequality

$$\left|\sum_{j=t+1}^{+\infty} k(j)p(t-j)\right| \leq \|p\|_0 \sum_{j=t+1}^{+\infty} |k(j)|$$

and assumption **A2** show that the sequence in (b) tends to 0 as $t \to +\infty$, i.e. belongs to Z.

In order to show (c), let $\epsilon > 0$ be arbitrary. Since $z \in Z$ there exists an integer $T = T(\epsilon) > 0$ such that

$$t \geq T(\epsilon) \Rightarrow |z(t)| \leq \frac{\epsilon}{\|k\|_1}.$$

For $t \geq T(\epsilon)$ we have

$$\begin{aligned}
\left|\sum_{j=0}^{t} k(j)z(t-j)\right| &= \left|\sum_{i=0}^{t} k(t-j)z(j)\right| \\
&\leq \left|\sum_{j=0}^{T(\epsilon)-1} k(t-j)z(j)\right| + \left|\sum_{j=T(\epsilon)}^{t} k(t-j)z(j)\right| \\
&\leq \|z\|_0 \sum_{j=0}^{T(\epsilon)-1} |k(t-j)| + \frac{\epsilon}{\|k\|_1}\sum_{j=T(\epsilon)}^{t} |k(t-j)| \\
&= \|z\|_0 \sum_{j=t-T(\epsilon)+1}^{t} |k(j)| + \frac{\epsilon}{\|k\|_1}\sum_{j=0}^{t-T(\epsilon)} |k(j)| \\
&\leq \|z\|_0 \sum_{j=t-T(\epsilon)+1}^{+\infty} |k(j)| + \epsilon.
\end{aligned}$$

This implies that

$$0 \leq \limsup_{t\to+\infty} |\textstyle\sum_{j=0}^{t} k(j)z(t-j)| \leq \epsilon$$

and, since $\epsilon > 0$ is arbitrary, that

$$\lim_{t\to+\infty} \sum_{j=0}^{t} k(j)z(t-j) = 0.$$

This proves (c). ◇

From Lemma 1 we obtain a direct sum decomposition of the linear term on the right hand side of equation (2) as follows

$$\sum_{j=0}^{t} k(j)x(t-j) = \sum_{j=0}^{\infty} k(j)p(t-j) + \left(-\sum_{j=t+1}^{\infty} k(j)p(t-j) + \sum_{j=0}^{t} k(j)z(t-j) \right)$$

In order to obtain the direct sum decomposition of the entire right hand side of equation (2) we need to make some assumptions about the operator r. Let $\Omega(n)$ be an open neighborhood of the origin in $A(n) = P(n) \oplus Z$. Assume that

$$\textbf{A3}: \begin{cases} \text{r}: \Omega(\text{n}) \to \text{A(n) is a continuous operator} \\ \text{and satisfies } \| \text{r(x)} \| = \text{o} (\| \text{x} \|) \text{ near x} = 0. \end{cases}$$

Thus, $r(x)$ is a sequence in $A(n)$ and $r(x)(t)$ is the t^{th} term, i.e. $r(x) = \{r(x)(t)\}_{t=0}^{+\infty}$. Under assumption A3, for each $x \in A(n)$ we can write $r(x) = r_p(x) + r_z(x)$ where $r_p(x) \in P(n)$ and $r_z(x) \in Z$. The projections $\Pi_p: A(n) \to P(n)$ and $\Pi_z: A(n) \to Z$ defined by $\Pi_p\, x = p$ and $\Pi_z\, x = z$ are linear and bounded. Since $r_p(x) = \Pi_p\, r(x)$ and $r_z(x) = \Pi_z\, r(x)$ we have that

$$r_p: \Omega(n) \to P(n) \text{ is continuous with } \|r_p(x)\|_0 = o\,(\|x\|) \text{ near } x = o$$

$$r_z: \Omega(n) \to Z \text{ is continuous with } \|r_z(x)\|_0 = o\,(\|x\|) \text{ near } x = 0$$

Finally, we also assume

$$\textbf{A4}: x = p + z \in \Omega(n) \Rightarrow r(x) - r(p) \in Z.$$

Lemma 2 *Assume* **A3** *holds. Then* **A4** *holds if and only if*

$$r_p(x) = r_p(p) \text{ for all } x = p + z \in \Omega(n). \tag{3}$$

Proof. Assume (3) holds and choose any $x \in \Omega(n)$. Then

$$r(x) = r_p(x) + r_z(x)$$

$$= r_p(p) + r_z(x).$$

From

$$r(x) - r(p) = r_p(p) + r_z(x) - (r_p(p) + r_z(p))$$
$$= r_z(x) - r_z(p)$$

we see that $r(x) - r(p) \in Z$. Hence **A4** holds.

Conversely, suppose that **A4** holds. For any $x \in \Omega(n)$ we have

$$r(x) - r(p) = r_p(x) + r_z(x) - (r_p(p) + r_z(p)) \in Z.$$

This implies that

$$r_p(x) - r_p(p) \in Z$$

and hence that $r_p(x) - r_p(p) = 0$. ◇

Suppose that $x = p + z \in \Omega(n)$ solves the nonlinear discrete renewal equation (2). From the assumptions and lemmas above we see that $p \in P(n)$ must be an n-periodic solution of the equation

$$p(t+1) = \sum_{j=0}^{\infty} k(j)p(t-j) + r_p(p)(t),\ t = 0, \pm 1, \pm 2, \ldots \tag{4}$$

and $z \in Z$ must be a solution of the equations

$$\begin{aligned} z(t+1) &= \left\{ g(t) - \sum\nolimits_{j=t+1}^{\infty} k(j)p(t-j) + r_z(p+z)(t) \right\} \\ &\quad + \sum\nolimits_{j=0}^{t} k(j)z(t-j),\ t = 0, 1, 2, \\ z(0) &= x_0 - p(0) \end{aligned} \tag{5}$$

By adding the equations (4) and (5) the converse is easily seen to be true. This proves the main theorem of this section.

