Funkcialaj Ekvacioj, **15** (1972), 101-117

Some Stability Criteria for Linear Systems of Volterra Integral Equations

By J. M. Bownds and J. M. Cushing

(Rensselaer Polytechnic Institute & IBM Thomas J. Watson Research Center)

Criteria are given for the stability, uniform stability, and asymptotic stability of a linear system of Volterra integral equations whose kernel is either a Pincherle-Goursat type kernel (called a PG kernel), a perturbation of a PG kernel, or a kernel dominated by a PG kernel. The approach taken depends on a representation formula for the fundamental matrix of a linear system with PG kernel which involves the fundamental solution of a certain associated ordinary differential equation. The stability criteria all are related directly to the given kernel of the system, either explicitly so, or implicitly through the associated ordinary differential equation.

1. Introduction.

In a previous paper [1] we have indicated some parallels and distinctions which may be drawn between the classical theory of Liapunov stability for systems of ordinary differential equations and a similar theory for systems of Volterra integral equations. As indicated in that paper, stability theory for such integral equations has been approached in various ways in the literature, and one finds important contributions, for example, in [2-13] and [20-23]. The primary distinction made in [1] was the introduction of a concept of uniform stability in studying perturbed linear systems which, together with a characterization of various stabilities on suitable normed spaces of initial functions for linear systems, leads to a natural generalization of the standard stability results for perturbed systems of ordinary differential equations. Knowing conditions under which stability is preserved under perturbations (see [1]) is, of course, not sufficient for deciding the stability of a given perturbed system; one must know in advance the stability properties of the unperturbed linear system. Consequently, for this reason as well as a matter of interest concerning linear systems in and of themselves, it is seen to be important to establish as many results or techniques as possible which imply the stability of linear systems or provide a means of studying their stability properties. The purpose of this paper is to offer some results along these lines. For other results on linear equations see [7, 8, 10, 12, 13]. Our approach and results seem to be completely independent of those found in the literature; the significant features here are that the stability properties are related directly to the kernel of the integral equation itself and not to the resolvent kernel or to the fundamental matrix (see [1] and below) and that use is made of what seems to be a new representation formula for integral equations having kernels which are of Pincherle-Goursat type. Although we restrict ourselves to continuous Kernels, our results apply to continuous iterates of singular kernels.

We mention that for the basic existence, uniqueness, continuity, and comparison theorems, reference is made to [14].

We consider systems of the form

$$(\mathrm{I}) \qquad u(x)=\varphi(x)+\int_a^x K(x,t)u(t)dt$$

where K is a $k\times k$ matrix which is assumed continuous for $x_0\leqq t\leqq x<+\infty$, $a\geqq x_0$, and φ, u are k-vectors which are at least continuous on $[x_0, +\infty)$. For completeness the definitions of the three types of stability to be considered will be repeated here. The significant features of these definitions are as follows: the inclusion of the possibility for different norms on the initial function φ and a definition of uniform stability which plays a major role in any stability preservation under perturbation [15, 1]. Let $\|u\|_{0,a}=\sup_{x\geqq a}|u(x)|$ and suppose N is a normed space of functions defined on $x\geqq x_0$ with norm $\|\cdot\|$. If to any $\varepsilon>0$ there exists a $\delta=\delta(a,\varepsilon)>0$ such that $\|\varphi\|\leqq\delta$, $\varphi\in N$, implies that the solution of (I) exists for all $x\geqq a$ and $\|u\|_{0,a}\leqq\varepsilon$, then (I) is called *stable on* N. If δ can be found independently of $a\geqq x_0$, then (I) is called *uniformly stable* (U. S.) *on* N. If (I) is stable on N and in addition has the property that for every $a\geqq x_0$ there exists a $\delta=\delta(a)>0$ such that $\|\varphi\|\leqq\delta$, $\varphi\in N$, implies $|u(x)|\to 0$ as $x\to+\infty$, then (I) is called *asymptotically stable* (A. S.) *on* N.

It is to be emphasized that the solution $u(x)$ need not necessarily be a member of the space N.

In [1] stability theorems are proved for (I) on the normed spaces

$$N_0=\{\varphi\in C[x_0,+\infty) : \|\varphi\|=\|\varphi\|_{0,x_0}\}$$
$$N_1=\{\varphi\in C^1[x_0,+\infty) : \|\varphi\|=\|\varphi\|_{0,x_0}+\|\varphi'\|_{0,x_0}\}$$
$$N_2=\left\{\varphi\in C^1[x_0,+\infty) : \|\varphi\|=\|\varphi\|_{0,x_0}+\int_{x_0}^{+\infty}|\varphi'|ds\right\}$$
$$N_3=\{\varphi : \varphi\equiv\text{constant}, \|\varphi\|=|\varphi|\}.$$

All of these spaces arise naturally by our approach. The space N_0 would seem to be the most interesting space since it is the largest and, hence, stability on N_0 implies stability on any of the other spaces. In so far as the preservation of stability under perturbations is concerned, it was shown in [1] that uniform stability on N_3 is very important for (I). Moreover, in [1] it was shown that uniform stability for (I) is equivalent on N_3 and N_2. For these reasons (and the fact that similar results to those below for the other spaces N_2 and N_1 can easily and obviously be derived using our approach) we study stability only on N_0.

