Journal of Difference Equations and Applications,
2002 Vol. 8 (12), pp. 1119–1120

Taylor & Francis
Taylor & Francis Group

A Periodically Forced Beverton–Holt Equation

J.M. CUSHING[a,*] and SHANDELLE M. HENSON[b]

[a]*Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA;* [b]*Department of Mathematics, Andrews University, Berrien Springs, MI 49104, USA*

(Received 23 January 2002; In final form 8 February 2002)

For $r > 1$ and $K > 0$ the difference equation

$$x_{t+1} = \frac{rK}{K + (r-1)\,x_t} x_t, \quad t = 0, 1, 2, \ldots$$

has a unique positive equilibrium K and all solutions with $x_0 > 0$ approach K as $t \longrightarrow \infty$. This equation (known as the Beverton–Holt equation) arises in applications to population dynamics, and in that context K is the “carrying capacity” and r is the “inherent growth rate”. A modification of this equation that arises in the study of populations living in a periodically (seasonally) fluctuating environment replaces the constant carrying capacity K by a periodic sequence K_t of positive carrying capacities.

*Corresponding author. E-mail: cushing@math.arizona.edu
ISSN 1023-6198 print/ISSN 1563-5120 online
DOI: 10.1080/1023619021000053980

Thus, we have a periodically forced Beverton–Holt equation

$$x_{t+1} = \frac{rK_t}{K_t + (r-1)x_t} x_t \tag{1}$$

in which the sequence $K_0, K_1, \ldots$ of positive numbers is periodic with a base period p, i.e. $K_{t+p} = K_t > 0$ for all $t \geq 0$ and a (minimal) integer $p \geq 1$. Keep the inherent growth rate $r > 1$ constant and consider the following assertions.

(a) Equation (1) has a positive p-periodic solution $y_t > 0$, and it is globally attracting for $x_0 > 0$.
(b) If $p > 2$, the strict inequality $av(y_t) < av(K_t)$ holds. Here av denotes the average of a periodic cycle, e.g.

$$av\,(y_t) = \frac{1}{p} \sum_{t=0}^{p-1} y_t.$$

These assertions are of ecological interest because they imply a fluctuating habitat is deleterious to a population in the sense that the average population size, in the long run, is less in a periodically oscillating habitat than it is in a constant habitat with the same average.

As pointed out above, (a) holds when $p = 1$ (i.e. $K_t = K$ is a constant). However, when $p = 1$ assertion (b) is false, since in that case $y_t = K$ and hence $av(y_t) = av(K_t)$. On the other hand, it is known that both (a) and (b) are true for $p = 2$ [1]. We conjecture (a) and (b) are in fact true for all periods $p \geq 2$. However, it remains an open problem to prove (or disprove) these assertions for $p \geq 3$.

References

[1] Cushing, J.M. and Shandelle M., Henson, Global dynamics of some periodically forced, monotone difference equations, *Journal of Difference Equations and Applications* **7** (2001), 859–872.