Theorem 3 *Under the assumptions **A1–A4**, $x = p + z \in \Omega(n)$ is an asymptotic n-cycle solution of the nonlinear renewal equation (2) if and only if $p \in P(n)$ is an n-periodic solution of the "**limit equation**" (4) and $z \in Z$ is a solution of equations (5).*

Notice that the limit equation (4) for p is decoupled from the equation (5) for z and can therefore be treated as an independent equation to be solved for an n-cycle p. This is a direct result of **A4** and Lemma 2. Of course, $p = 0$ is a solution of (4). However, if $p = 0$, then equation (5) reduces to the original renewal equation (2) for a solution $x = z$ that tend to 0 as $t \to +\infty$. While our results below apply to this case, we are more interested in the case when the limit equation (4) has nontrivial n-cycles $p \in P(n)$, $p \neq 0$, (including the case of nontrivial 1-cycles or equilibria).

We can apply Theorem 3 to the nonlinear renewal equation (1) with parameter $\lambda \in R$ by replacing the kernel sequence $k(t)$ by $\lambda k(t)$, the forcing function $g(t)$ by $g(\lambda)(t)$, and the higher order term $r(x)(t)$ by $r(\lambda,x)(t)$. We obtain the result that the problem of finding asymptotically periodic solutions of the nonlinear renewal equation (1) is equivalent to finding periodic solutions $p \in P(n)$ of the associated limit equation

$$p(t+1) = \lambda \sum_{j=0}^{\infty} k(j)p(t-j) + r_p(\lambda, p)(t), t = 0, \pm 1, \pm 2, \ldots \tag{6}$$

and asymptotically zero solutions $z \in Z$ of the equation

$$\begin{aligned} z(t+1) &= \left\{g(\lambda)(t) - \lambda \textstyle\sum_{j=t+1}^{\infty} k(j)p(t-j) + r_z(\lambda, p+z)(t)\right\} \\ &\quad + \lambda \textstyle\sum_{j=0}^{t} k(j)z(t-j), \quad t = 0, 1, 2, \ldots \\ z(0) &= x_0 - p(0) \end{aligned} \tag{7}$$

Here it is assumed that

$$\mathbf{A5}: \begin{cases} \mathrm{r} = \mathrm{r}(\lambda, \mathrm{x}): \mathrm{R} \times \Omega(\mathrm{n}) \to \mathrm{A(n)} \text{ satisfies } \mathbf{A3\text{–}A4} \text{ for each } \lambda \in \mathrm{R}, \\ \text{uniformly in } \lambda \text{ on compact subintervals of R and } \mathrm{g} = \mathrm{g}(\lambda): \mathrm{R} \to \mathrm{A(n)} \\ \text{satisfies } \mathbf{A1} \text{ for each } \lambda \in \mathrm{R}. \end{cases}$$

3. CYCLES OF THE LIMIT EQUATION

In this section we consider the existence of nontrivial n-cycle solutions in $P(n)$ of the limit equation (6). We begin with a general theory for linear limit equations. The kernel sequence k is assumed to satisfy **A2** throughout.

3.1 Linear Limit Equations

The Banach space $P(n)$ of n-cycles is finite dimensional and a complex basis is given by the sequences $\{\rho_{jn}^t\}_{t=-\infty}^{\infty}$ where the ρ_{jn}, for $j = 0,1,\ldots,n-1$, are the n^{th} roots of unity. The n sequences $\{\rho_{jn}^t\}_{t=-\infty}^{\infty}$ are mutually orthogonal with respect to the inner product $\langle p,q \rangle = \sum_{t=0}^{n-1} p(t)q^*(t)$. To see this we note that

$$\langle \rho_{jn}^t, \rho_{sn}^t \rangle = \sum_{t=0}^{n-1} [exp\,(2\pi i(j-s)/n)]^t$$

If $j \neq s$, then the n^{th} root of unity $\xi = \exp\,(2\pi i(j-s)/n) \neq 1$ satisfies the equation

$$0 = \xi^n - 1 = (\xi - 1)\left(\sum_{t=0}^{n-1} \xi^t\right) = (\xi - 1)\,\langle \rho_{jn}^t, \rho_{sn}^t \rangle$$

and hence $\langle \rho_{jn}^t, \rho_{sn}^t \rangle = 0$. On the other hand, $\langle \rho_{jn}^t, \rho_{jn}^t \rangle = n$. Any sequence $p \in P(n)$ can be written

$$p(t) = \sum_{s=0}^{n-1} c_s \rho_{sn}^t$$
$$c_s = \frac{1}{n} \langle p(t), \rho_{sn}^t \rangle \tag{8}$$

Consider first the *homogeneous* linear equation

$$p(t+1) = \sum_{j=0}^{\infty} k(j) p(t-j) \tag{9}$$

Substituting (8) into this equation we see that there is a nontrivial n-cycle solution if and only if the coefficients c_s are not all equal to zero and satisfy the equation

$$\sum_{s=0}^{n-1} c_s \rho_{sn}^{t+1} = \sum_{j=0}^{\infty} k(j) \sum_{s=0}^{n-1} c_s \rho_{sn}^{t-j}$$

or

$$\sum_{s=0}^{n-1} \{c_s \rho_{sn}\} \rho_{sn}^t = \sum_{s=0}^{n-1} \{c_s \sum_{j=0}^{\infty} k(j) \rho_{sn}^{-j}\} \rho_{sn}^t.$$

This is equivalent to

$$c_s \rho_{sn} = c_s \sum_{j=0}^{\infty} k(j) \rho_{sn}^{-j}$$

or

$$c_s (\rho_{sn} - \hat{k}(\rho_{sn})) = 0, s = 0, 1, \ldots, n-1$$

where

$$\hat{k}(z) \doteq \sum_{j=0}^{\infty} k(j) z^{-j}$$

is the z-transform of the kernel sequence k. From this follows the next lemma.