It was shown in [1] that if $U(s, x)$ is a $k \times k$ continuous matrix which solves the matrix equation

$$\text{(U)} \qquad U(s, x) = I + \int_s^x K(x, t)\, U(s, t)\, dt,$$

for $a \leqq s \leqq x < +\infty$, then the solution to (I) has the "variation of constants" representation

$$\text{(VC)} \qquad u(x) = U(a, x)\varphi(a) + \int_a^x U(s, x)\varphi'(s)\, ds$$

or, if $\partial U/\partial s$ is continuous,

$$u(x) = \varphi(x) - \int_a^x \frac{\partial U}{\partial s}(s, x)\varphi(s)\, ds.$$

The matrix $U(s, x)$ is called the "fundamental matrix" for (I). The relationship between this equation and the representation formulas found in the literature is described in [1]. This particular representation will be significant below.

The main approach to be taken here is via a consideration of the so-called Pincherle-Goursat (PG) kernels [16]; that is, kernels which have one or more decompositions of the form

$$\text{(PG)} \qquad K(x, t) = \sum_{n=1}^{p} A_n(x) B_n(t).$$

These kernels, in the theory of Fredholm integral equations, are also referred to as degenerate [17] or kernels of finite rank [18]. In particular, for such kernels, it is shown in § 2 that the solution $U(s, x)$ of the fundamental equation (U) has a representation in terms of a fundamental matrix for an associated system of ordinary differential equations. This in itself is of interest and, as indicated below, allows for a closed form solution to (I) to be exhibited whenever the associated ordinary differential equation can be solved explicitly; for example, in the special case of a scalar equation with a multiplicative kernel. From the point of view of establishing stability results, this representation for $U(s, x)$ is important for the obvious reason that we may explicitly utilize known stability criteria for differential equations to establish similar results for (I). This is done in § 2 for PG kernels. More generally, in § 3, theorems are stated for what we will call PG *dominated kernels*; that is, kernels which are bounded in norm by a scalar PG kernel. The fourth part of this paper gives some results assuming the kernel in (I) is actually only a perturbation of a PG kernel; these results are immediate applications of theorems from [1], but are, unfortunately, quite restrictive. An attempt to ease these restrictions will be the topic of future work by the authors.

2. PG kernels.

(A) *A representation formula and preliminary lemmas.* Recall that $K(x,t)$ is a $k\times k$ continuous matrix on $x_0\leqq t\leqq x<+\infty$, and assume that there exist $2p$, $k\times k$ continuous matrices $A_n(x), B_n(t), n=1,2,\cdots,p$, such that (PG) holds.

It will be shown below that for such kernels, the solution to (U) may be expressed in terms of a fundamental matrix for what we will call *the associated system* of differential equations. Depending on the nature of the actual decomposition, the associated system will, in general, be larger in dimension than the original system of integral equations. We denote the associated system by

$$\text{(ADE)} \qquad y'(x)=M(x)y(x), \quad x\geqq x_0,$$

where

$$M(x)=\begin{pmatrix} B_1(x)A_1(x)\cdots\cdots B_1(x)A_p(x) \\ \vdots \qquad\qquad \vdots \\ B_p(x)A_1(x)\cdots\cdots B_p(x)A_p(x)\end{pmatrix}.$$

Notice that $y(x)$ is a kp-vector and $M(x)$ is of dimension $kp\times kp$. Let $Y(x,t)$ denote the fundamental matrix $(kp\times kp)$ of (ADE) for which $Y(t,t)=I$. We may then write

$$\text{(Y)} \qquad Y(x,t)=(\bar{Y}_{nm}(x,t)), \quad n,m=1,2,\cdots,p,$$

where $\bar{Y}_{nm}(x,t)$ is the $nm^{\underline{\text{th}}}$, $k\times k$ block matrix.

Lemma 2.1. *The solution of* (U) *(the "fundamental matrix" for the system* (I)), *is given by*

$$\text{(RU)} \qquad U(s,x)=I+\sum_{n,m=1}^{p}\int_s^x A_n(x)\bar{Y}_{nm}(x,t)B_m(t)dt,$$

where $\bar{Y}_{nm}$ is defined by (Y).

Proof. The proof proceeds by directly verifying that (RU) is the solution to (U) subject to the assumption in (PG). Hence, substituting (RU) into the right-hand side of (U), we obtain

$$(2.1)\quad \begin{cases} I+\displaystyle\int_s^x\sum_{i=1}^{p}A_i(x)B_i(t)\Big[I+\sum_{m,n=1}^{p}\int_{z=s}^{t}A_m(t)\bar{Y}_{mn}(t,z)B_n(z)dz\Big]dt \\ =I+\displaystyle\int_s^x\sum_{i=1}^{p}A_i(x)B_i(t)dt \\ \quad +\displaystyle\sum_{i,m,n=1}^{p}\int_{t=s}^{x}\int_{z=s}^{t}A_i(x)B_i(t)A_m(t)\bar{Y}_{mn}(t,z)B_n(z)dzdt.\end{cases}$$

Changing the order of integration, we find that the double integral becomes

$$\int_{z=s}^{x}\int_{t=z}^{x} A_i(x)B_i(t)A_m(t)\bar{Y}_{mn}(t,z)B_n(z)dtdz. \tag{2.2}$$

Also, since $Y(x,t)=(\bar{Y}_{nm}(x,t))$ is a fundamental matrix of (ADE), it follows that each column is a solution to (ADE). As a consequence, for each $i,k=1,\cdots,p$,

$$\frac{\partial}{\partial x}\bar{Y}_{ik}(x,t)=\sum_{m=1}^{p}B_i(x)A_m(x)\bar{Y}_{mk}(x,t), \tag{2.3}$$

with

$$\bar{Y}_{ik}(t,t)=\delta_{ik}I. \tag{2.4}$$

We may, of course, combine (2.3) with (2.4) and write

$$\bar{Y}_{ik}(x,t)=\delta_{ik}I+\int_{z=t}^{x}B_i(z)\sum_{m=1}^{p}A_m(z)\bar{Y}_{mk}(z,t)dz.$$

Now we observe that

$$\begin{aligned}&\sum_{i,m,n=1}^{p}\int_{t=s}^{x}\int_{z=s}^{t}A_i(x)B_i(t)A_m(t)\bar{Y}_{mn}(t,z)B_n(z)dzdt\\&=\sum_{i,n=1}^{p}\int_{z=s}^{x}A_i(x)\left[\int_{t=z}^{x}B_i(t)\sum_{m=1}^{p}A_m(t)\bar{Y}_{mn}(t,z)dt\right]B_n(z)dz\\&=\sum_{i,n=1}^{p}\int_{z=s}^{x}A_i(x)\left[\bar{Y}_{in}(x,z)-\delta_{in}I\right]B_n(z)dz\\&=\sum_{i,n=1}^{p}\int_{z=s}^{x}A_i(x)\bar{Y}_{in}(x,z)B_n(z)dz-\sum_{i=1}^{p}\int_{z=s}^{x}A_i(x)B_i(z)dz.\end{aligned}$$

Returning to (2.1), we have using (RU),

$$\begin{aligned}&I+\int_{s}^{x}\sum_{i=1}^{p}A_i(x)B_i(t)dt\\&\quad+\sum_{i,n=1}^{p}\int_{z=s}^{x}A_i(x)\bar{Y}_{in}(x,z)B_n(z)dz-\sum_{i=1}^{p}\int_{s}^{x}A_i(x)B_i(z)dz\\&=I+\sum_{i,n=1}^{p}\int_{z=s}^{x}A_i(x)\bar{Y}_{in}(x,z)B_n(z)dz\\&=U(s,x).\end{aligned}$$

This completes the proof of Lemma 2.1.

Remark. In the scalar case ($k=1$) and for a scalar, multiplicative kernel ($p=1$), the scalar representation formula (RU) becomes

$$U(s,x)=1+\int_{s}^{x}A(x)Y(x,t)B(t)dt,$$

where $Y(x, t)$ is the fundamental (scalar) solution to the scalar equation $y'(x) = B(x)A(x)y(x)$ which satisfies $Y(t, t) = 1$. But in this simple case

$$Y(x, t) = \exp\left(\int_t^x B(z)A(z)dz\right).$$

Hence,

$$U(s, x) = 1 + \int_s^x A(x)B(t) \exp\left(\int_t^x B(z)A(z)dz\right)dt.$$

Consequently, the scalar equation (I) has the solution

$$u(x) = \varphi(x) + A(x)\int_a^x \varphi(s)B(s) \exp\left(\int_s^x B(z)A(z)dz\right)ds.$$

We will require all or part of the following lemmas. The proof of Lemma 2.2 may be found in Coppel [15]. Lemmas 2.3 and 2.4 are found in [1] (Theorems 2.1 and 4.3 respectively), and Lemma 2.5 is due to Sato [14]. We always assume $U(s, x)$ is the continuous solution to (U) and $Y(x, t)$ is the fundamental matrix for (ADE) for which $Y(t, t) = I$. Also, if M is a matrix, we use the norm

$$\|M\| = \sup_{|\xi|=1} |M\xi|.$$

Lemma 2.2. *The system* (ADE) *is*:

(i) *stable if and only if there exists a constant* $M(t) > 0$ *such that* $\|Y(x, t)\| \leqq M(t)$ *for* $x_0 \leqq t \leqq x$;

(ii) *uniformly stable* (U.S.) *if and only if* M *in* (i) *is independent of* t;

(iii) *asymptotically stable* (A.S.) *if and only if* $\|Y(x, t)\| \to 0$ *as* $x \to +\infty$ *for each* $t \geqq x_0$;

(iv) *uniformly asymptotically stable* (U.A.S.) *if and only if there exists positive constants* M, α *such that*

$$\|Y(x, t)\| \leqq M \exp[-\alpha(x-t)], \quad x_0 \leqq t \leqq x.$$

Lemma 2.3. *The system* (I) *is*:

(i) *stable on* N_3 *if and only if there exists a constant* $M(a) > 0$ *such that* $\|U(s, x)\| \leqq M(a)$, $a \leqq s \leqq x$;

(ii) U.S. *on* N_3 *if and only if* M *in* (i) *is independent of* a;

(iii) A.S. *on* N_3 *if and only if* $\|U(s, x)\| \to 0$ *as* $x \to +\infty$ *for each* $s \geqq a$.

Lemma 2.4. *Suppose* $U(s, x)$ *possesses a continuous partial in* s *for* $x_0 \leqq s \leqq x$.