Lemma 4 *Assume $k(t)$ satisfies **A2**. The homogeneous equation (9) has a nontrivial periodic cycle solution if and only if the "**characteristic equation**"*

$$z - \hat{k}(z) = 0$$

has a root $z = \exp(i\theta)$ on the unit complex circle for which $\theta/2\pi$ is equal to a rational number. Specifically, if $\theta/2\pi = m/n$ and m, n are to lowest terms then (9) has a nontrivial n-cycle.

Consider now the *nonhomogeneous* linear equation

$$p(t+1) = h(t) + \sum_{j=0}^{\infty} k(j)p(t-j) \tag{10}$$

where the "*forcing sequence*" $h = \{h(t)\}_{t=-\infty}^{+\infty}$ is n-periodic, i.e. $h \in P(n)$. A substitution of (8) into this equation results in the equations

$$c_s\,(\rho_{sn} - \hat{k}(\rho_{sn})) = h_s,\ s = 0, 1, \ldots, n-1$$

for the coefficients c_s where the h_s are the coefficients associated with the forcing sequence h, i.e.

$$h(t) = \sum_{s=0}^{n-1} h_s \rho_{sn}^t.$$

Define the set

$$S_n \doteq \{s : |\ \rho_{sn} - \hat{k}(\rho_{sn}) = 0\}.$$

The Fredholm-type alternative in the next lemma follows immediately.

Lemma 5 *Assume $k(t)$ satisfies* **A2** *and $h \in P(n)$. Write*

$$h(t) = \sum_{s=0}^{n-1} h_s \rho_{sn}^t.$$

(a) If the homogeneous equations (9) has no nontrivial n-cycle solutions, then the nonhomogeneous equation (10) has a unique n-cycle solution

$$p(t) = \sum_{s=0}^{n-1} \left(\frac{h_s}{\rho_{sn} - \hat{k}(\rho_{sn})} \right) \rho_{sn}^t$$

(b) Suppose the homogeneous equations (9) has nontrivial n-cycle solutions. Then the nonhomogeneous equation (10) has n-cycle solutions if and only if the forcing sequence h is orthogonal to all of the nontrivial n-cycle solutions of (9). For such h the nonhomogeneous equation (10) has infinitely many n-cycle solutions given by

$$p(t) = \Sigma_{s\in S_n} c_s \rho_{sn}^t + \Sigma_{s\notin S_n} \left(\frac{h_s}{\rho_{sn} - \hat{k}(\rho_{sn})} \right) \rho_{sn}^t$$

c_s arbitrary for all $s \in S_n$

only one of which

$$p(t) = \sum_{s \notin S_n} \left(\frac{h_s}{\rho_{sn} - \hat{k}(\rho_{sn})} \right) \rho_{sn}^t$$

is orthogonal to the nontrivial homogeneous n-cycles of the homogeneous equation (9).

3.2 Nonlinear Limit Equations

We will restrict our attention to $n = 2$ and $n = 1$ cycles. Assume that $r = r(\lambda, x) : R \times \Omega(n) \to A(n)$ and $g = g(\lambda)$ satisfy **A5**. We are interested in the existence of nontrivial periodic solutions of the nonlinear limit equation (6). Here the higher order term $r_p = r_p(\lambda, x) : R \times \Omega(n) \to P(n)$ is continuous and satisfies $|r_p(\lambda, p)|_0 = o(|p|_0)$ near $p = 0$ uniformly on compact λ intervals. By Lemma 3 we need only consider r_p restricted to an open neighborhood $\Omega_p(n)$ of $p = 0$ in $P(n)$. Specifically, set

$$\Omega_p(n) = \{p \in P(n) \mid (p, 0) \in \Omega(n)\}.$$

First of all, to find 2-cycle solutions $p \in P(2)$ of equation (6) we substitute

$$p(t) = c_1 + c_2(-1)^t$$

into the equation, equate coefficients of the independent cycles $\{1\}$ and $\{(-1)^t\}$, and thereby obtain a system of two nonlinear algebraic equations

$$\begin{aligned} c_1 &= \lambda\hat{k}(1)c_1 + h_1(\lambda, c_1, c_2) \\ -c_2 &= \lambda\hat{k}(-1)c_2 + h_2(\lambda, c_1, c_2) \end{aligned} \tag{11}$$

for the real coefficients

$$c_1 = c_1(\lambda), c_2 = c_2(\lambda).$$

Here h_1 and h_2 are the coefficients of the 2-cycle $r_p(\lambda, p)$, i.e. h_1 and h_2 are defined by

$$r_p\,(\lambda, c_1 + c_2(-1)^t) = h_1(\lambda, c_1, c_2) + h_2(\lambda, c_1, c_2)(-1)^t$$

The equations (11) can be written in the form

$$\mathbf{c} = \lambda\mathbf{L}\mathbf{c} + \mathbf{h}(\lambda, \mathbf{c}) \tag{12}$$

where

$$\mathbf{L} = \begin{pmatrix} \hat{k}(1) & 0 \\ 0 & -\hat{k}(-1) \end{pmatrix}$$

$$\mathbf{h}(\lambda, \mathbf{c}) = \begin{pmatrix} h_1(\lambda, c_1, c_2) \\ -h_2(\lambda, c_1, c_2) \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

A fundamental tenet of bifurcation theory is that nonzero solutions of equation (12) can bifurcate from $\mathbf{c} = \mathbf{0}$ only at the eigenvalues of the linearized equation (this follows essentially from the implicit function theorem), which in this case are the two real numbers $\lambda = 1/\hat{k}(1)$ and $-1/\hat{k}(-1)$. The associated eigenvectors are the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively. Since we are interested in the bifurcation of 2-cycles (and hence a nonzero component c_2) we are interested in the second of these two eigenvalues.