(i) *If there exists a constant* $M > 0$ *such that*

$$\int_{x_0}^x \left\|\frac{\partial U}{\partial s}(s, x)\right\| ds \leqq M, \quad x \geqq x_0,$$

then (I) *is* U.S. *on* N_0;

(ii) *If*

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial s}(s, x)\right\|ds \to 0 \quad as\ x \to +\infty,$$

then (I) *is* A.S. *on* $N_0 \cap \{\varphi \in C[x_0, +\infty) : |\varphi(x)| \to 0,\ x \to +\infty\}$.

Lemma 2.5. *If $u(x)$ is the solution to* (I) *and $w(x)$ satisfies the inequality*

$$w(x) \leqq \varphi(x) + \int_a^x K(x, t)w(t)dt,$$

then $w(x) \leqq u(x)$, $a \leqq x$, provided $K(x, t) \geqq 0$, $x \geqq t \geqq a$.

(B) *Some examples.* The stability theorems for (I) in part (C) below will all require that (ADE) be at least uniformly stable. In additon, certain assumptions will be made regarding the nature of the PG kernel itself. A natural question arises as to whether there is actually an inherent connection between the stability of (I) and of (ADE). If such a connection exists, of course, additional assumptions on $K(x, t)$ could immediately be suspected as being unnecessary.

In Example 1, we see that (I) may be unstable on N_0 even though (ADE) is uniformly stable. On the other hand, in Example 2, we will see that (I) may be uniformly stable while (ADE) is unstable. Hence, if we assume we that (ADE) is uniformly stable we must make some additional assumption if wish to prove that (I) is stable. Finally, in Example 3, we see that the particular additional assumption which we make is not so strong as to imply that (ADE) is actually U.S. In fact, we show that the added restriction (ii) in Theorem 2.1 may hold while both (ADE) and (I) are unstable. In each example we consider the scalar case ($k=1$) with a multiplicative kernel ($p=1$).

Example 1. Consider the equation

$$u(x) = \varphi(x) + \int_0^x \sum_{n=1}^{\infty} a_n(x)u(t)dt \tag{2.5}$$

where, for each positive integer n,

$$a_n(x) = \begin{cases} 0, & 0 \leqq x \leqq n - n^{-1}2^{-n} \\ n^2 2^n[x - (n - n^{-1}2^{-n})], & n - n^{-1}2^{-n} \leqq x \leqq n \\ -n^2 2^n[x - (n + n^{-1}2^{-n})], & n \leqq x \leqq n + n^{-1}2^{-n} \\ 0, & x \geqq n + n^{-1}2^{-n}. \end{cases}$$

It is clear that $\sum_{n=1}^{\infty} a_n(x)$ is unbounded on $[0, +\infty)$, but

$$\int_0^{+\infty} \sum_{n=1}^{\infty} a_n(x)dx = \sum_{n=1}^{\infty} 2^{-n} = 1.$$

In this case, the associated (scalar) differential equation is

$$y'(x)=\sum_{n=1}^{\infty}a_n(t)y(t), \tag{2.6}$$

the fundamental solution for which is given by and satisfies

$$\|Y(x,t)\|=\exp\left(\int_t^x\sum_{n=1}^{\infty}a_n(s)ds\right)\leqq\exp\left(\int_0^{+\infty}\sum_{n=1}^{\infty}a_n(s)ds\right)=e,$$

which implies, by Lemma 2.2 (ii), that (2.6) is U.S.

However, the representation (RU) implies (see Remark following Lemma 2.1), that the fundamental matrix for (2.5) satisfies

$$U(s,x)=1+\int_s^x\sum_{n=1}^{\infty}a_n(x)\exp\left(\int_z^x\sum_{n=1}^{\infty}a_n(s)ds\right)dz$$
$$\geqq 1+\sum_{n=1}^{\infty}a_n(x)\int_s^x 1\,dz=1+\sum_{n=1}^{\infty}a_n(x)(x-s)$$

Since $\sum a_n(x)$ is unbounded, Lemma 2.3 implies that (2.5) is unstable on N_3 and hence unstable on N_0.

Example 2. Consider the scalar equation

$$u(x)=\varphi(x)+\int_a^x x^{-2}tu(t)dt,\quad x\geqq a\geqq 1. \tag{2.7}$$

Then the associated (scalar) differential equation is

$$y'(x)=x^{-1}y(x),\quad x\geqq 1, \tag{2.8}$$

which has as its fundamental solution $Y(x,t)=xt^{-1}$. The unboundedness of this solution in x for each $t\geqq 1$ implies that (2.8) is unstable. On the other hand, from the remark in part (A) we find that $U(s,x)=1+x^{-1}(x-s)$ and, hence, $\partial U/\partial s=-x^{-1}$, $1\leqq s\leqq x$. It follows from Lemma 2.4 and the inquality

$$\int_1^x\left\|\frac{\partial U(s,x)}{\partial s}\right\|ds=1-x^{-1}\leqq 1$$

that (2.7) is U.S. on N_0.

Example 3. Consider the scalar equation

$$u(x)=\varphi(x)+\int_1^x x^{-1}u(t)dt,\quad x\geqq 1. \tag{2.9}$$

The associated differential equation is given by (2.8) so that as in Example 2 this integral equation has an unstable (ADE). Also, (2.9) is unstable on N_3 (and hence on N_0), for by the remark in part (A), $U(s,x)=1+\log xs^{-1}$ which is unbounded in $x\geqq s$ for each $s\geqq 1$. We notice here that $A_1(x)=x^{-1}$, $B_1(t)=1$, and so

$$|A_1(x)|\int_s^x |B_1(t)|dt = x^{-1}\int_s^x dt = x^{-1}(x-s) \leqq 1, \tag{2.10}$$

for all $1 \leqq s \leqq x < +\infty$. This observation will have specific meaning below since (2.10) corresponds to hypothesis (ii) in Theorem 2.1.