If we assume that $\hat{k}(1) \neq -\hat{k}(-1)$, then for $\lambda = -1/\hat{k}(-1)$ the linear operator $\mathbf{L}$ has a geometrically simple characteristic value of -1. This implies that for this value of λ the linearized limit equation

$$p(t+1) = \lambda \sum_{j=0}^{\infty} k(j)p(t-j) \tag{13}$$

has one independent nontrivial 2-cycle solution. These facts, together with $|\mathbf{h}(\lambda, \mathbf{c})| = o(|\mathbf{c}|)$, imply that a bifurcation result of Rabinowitz ([10], Corollary 1.12) is applicable to the equations (11). This theorem implies the existence of a continuum of nontrivial solutions $(\lambda, \mathbf{c}) \in R \times R^2$ of (11) that bifurcates from $(\lambda, \mathbf{c}) = (-1/\hat{k}(-1), \mathbf{0})$ and connects either to the boundary of the domain of $\mathbf{h}(\lambda, \mathbf{c})$ or to the other bifurcation point $(\lambda, \mathbf{c}) = (1/\hat{k}(1), \mathbf{0})$. Locally the continuum is tangential to the eigenvector $\mathbf{e}_2$ and hence consists of 2-cycles that are not 1-cycles. We summarize these results in the following theorem.

Theorem 6 *Assume that r and g satisfy* **A5** *with $n = 2$ and that k satisfies* **A2**. *Assume further that $\hat{k}(-1) \neq 0$ and $\hat{k}(1) \neq -\hat{k}(-1)$. There exists a continuum $C(2)$ of pairs $(\lambda, p) \in R \times P(2)$ where p is a nontrivial 2-cycle solution of the limit equation (6). The continuum $C(2)$ bifurcates from $p = 0$ at $\lambda = -1/\hat{k}(-1)$ and connects either to the boundary $\partial(R \times \Omega_p(2))$ or to the point $(\lambda, p) = (1/\hat{k}(1), 0)$ (if $\hat{k}(1) \neq 0$).*

If $\Omega_p(2)$ is unbounded then the boundary $\partial(R \times \Omega_p(2))$ contains the point at ∞. Thus, in this case, it is possible that the bifurcating continuum $C(2)$ "connects to ∞", i.e. is unbounded in $R \times \Omega_p(2)$. In particular, this is true if $P(2) = \Omega_p(2)$, i.e. r_p is globally defined on $P(2)$, and $\hat{k}(1) = 0$.

While in a neighborhood of the bifurcation point the continuum $C(2)$ consists of 2-cycles that are not 1-cycles (equilibria), the entire continuum might not consist of such 2-cycles. This, for example, would not be the case when the second alternative (that the continuum connects to the point $(\lambda, p) = (1/\hat{k}(1), 0)$). Often in applications the second alternative for the bifurcating continuum can be ruled out by making use of special features of the equations. Here is an example.

Corollary 7 *Suppose the assumptions of Theorem 6 hold and that $r_p(0, p) = 0$ for all $p \in \Omega_p(2)$. If $\hat{k}(1)$ and $-\hat{k}(-1)$ are nonzero and have opposite signs, then the bifurcating continuum $C(2)$ in Theorem 6 connects to the boundary $\partial(R \times \Omega_p(2))$. In particular, if $\Omega_p(2) = P(2)$ the continuum $C(2)$ is unbounded in $R \times P(2)$.*

Proof. If the continuum connects both points $(\lambda, p) = (-1/\hat{k}(-1), 0)$ and $(1/\hat{k}(1), 0)$ then there would exist a nontrivial 2-cycle solution p of equation (6) for the intermediate value $\lambda = 0$. However, $r_p(0, p) = 0$ and equation (6) with $\lambda = 0$ imply $p = 0$, a contradiction.

A similar approach to that above can be taken to obtain a bifurcating continuum of 1-cycles (equilibria) of equation (6). A 1-cycle $p(t) \equiv c$ solves equation (6) if and only if c satisfies the equation $c = \lambda\hat{k}(1)c + r_p(\lambda, c)$. This equation has the form (12) with $\mathbf{L}c = \hat{k}(1)c$ and $\mathbf{h}(\lambda, c) = r_p(\lambda, c)$, to which the Rabinowitz results apply. (Since $\mathbf{L}$ has only one eigenvalue in this case, the second alternative is ruled out.) We obtain the following theorem. ◇

Theorem 8 *Assume that r and g satisfy* **A5** *with $n = 1$ and that k satisfies* **A2**. *Assume further that $\hat{k}(1) \neq 0$. There exists a continuum $C(1)$ of pairs $(\lambda, p) \in R \times P(1)$ where p is a nontrivial equilibrium solution of the limit equation (6). The continuum $C(1)$ bifurcates from $p = 0$ at $\lambda = 1/\hat{k}(1)$ and connects to $\partial(R \times \Omega_p(1))$. In particular, if $\Omega_p(1) = P(1)$ then $C(1)$ is unbounded in $R \times P(1)$.*

The critical bifurcation values $\lambda = -1/\hat{k}(-1)$ and $\lambda = 1/\hat{k}(1)$ are those values at which the linearized limit equation (13) has nontrivial 1-cycle (equilibrium) or 2-cycle solutions respectively.

4. CYCLES AND ASYMPTOTIC CYCLES

We return now to the nonlinear, discrete renewal equation (2). In this section we consider the existence of both periodic and asymptotically periodic solutions of this equation.

A solution $x = p + z \in A(n)$ of equation (2) is n-periodic if and only if $z = 0$. By Theorem 3 $z = 0$ will then solve the equations in (5), which happens if and only if the forcing function and the initial condition are given by

$$g(t) = \textstyle\sum_{j=t+1}^{\infty} k(j)p(t-j) - r_z(p)(t)$$

$$x_0 = p(0) \tag{14}$$

By Theorem 3, $p \in P(n)$ must solve the limit equation (4). One such solution is of course $p = 0$, in which case $g = 0$, $x_0 = 0$ and hence $x = 0$. On the other hand, if $0 \neq p \in P(n)$ is a nontrivial periodic solution of the limit equation (4), then it will also be a solution of the renewal equation (2) if the forcing function and the initial condition are chosen by the formulas (14). We arrive at the following basic result.

Theorem 9 *A periodic sequence $x = p \in P(n)$, $p \neq 0$, solves the nonlinear renewal equation (2) if and only if p solves the limit equation (4) and the forcing function and the initial condition are given by (14).*

This theorem relates the existence of nontrivial n-periodic solutions of a nonlinear renewal equation to that of its limit equation. In the preceding section we considered the existence of nontrivial periodic solutions of limit equations as a bifurcation phenomenon, for the cases $n = 2$ and $n = 1$. Theorem 6, together with Theorem 9 above, gives a bifurcation-like result for nontrivial 2-cycles of the renewal equation (1).

Theorem 10 *Assume that r and g satisfy **A5** with $n = 2$ and that k satisfies **A2**. Assume also that $\hat{k}(-1) \neq 0$ and $\hat{k}(1) \neq -\hat{k}(-1)$. Then there exists a continuum of $(\lambda, p, g, x_0) \in R \times \Omega_p(2) \times Z \times R$ where $x(t) = p(t)$ is a nontrivial 2-cycle solution of the discrete renewal equation (1) with parameter value λ, with forcing function given by*

$$g(\lambda)(t) = \lambda \textstyle\sum_{j=t+1}^{\infty} k(j)p(t-j) - r_z(\lambda,p)(t) \tag{15}$$

and with initial condition

$$x_0 = p(0). \tag{16}$$

This continuum bifurcates from the $(\lambda, p, g, x_0) = (-1/\hat{k}(-1), 0, 0, 0)$. It exists globally in the sense that the pairs (λ, p) taken from the continuum connect either to the boundary $\partial\ (R \times \Omega_p(2))$ or to the point $(\lambda, p) = (1/\hat{k}(1), 0)$ (if $\hat{k}(1) \neq 0$).

With regard to the alternative in the last sentence of this theorem, Corollary 7 implies that the continuum connects to the boundary $\partial\ (R \times \Omega_p(2))$ if $r_p(0, p) = 0$ for all $p \in \Omega_p(2)$ and if $\hat{k}(1)$ and $-\hat{k}(-1)$ are nonzero and have opposite signs.

Theorem 8, together with Theorem 9 above, gives a bifurcation-like result for nontrivial equilibria of the renewal equation (1).

Theorem 11 *Assume that r and g satisfy **A5** with $n = 1$ and that k satisfies **A2**. Assume also that $\hat{k}(1) \neq 0$. Then there exists a continuum of $(\lambda, p, g, x_0) \in R \times \Omega_p(1) \times Z \times R$ where $x(t) = p$ is a nontrivial equilibrium solution of the discrete renewal equation (1) with parameter value λ, with forcing function given by (15), and with initial condition given by (16). This continuum bifurcates from the $(\lambda, p, g, x_0) = (1/\hat{k}_0(1), 0, 0, 0)$. It exists globally in the sense that the pairs (λ, p) taken from the continuum connect to the boundary $\partial\ (R \times \Omega_p(1))$.*

Periodic solutions are, of course, asymptotically periodic. There can also exist other asymptotically periodic solutions of equation (2) for appropriate forcing functions and initial conditions. We describe below a result that characterizes the set of forcing functions and initial conditions (g, x_0) lying in a $Z \times R$ neighborhood of the periodic cycle produced by the forcing function and initial condition (14).

Let F denote the set of pairs $(f, y_0) \in Z \times R$ for which the linear renewal equation

$$y(t+1) = f(t) + \textstyle\sum_{j=0}^{t} k(j)y(t-j),\ t = 0, 1, 2, \ldots$$
$$y(0) = y_0 \tag{17}$$

has a solution in $y \in Z$. The set F contains at least the pair $(0, 0)$ and is easily seen to be a linear subspace of $Z \times R$. The solution operator S defined by $S(f,y_0) = y$ is continuous (bounded) as a linear operator mapping F to Z.

Given a $p \in P(n)$, define the operator $G(p)$: $F \rightarrow Z \times R$ by

$$G(p)\colon (f,y_0) \rightarrow \left(f(t) + \sum_{j=t+1}^{\infty} k(j)p(t-j) - r_z(p + S(f,y_0)),\, y_0 + p(0) \right)$$

Theorem 12 *Assume k satisfies* **A2** *and that r satisfies* **A3–A4**. *Then $x = p + z \in A(n)$ is an asymptotically n-periodic solution of the nonlinear renewal equation (2) if and only if the pair (g,x_0) lies in the range of the operator $G(p)$ where*

$p \in P(n)$ solves the limit equation (6)

and

$z = S(f, y_0)$ where (f,y_0) is a pre-image of (g, x_0)

(i.e., $(g, x_0) = G(p)(f, y_0)$).

Proof. Suppose $x = p + z \in A(n)$ is an asymptotically n-periodic solution of the nonlinear renewal equation (2). By the decomposition Theorem 3 we know that $p \in P(n)$ solves the limit equation (4) and that $z \in Z$ solves the (5). If we define $(f,y_0) \in Z \times R$ by the expressions

$$f(t) = g(t) - \textstyle\sum_{j=t+1}^{\infty} k(j)p(t-j) + r_z(p+z)(t)$$
$$y_0 = x_0 - p(0) \tag{18}$$

then it follows that $y = z$ solves the linear equation (17). Thus $z = S(f, y_0)$. It follows that

$$g(t) = f(t) + \textstyle\sum_{j=t+1}^{\infty} k(j)p(t-j) - r_z(p + S(f,y_0))(t)$$
$$x_0 = y_0 + p(0)$$

or in other words that $(g, x_0) = G(p)(f, y_0)$.