(C) *Stability theorems requiring the uniform stability of* (ADE). It will be seen that with the use of the representation (RU), together with appropriate additional assumptions, we may relate the uniform stability of (ADE) to the stability of (I).

Theorem 2.1. *Assume that* $K(x, t)$ *is a* PG *kernel,* $\partial U(s, x)/\partial s$ *is continuous, and that*

(i) (ADE) *is* U.S.;

(ii) *there exists a constant* $L>0$ *such that*

$$\sum_{n=1}^{p} ||A_n(x)|| \int_{x_0}^{x} \sum_{m=1}^{p} ||B_m(t)|| dt \leqq L, \quad x_0 \leqq x < +\infty.$$

Then (I) *is* U.S. *on* N_0.

Remark. Examples 1 and 3 indicate the mutual independence and apparent necessity of both (i) and (ii) in the sense that the theorem is false if either assumption is dropped.

Proof. From (RU), we obtain the estimate

$$\left\|\frac{\partial U}{\partial t}(t, x)\right\| \leqq \sum_{n,m=1}^{p} ||A_n(x)||\ ||\bar{Y}_{nm}(x, t)||\ ||B_m(t)||,$$

which yields

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t, x)\right\| dt \leqq \sum_{n=1}^{p} ||A_n(x)|| \int_{x_0}^{x} \sum_{m=1}^{p} ||\bar{Y}_{nm}(x, t)||\ ||B_m(t)|| dt.$$

Now, since (ADE) is U.S., there exists a constant $M>0$ such that $||\bar{Y}_{nm}(x, t)|| \leqq ||Y(x, t)|| \leqq M$, $t \leqq x$. So, using (ii), we have

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t, x)\right\| dt \leqq ML,$$

which, by Lemma 2.4, implies that (I) is U.S. on N_0.

The assumption in (ii) can be weakened at the expense of a strengthing of (i). In particular we have the following result.

Theorem 2.2. *In Theorem* 2.1 *replace* (i) *and* (ii) *respectively by the assumptions* :

(i) (ADE) *is* U.A.S.;

(ii) *there exists a constant* $L>0$ *such that*

$$\sum_{n=1}^{p} ||A_n(x)|| e^{-\beta x} \int_{x_0}^{x} \sum_{m=1}^{p} ||B_m(t)|| e^{\beta t} dt \leqq L, \quad x_0 \leqq x,$$

for some sufficiently small constant $\beta>0$.

Then (I) *is* U.S. *on* N_0.

Proof. Since (ADE) is U.A.S., Lemma 2.2 implies the existence of positive constants M, α such that $\|\bar{Y}_{nm}(x,t)\|\leq\|Y(x,t)\|\leq M\exp(-\alpha(x-t))$, for $x_0\leq t\leq x<+\infty$. Now, differentiation of (RU) leads to

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt\leq M\sum_{n=1}^{p}\|A_n(x)\|\int_{x_0}^{x}\sum_{m=1}^{p}\|B_m(t)\|e^{-\alpha(x-t)}dt$$

$$\leq M\sum_{n=1}^{p}\|A_n(x)\|\int_{x_0}^{x}\sum_{m=1}^{p}\|B_m(t)\|e^{-\beta(x-t)}dt,$$

for $0<\beta<\alpha$. Hence,

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt\leq ML,$$

which, by Lemma 2.4, proves uniform stability on N_0.

For the asymptotic stability of (I) with a PG kernel, assuming at least the uniform stability of (ADE), we have the following theorem.

Theorem 2.3. *Assume* $K(x,t)$ *is a* PG *kernel and* $\partial U(s,x)/\partial s$ *is continuous.*

(i) *If* (ADE) *is* U.S. *and if*

$$\sum_{n=1}^{p}\|A_n(x)\|\int_{x_0}^{x}\sum_{m=1}^{p}\|B_m(t)\|dt\to 0\quad as\ x\to+\infty,$$

then (I) *is* A.S. *on* $N_0\cap\{\varphi\in C[x_0,+\infty):|\varphi(x)|\to 0\ as\ x\to+\infty\}$.

(ii) *If* (ADE) *is* U.A.S. *and if*

$$\sum_{n=1}^{p}\|A_n(x)\|e^{-\beta x}\int_{x_0}^{x}\sum_{m=1}^{p}\|B_m(t)\|e^{\beta t}dt\to 0\quad as\ x\to+\infty$$

and sufficiently small $\beta>0$, *then* (I) *is* A.S. *on* $N_0\cap\{\varphi\in C[x_0,+\infty):|\varphi(x)|\to 0$ *as* $x\to+\infty\}$.

Proof. From (RU), as before, we have the estimate

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt\leq M\sum_{n=1}^{p}\|A_n(x)\|\int_{x_0}^{x}\sum_{m=1}^{p}\|B_m(t)\|dt\to 0$$

as $x\to+\infty$. So, by Lemma 2.4, (I) is A.S. on the space asserted in (i). Part (ii) is proved similarly using the estimates of the proof of the preceding theorem.