Conversely, suppose $p \in P(n)$ solves the limit equation (4) and $z = S(f, y_0)$ where $(g, x_0) = G(p)(f, y_0)$. Then by the definition of the solution operator S, z solves the linear equation (17) with forcing function f and initial condition that satisfy (18). When (18) is substituted into (17) we find that z solves equation (5). The decomposition Theorem 3 implies that $x = p + z$ solves the nonlinear renewal equation (2) ◇.

This theorem shows that the set of pairs (g, x_0) for which the nonlinear renewal equation (2) has a solution asymptotic to a given n-periodic solution p of the limit equation (4) is identical to the range of the operator $G(p)$: $F \to Z \times R$. The next theorem gives conditions under which the operator $G(p)$ is a homeomorphism in a neighborhood of the point $(f, y_0) = (0, 0) \in Z \times R$, i.e. under which $\partial_{(f,y_0)}G(0)$ is invertible. The proof is omitted since it is virtually identical to the proof of Corollary 4.3 in [4] (using z-transforms [9] in place of Laplace transforms).

Recall that $\hat{k}(z)$ denotes the z-transform of the kernel sequence $k(j)$. We need the following assumption.

A6: $\left\{ \begin{array}{l} \hat{k}(z) \text{ has no roots on the unit circle } |z| = 1 \text{ and a finite} \\ \text{number } \nu \text{ of roots } z_i \text{ satisfying } |z| > 1. \text{ Each root } z_i \\ \text{has finite (algebraic) multiplicity } m_i > 0 \end{array} \right.$

Define the expression

$$\hat{k}_-(z) = z - (z - \hat{k}(z)) \prod_{i=1}^{\nu} \left(\frac{z + z_i}{z - z_i} \right)^{m_i}$$

As in [4] it can be proved from the assumption

A7: $\left\{ \begin{array}{l} \hat{k}_-(z) \text{ is the z-transform of a sequence } k_-(j) \\ \text{satisfying } \| k_- \|_1 \doteq \sum_{j=0}^{+\infty} |k_-(j)| < +\infty \end{array} \right.$

that the solution operator S is continuous (bounded) on F and that the following theorem holds.

Theorem 13 *Assume k satisfies **A2**, **A6**, and **A7**. Assume that r satisfies **A3–A4** and $r_z = r_z(x)$: $\Omega(n) \to Z$ is continuously Fréchet differentiable in x and $\partial_x r_z(0) = 0$. Then for each sufficiently small solution $p \in P(n)$ of the limit equation (4) the operator G is a homeomorphism from a neighborhood of $(f, y_0) = (0, 0) \in F$ to a neighborhoodof $G(p)(0, 0) = \left(\sum_{j=t+1}^{\infty} k(j)p(t-j) - r_z(p), p(0)\right) \in Z \times R$.*

This theorem shows that the "size" of the set of forcing functions and initial conditions (g, x_0) that give rise to asymptotically n-periodic solutions of the nonlinear renewal equation (2) is, locally near the n-periodic solution producing pair (g, x_0) given by (14), the same as that linear subspace Z for the associated linear renewal equation (17). The theorem only applies for n-periodic solutions p of sufficiently small norm $|p|_0$. It applies, however, to those 2-periodic and equilibrium solutions lying on the bifurcating continua of Theorems 6 and 8 lying near the bifurcation point. Thus, there will actually be a bifurcating family of asymptotically periodic solutions in the case of these two theorems. Note that if the limit equation (6) is autonomous in the sense that integer translates of solutions are solutions then Theorem 13 can be applied to each translate to obtain a bifurcating family of asymptotically periodic solutions.

5 AN APPLICATION

We consider an application to a discrete renewal equation that serves as a model for the dynamics of a biological population. Suppose the population is divided into age classes of equal (unit) length. Let $b(t) \geq 0$ denote the birth rate per unit time, i.e. the number of individuals born (and surviving to the next census) per unit time into the youngest age class. If $m(j) \geq 0$ is the number of offspring of an individual of age j, $\pi(j) \in [0, 1]$ is the probability that a newborn reaches age j, and $s \in [0, 1]$ is the probability that a newborn survives to the first census, then $sm(j)\pi(j)b(t-j)$ is the number of newborn contributed by all individuals of age j during the time interval $(t, t+1]$ that survive to be counted at time $t+1$. Thus, if $b_{in}(t)$ denotes the number of newborns produced by the initial population, we have

$$b(t+1) = b_{in}(t) + \sum_{j=0}^{t} sm(j)\pi(j)b(t-j), \quad t = 0, 1, 2, \ldots$$

This is a linear discrete renewal equation. (This equation can also be derived from the well known Leslie matrix model for age-structured populations). Fertility and survival rates are not, in general, independent of population size, however. If a vital statistic such as $m(j)$ or $\pi(j)$ depends on population density in some way then the equation becomes nonlinear.

We will consider, in our application, one special case called the "Easterlin hypothesis", namely that the fertility of an individual depends on the density of its own age class. This assumption means $m(j)$ is a function of $b(t-j)$ and since the usual assumption is that density effects are deleterious $m(j)$ should be taken as a decreasing function of $b(t-j)$. Thus, we consider

$$m(j) = \mu(j)f(b(t-j)), \quad f(0) = 1$$

where $f\colon R \to R$ is a nonnegative, twice continuously differentiable, decreasing function of its argument on a neighborhood of the origin. Here $\mu(j) \geq 0$ is the "inherent" age specific fertility rate (i.e. the fertility rate at low densities) which we normalize by writing

$$\mu(j) = \beta\phi(j)$$

$$\phi(j) \geq 0, \quad \sum_{j=0}^{\infty} \phi(j) = 1 \tag{19}$$

The constant $\beta > 0$ is the "inherent net reproductive value", i.e. the expected number of offspring per individual per life time [5].