At this point, as a matter of observation, we mention that it possible to prove stability theorems on the smaller space N_3 which do not require any differentiability for $U(s,x)$ and which have slightly weaker hypotheses. One would use Lemma 2.3 and methods very similar to those above.

(D) *Stability criteria relating directly to the* PG *kernel.* In his book

Coppel [15] has stated useful criteria for determining the stability of linear systems of ordinary differential equations. Since we have the representation (RU) and the above lemmas, it is natural to attempt to establish similar criteria for (I). We note here that it is possible to directly generalize these criteria for Volterra-Stieltjes integral equations in which $K(x,t)=K(t)$ using a recent inequality of Martin [19]. These results would generalize those for the classical initial value problem but would not necessarily supply stability criteria for (I).

As in Coppel [15] and Martin [19], for a matrix A, define

$$\mu(A)=\lim_{h\to 0+}\frac{\|I+hA\|-1}{h}. \tag{2.11}$$

It is important to note that $\mu(A)$ is a continuous function of t if A is, and $\mu(A)$ could be negative.

The following lemma extends known results for vectors to corresponding results for matrices. The proof is a straightforward generalization of the proof in [15] with vector norms being replaced by matrix norms and so will not be given here.

Lemma 2.6. *Let* $Y(x,t)$ *be the fundamental matrix for* (ADE) *satisfying* $Y(t,t)=I$. *Then*

$$\|Y(x,t)\|\leqq\exp\left(\int_t^x\mu(M(s))ds\right), \tag{2.12}$$

where μ *is defined by* (2.11).

We use this fact to relate directly boundedness criteria on the kernel in (I) to the stability of the solution; we no longer assume anything about the associated differential equation. The theorems below list sufficient conditions for the stability of the integral equation (I) with a PG kernel, but do not exclude the instability of the associated equation (ADE).

Theorem 2.4. *Suppose* U *has a continuous partial in* s *and* $K(x,t)$ *is a* PG *kernel. If there exists a constant* $L>0$ *such that*

$$\sum_{n=1}^{p}\|A_n(x)\|\int_{x_0}^{x}\sum_{m=1}^{p}\|B_m(t)\|\exp\left(\int_t^x\mu(M(z))dz\right)dt\leqq L \tag{2.13}$$

then (I) *is* U.S. *on* N_0.

Proof. From (RU), we have the estimate

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt\leqq\sum_{m,n=1}^{p}\|A_n(x)\|\int_{x_0}^{x}\|\bar{Y}_{nm}(x,t)\|\ \|B_m(t)\|dt;$$

and, using Lemma 2.6 together with the obvious fact that $\|\bar{Y}_{nm}\|\leqq\|Y\|$, we see that

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t, x)\right\|dt \leqq \sum_{m,n=1}^{p}\|A_n(x)\|\int_{x_0}^{x}\|B_m(t)\|\exp\left(\int_t^x \mu(M(z))dz\right)dt.$$

The assumption in (2.13) then completes the proof.

The next theorem, concerning asymptotic stability, is to be compared with Theorem 2.3 with reference to the remarks in part (B).

Theorem 2.5. *With the same assumptions as in Theorem* 2.4, *replace* (2.13) *with the assumption*

$$\sum_{n=1}^{p}\|A_n(x)\|\int_{x_0}^{x}\sum_{m=1}^{p}\|B_m(t)\|\exp\left(\int_t^x \mu(M(z))dz\right)dt \to 0$$

as $x\to+\infty$, *then* (I) *is* A.S. *on* $N_0\cap\{\varphi\in C[x_0,+\infty) : |\varphi(x)|\to 0$ *as* $x\to+\infty\}$.

Proof. This theorem follows immediately from the estimate derived in the proof of the preceding Theorem 2.4. and Lemma 2.4.

We conclude this section by mentioning the fact that our representation formula (RU) together with our characterization of stability for (I) on N_3 (Lemma 2.3) may be used in an obvious way to establish the *instability* of (I) with a PG kernel provided enough is known about the associated equation (ADE). Specifically, we can state the following result.

Theorem 2.6. *If*

$$\liminf_{x\to+\infty}\left\|\sum_{n,m=1}^{p}\int_s^x A_n(x)\bar{Y}_{nm}(x, t)B_m(t)dt\right\| = +\infty$$

for each fixed $s\geqq x_0$, *then* (I) *is unstable on* N_3 *and therefore unstable on* N_0.

Proof. With this assumption, (RU) implies that $\|U(s, x)-I\|$ is unbounded in x for each $s\geqq x_0$. Hence, $\|U(s, x)\|$ is unbounded in x for each $s\geqq x_0$ for which the instability of (I) on N_3 follows by the characterization in Lemma 2.3.

3. PG-dominated kernels.

We now consider the linear integral equation (I) in which $K(x, t)$ is not necessarily a PG kernel. We say that $K(x, t)$ is PG-*dominated* if there exist scalar functions $a_n(x), b_n(t), n=1, 2, \cdots, p$, such that

$$\|K(x, t)\| \leqq \sum_{n=1}^{p} a_n(x)b_n(t), \quad x_0\leqq t\leqq x<+\infty.$$

Clearly, a PG kernel is also PG-dominated with $a_n(x)=\|A_n(x)\|$ and $b_n(t)=\|B_n(t)\|$ for $n=1, 2, \cdots, p$. For this reason, it may appear that the theorems to follow are generalizations of those stated in §2. This, however, is not the case as a formal remark later will point out.