It is a natural assumption that $b_{in}(t) \geq 0$ has compact support. In any case, we assume that

$$\lim_{t \to +\infty} b_{in}(t) = 0. \tag{20}$$

With these assumptions our model equations become

$$b(t+1) = b_{in}(t) + \beta \sum_{j=0}^{t} \phi(j) f(b(t-j))\, b(t-j), t = 0, 1, 2, \ldots$$

$$b(0) = b_0 \geq 0. \tag{21}$$

This equation has the form of (1) with

$$x(t) = b(t),\ \lambda = \beta,\ g(\lambda)(t) = b_{in}(t),\ k(j) = \phi(j)$$

and

$$r(\lambda, x)(t) = -\lambda \sum_{j=0}^{t} \phi(j)\, (1 - f(x(t-j)))\, x(t-j).$$

5.1 Equilibria

We begin by looking for equilibrium solutions. Clearly (21) has the "trivial" equilibrium $b(t) = 0$ for forcing function and initial condition given by $(b_{in}(t), b_0) = (0,0)$. To find a bifurcating continuum of nontrivial (positive) equilibria we will utilize Theorem 11.

Under the assumptions (19) and (20) the assumptions **A1** and **A2** hold. It is easy to see that on a neighborhood of the origin in $A(n)$ the higher order term r satisfies **A3** (uniformly on compact $\lambda = \beta$ intervals). For the periodic part of r we have

$$r_p(\lambda, x)(t) = -\lambda \sum_{j=0}^{\infty} \phi(j)\, (1 - f(p(t-j)))\, p(t-j), \tag{22}$$

an operator that is easily seen to satisfy **A4**. Also we have

$$r_z(\lambda, x)(t) = -\lambda \sum_{j=0}^{t} \phi(j)\, (1 - f(x(t-j)))\, x(t-j) + \lambda \sum_{j=0}^{\infty} \phi(j)$$
$$(1 - f(p(t-j)))\, p(t-j)$$

or

$$r_z(\lambda, x)(t) = -\lambda \sum_{j=0}^{t} \phi(j)\, [f(p(t-j))\, p(t-j) - f(x(t-j))\, x(t-j)]$$

$$+ \lambda \sum_{j=0}^{t} \phi(j) z(t-j) + \lambda \sum_{j=t+1}^{\infty} \phi(j)\, (1 - f(p(t-j)))\, p(t-j).$$

That $r_z(\lambda, x)(t) \in Z$ follows from

$$f(p(j))p(j) - f(x(j))x(j) \in Z$$

and the following lemma. That r_z is also continuous (bounded) on a neighborhood of the origin in $A(n)$ (uniformly on compact $\lambda = \beta$ intervals) is easy to see.

Lemma 14 *If $k(j)$ satisfies* **A2** *and if $z \in Z$, then* $\sum_{j=0}^{t} k(j)z(t - j) \in Z$

Proof. Given an arbitrary $\epsilon > 0$ let $T = T(\epsilon) \geq 1$ be an integer so large that $t \geq T(\epsilon)$ implies

$$|z(t)| \leq \frac{\epsilon}{2\|k\|_1} \quad and \quad \sum_{j \geq T(\epsilon)}^{\infty} |k(j)| \leq \frac{\epsilon}{2\|z\|_0}.$$

For $t \geq T(\epsilon)$ we have from

$$\sum_{j=0}^{t} k(j)z(t-j) = \sum_{j=0}^{T(\epsilon)-1} k(j)z(t-j) + \sum_{j=T(\epsilon)}^{t} k(j)z(t-j)$$

the inequalities

$$\left| \sum_{j=0}^{t} k(j)z(t-j) \right| \leq \frac{\epsilon}{2\|k\|_1} \sum_{j=0}^{T(\epsilon)} |k(j)| + \|z\|_0 \sum_{j=T(\epsilon)}^{t} |k(j)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

which in turn imply

$$0 \leq \limsup_{t \to +\infty} \left| \sum_{j=0}^{t} k(j)z(t-j) \right| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $lim_{t \to +\infty} \sum_{j=0}^{t} k(j)z(t - j) = 0.$ ◇

We have verified that r satisfies assumptions **A3** (uniformly on compact $\lambda = \beta$ intervals). To show that r satisfies the required assumption **A5** we have only left to show that r satisfies **A4**. For any real $\lambda = \beta$ it follows from the definition (22) of r_p that (3) holds and therefore, **A4** follows by Lemma 2.

Note that by the normalization (19)

$$\hat{k}(1) = \hat{\phi}(j) = \sum_{j=0}^{\infty} \phi(j) = 1.$$

We can now apply Theorem 11 to the model equation (21) to obtain a continuum of $(\beta, b(t), b_{in}(t), b_0) \in R \times C(1) \times Z \times R$ that bifurcates from $(1, 0, 0, 0)$ such that

$b(t)$ is a nontrivial equilibrium solution of (21) with forcing function $b_{in}(t)$, initial condition b_0, and inherent net reproductive value β. Since r is globally defined, $\Omega_p(1) = P(1)$ and the components $(\beta, b(t))$ from the continuum connect to ∞ in $R \times C(1)$.

Note that the bifurcation occurs at the critical value $\beta = 1/\hat{k}(1) = 1$ where each individual is expected to exactly replace itself during its life time.

We know from the theory in Sec. 4 that the nontrivial equilibria $b(t) = b_\infty \neq 0$ from the bifurcating continuum are solutions of the limit equation

$$b(t+1) = \beta \sum_{j=0}^{\infty} \phi(j) f(b(t-j)) b(t-j), \; t = 0, 1, 2, \ldots$$

Therefore, in this example, an algebraic equation for the nontrivial equilibria is obtainable, namely

$$1 = \beta f(b_\infty)$$

Thus, $b_\infty = f^{-1}(1/\beta)$ for $\beta > 1$. It follows from (21) that $b_{in}(t) = 1 - \sum_{j=0}^{t} \phi(j)$. The bifurcating continuum $C(1)$ in this example can be explicitly given by the formulas

$$(\beta, b(t), b_{in}(t), b_0) = \left(\beta, f^{-1}(1/\beta), 1 - \sum_{j=0}^{t} \phi(j), f^{-1}(1/\beta) \right) \textit{for } \beta > 1. \tag{23}$$

5.2 2-cycles

To illustrate the use of Theorem 10 we consider a special case of the model equation (21) in which the nonlinearity f in the fertility rate has the Ricker form

$$f(x) = e^{-cx}, \; c > 0.$$

In this case the bifurcating continuum $C(1)$ of equilibria (23) is

$$(\beta, b(t), b_{in}(t), b_0) = \left(\beta, \ln \beta, 1 - \sum_{j=0}^{t} \phi(j), \ln \beta \right) \textit{for } \beta > 1.$$

If equation (21) is "centered" on these equilibria by defining

$$x(t) = b(t) - \ln \beta$$

one obtains the equation of the renewal form (1) with

$$\lambda = 1 - \ln \beta$$

$$g(\lambda)(t) = b_{in}(t) + \sum_{j=t+1}^{\infty} \phi(j)$$

$$k(j) = \phi(j)$$

$$r(\lambda)(x) = \sum_{j=0}^{\infty} \phi(j)(e^{-x(t-j)} - 1)x(t-j).$$

It is straightforward to show that **A5** is satisfied. Theorem 10 implies the existence of a bifurcating continuum of nontrivial 2-cycles provided the fertility kernel satisfies the conditions

$$\hat{\phi}(-1) = \sum_{j=0}^{\infty} \phi(j)(-1)^{-j} \neq 0, -1 \tag{24}$$

hold. The critical bifurcation value $\lambda = -1/\hat{\phi}(-1)$ corresponds to the critical inherent net reproductive value given by

$$\beta_{cr} = exp\left(1 + \frac{1}{\hat{\phi}(-1)}\right).$$

As an example, consider the case when fertility is a geometrically decreasing function of age, that is to say when $\phi(j)$ is proportional to a^j for some constant a satisfying $0 < a < 1$. The normalization (19) then specifies

$$\phi(j) = (1 - a)a^j.$$

The z-transform $\hat{\phi}(z) = (1 - a)z(z - a)^{-1}$ yields $\hat{\phi}(-1) = (1 - a)(1 + a)^{-1} > 0$ and the required conditions (24) hold. The critical bifurcation value in this case is

$$\beta_{cr} = exp\left(\frac{2}{1 - a}\right).$$

As a second example, consider a case of delayed fertility due to a maturation period, after which fertility is again a geometrically decreasing function of age, as described by the fertility kernel

$$\phi_m(j) = \begin{cases} 0 & for\, j = 0, 1, \ldots, m - 1 \\ (1 - a)a^{j-m} & for\, j = m, m + 1, \ldots \end{cases}$$

Here the integer $m \geq 1$ is the maturation age. This fertility kernel satisfies the normalization (19). Moreover, the z-transform is $\hat{\phi}_m(z) = (1 - a)z^{1-m}(z - a)^{-1}$ and hence $\hat{\phi}_m(-1) = (-1)^m(1 - a)(1 + a)^{-1}$, which is easily seen to satisfy the conditions (24) for all integers $m \geq 1$. The critical bifurcation value of $\lambda = 1 - \ln \beta$ in this case is determined by

$$\lambda_{cr} = -\frac{1}{\hat{\phi}(-1)} = (-1)^{m+1}\frac{1 + a}{1 - a}.$$

This gives the critical inherent net reproductive value of

$$\beta_{cr} = exp\,(\frac{2}{1-a})\ when\ m = 2, 4, 6, \ldots$$

For $m = 1, 3, 5, \ldots$ no 2-cycle bifurcation occurs in this case.

Since f is globally defined the bifurcating continuum of 2-cycles connects to ∞ (i.e. is unbounded) in $R \times P(2)$. Thus, either the set of parameter values β (the "spectrum") or set of 2-cycles (or more specifically the norms of the 2-cycles) from the bifurcating continuum must be unbounded (or both). It follows directly from the model equation (21) that if the spectrum β values bounded then so is the set of 2-cycle norms. This contradiction implies that in fact the spectrum must be an unbounded interval. Moreover, it cannot contain the point $\beta = 0$ for this would imply there exists a nontrivial 2-cycle solution of (21) when $\beta = 0$, which is clearly impossible. Thus, the spectrum is an interval containing the bifurcation point $\beta_{cr} > 1$ which is unbounded above. In particular, it follows that 2-cycle solutions exist at least for all $\beta > \beta_{cr}$.

We have seen in these examples a common scenario in population models, namely a primary bifurcation of nontrivial equilibria (from the extinction state represented by the trivial equilibrium) at a critical value of the inherent net reproductive value β equal to 1 followed (possibly) by a secondary bifurcation to a nontrivial 2-cycle at a larger critical value of β [1], [2], [3].

6. CONCLUDING REMARKS

In the paper we have dealt with the existence of n-periodic and asymptotically periodic solutions of the discrete renewal equation (1). We did by using bifurcation theory techniques and relating equation (1) to its limit equation (6). The main bifurcation results, Theorems 10 and 11, are for the case of $n = 2$ (2-cycle) and $n = 1$ (equilibria) only. If the linearized limit equation (13) has a nontrivial n-cycle for some integer $n \geq 3$ it remains an open question as to the nature of the resulting bifurcation for the nonlinear renewal equation (1). Most likely there will bifurcate a continuum of aperiodic solutions (lying on an "invariant" loop in a suitable phase space) and asymptotically aperiodic solutions as is known to occur for maps in general [11].

In this paper we did not study any stability properties of the bifurcating cycles. Locally, near the bifurcation point, a natural conjecture is that stability is related to the direction of bifurcation, but this remains an open question.

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