We will use the following lemma.

Lemma 3.1. *Suppose $K(x,t)$ is PG-dominated and $\partial U(s,x)/\partial s$ is continuous. If there exists a constant $L>0$ such that*

$$\int_{x_0}^{x}\|K(x,t)\|dt\leqq L,\quad x\geqq x_0,$$

then

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt\leqq L\left[1+\int_{x_0}^{x}\sum_{n,m=1}^{p}a_n(x)b_n(t)y_{nm}(x,t)dt\right],\tag{3.1}$$

where $y_{nm}(x,t)$ is the scalar $(n,m)^{\text{th}}$ entry of the fundamental matrix $Y(x,t)$, $Y(t,t)=I$, for the associated system

$$y'(x)=M^*(x)y(x)\tag{ADE*}$$

with $M^*(x)=(a_n(x)b_m(x))$.

Proof. From (U), we obtain

$$\left\|\frac{\partial U}{\partial t}(t,x)\right\|\leqq\int_{t}^{x}\|K(x,z)\|\left\|\frac{\partial U}{\partial t}(t,z)\right\|dz+\|K(x,t)\|,$$

so that an integration together with an application of Fubini's theorem yields

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt\leqq\int_{x_0}^{x}\|K(x,t)\|dt+\int_{z=x_0}^{x}\|K(x,z)\|\int_{t=x_0}^{z}\left\|\frac{\partial U}{\partial t}(t,z)\right\|dtdz.$$

Now, using the integrability assumption on $K(x,t)$, as well as the assumption that $K(x,t)$ is PG-dominated, we find

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt\leqq L+\int_{z=x_0}^{x}\sum_{n=1}^{p}a_n(x)b_n(z)\int_{t=x_0}^{z}\left\|\frac{\partial U}{\partial t}(t,z)\right\|dtdz.$$

According to our variation of constants formula (VC), together with the representation (RU), the solution to

$$w(x)=L+\int_{x_0}^{x}\sum_{n=1}^{p}a_n(x)b_n(t)w(t)dt$$

is given by

$$w(x)=L\left[1+\int_{x_0}^{x}\sum_{n,m=1}^{p}a_n(x)b_m(t)y_{nm}(x,t)dt\right].$$

Hence, appealing to Lemma 2.5, we finally obtain the result (3.1).

We are now in a position to prove stability theorems on N_0.

(A) *Stability theorems requiring the uniform stability of* (ADE*).

Theorem 3.1. *Assume that $K(x,t)$ is PG-dominated and that $\partial U(s,x)/\partial s$ is continuous. Further assume that* (ADE*) *is* U.S. *If there exists a constant $L>0$ such that*

$$\sum_{n=1}^{p}|a_n(x)|\int_{x_0}^{x}\sum_{m=1}^{p}|b_m(t)|dt \leqq L, \quad x_0 \leqq x < +\infty,$$

then (I) *is* U.S. *on* N_0.

Proof. We have

$$\int_{x_0}^{x}\|K(x,t)\|dt \leqq \int_{x_0}^{x}\sum_{n=1}^{p}|a_n(x)|\,|b_n(t)|dt$$
$$\leqq \int_{x_0}^{x}\sum_{n,m=1}^{p}|a_n(x)|\,|b_m(t)|dt \leqq L.$$

Consequently, the above Lemma 3.1 implies that

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt \leqq L\left[1+\int_{x_0}^{x}\sum_{n,m=1}^{p}|a_n(x)|\,|b_m(t)|\,|y_{nm}(x,t)|dt\right].$$

Since (ADE*) is U.S., Lemma 2.2 implies the existence of a constant $M>0$ such that

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt \leqq L[1+ML].$$

An appeal to Lemma 2.4 then completes the proof.

Theorem 3.2. *Suppose* $K(x,t)$ *is* PG-*dominated,* $U(s,x)$ *has the usual differentiability, and* (ADE*) *is* U.A.S. *If there exist constants* $K>0$, $L>0$ *and a sufficiently small constant* $\beta>0$ *such that*

$$\int_{x_0}^{x}\sum_{n=1}^{p}a_n(x)b_n(t)dt \leqq L, \tag{3.2}$$

and

$$\sum_{n=1}^{p}|a_n(x)|e^{-\beta x}\int_{x_0}^{x}\sum_{m=1}^{p}|b_m(t)|e^{\beta t}dt \leqq K, \tag{3.3}$$

for $x_0 \leqq x < +\infty$, *then* (I) *is* U.S. *on* N_0.

Proof. The assumption (3.2) assures the integrability requirement in Lemma 3.1. Hence, we have (3.1). But, as before, from the U.A.S. of (ADE*) we have

$$|y_{nm}(x,t)| \leqq \|Y(x,t)\| \leqq Me^{-\alpha(x-t)},$$

for some $M, \alpha>0$, where $Y(x,t)$ is the usual fundamental matrix for (ADE*). Therefore, for $0<\beta<\alpha$, using (3.3), we find

$$\int_{x_0}^{x}\left\|\frac{\partial U}{\partial t}(t,x)\right\|dt \leqq L\left[1+\sum_{n=1}^{p}|a_n(x)|\int_{x_0}^{x}\sum_{m=1}^{p}|b_m(t)|Me^{-\beta(x-t)}dt\right]$$
$$\leqq L[1+MK].$$

The conclusions of the theorem then follow from Lemma 2.4.

(B) *Stability criteria relating directly to the kernel.* Here we prove a theorem which is similar to Theorem 2.4, but for more general kernels. The expense is an added restriction.

Theorem 3.3. *Assume $K(x,t)$ is* PG-*dominated, $\partial U(s,x)/\partial s$ is continuous, and that* (3.2) *holds. If, in addition, there exists a constant $M>0$ such that*

$$\sum_{n,m=1}^{p} |a_n(x)| \int_{x_0}^{x} |b_m(t)| \exp\left(\int_t^x \mu(M^*(z))dz\right)dt \leqq M, \tag{3.4}$$

then (I) *is* U.S. *on N_0.*

Proof. The assumption (3.2) implies, by Lemma 3.1, that (3.1) holds. Then (3.4) leads to

$$\int_{x_0}^{x} \left\| \frac{\partial U}{\partial t}(t,x) \right\| dt \leqq L[1+M],$$

which, as usual, produces the result.

Remarks. (1) We observe here that if the kernel in (I) is actually of PG type, then we have a choice of stability theorems: we may try to use theorems directly from §2 or we may take $a_n = \|A_n\|$, $b_n = \|B_n\|$, and try to use the theorems in this section. That there is, in fact, a difference in the two approaches can be seen by comparing, for example, Theorem 2.1 with Theorem 3.1. The system (ADE) in Theorem 2.1 is of dimension $kp\times kp$ and its coefficient matrix could contain negative entries which may, of course, have a profound effect on determining the uniform stability of that system. On the other hand, if we chose a_n, b_n as above, then we consider the smaller $p\times p$ system (ADE*). However, the coefficient matrix of this system has only non-negative entries which could, of course, make for much different asymptotic behavior than might be observed for (ADE).

(2) We have not stated any theorems on the A.S. of (I) with a PG-dominated kernel. However, it is clear how theorems similar to Theorems 2.3 and 2.5 could be derived; these would require, in place of the conditions (3.3) and (3.4), that the expressions in (3.3) and (3.4) tend to zero as $x \to +\infty$.

4. Perturbations.

In the theory of Fredholm integral equations, one major result is that any L_2 kernel may be approximated in the L_2 norm by a PG kernel [16]. The proof of this theorem depends heavily on the boundedness of the interval of integration, a luxury which is not present in discussing the asymptotic behavior of a Volterra equation. Of course it still may be possible to approximate uniformly $K(x,t)$ by a suitable PG kernel. One obvious possibility is that $K(x,t)$ be expressible as a series of multiplicative, integrable functions of x and t on

$x_0 \leqq t \leqq x < +\infty$. If this is the case, then use may be made of the previous theorems together with the main perturbation result found in [1].

For the purposes of this exposition, we will state our perturbation theorem from [1] in terms of a kernel which may be approximated by a PG kernel; the exact nature of this approximation is fully described in hypotheses H1 and H2 below.

Theorem 4.1. *Assume that $K(x, t)$ has the representation*

$$K(x,t)=\sum_{n=1}^{p} A_n(x)B_n(t)+R(x,t), \quad x_0 \leqq t \leqq x < +\infty,$$

where $A_n(x)$, $B_n(t)$ are as in § 2, *and $R(x, t)$ satisfies*

H1 : $$\int_{x_0}^{+\infty} \|R(x,x)\| dx < +\infty;$$

H2 : $$\int_{x_0}^{+\infty} \int_{x_0}^{s} \left\| \frac{\partial R}{\partial s}(s,t) \right\| dt ds < +\infty.$$

Further assume that the unperturbed equation

$$u(x)=\varphi(x)+\int_a^x \sum_{n=1}^{p} A_n(x)B_n(t)u(t)dt, \quad x_0 \leqq a \leqq x < +\infty, \tag{4.1}$$

is U.S. *on N_3. Then* (I) *is stable or uniformly stable on a normed space N of initial functions if the unperturbed equation is respectively stable or uniformly stable on that space N.*

Also from [1], we have the following result for A.S.

Theorem 4.2. *Assume that $K(x, t)$, $R(x, t)$ and the unperturbed equation* (4.1) *are as in Theorem* 4.1. *Then* (I) *is* A.S. *on any space $N \supseteqq N_3$ on which the unperturbed equation* (4.1) *is* A.S.

It is clear that theorems similar to Theorems 4.1 and 4.2 could be stated for perturbations of PG-dominated kernels also.

References

[1] J. M. Bownds and J. M. Cushing, Some stability theorems for system of Volterra integral equations, to appear in Journal of Applicable Analysis.

[2] J. J. Levin and J. A. Nohel, Perturbations of a nonlinear Volterra equation, Mich. Math. J., **12** (1965), 431–447.

[3] J. J. Levin, The qualitative behavior of a nonlinear Volterra equation, Proc. AMS, **16** (1965), 711–718.

[4] R. K. Miller, Asymptotic behavior of solutions of nonlinear Volterra equations, Bull. AMS, **72** (1966), 153–156.

[5] C. Corduneanu, Some perturbation problems in the theory of integral equations, Math. Systems Theory, **1** (1967), 143–155.

[6] R. K. Miller, On the linearization of Volterra integral equations, J. Math. Anal.

Appl., **32**, No. 1, (1968), 198–208.

[7] J. A. Nohel, Problems in qualitative behavior of solutions of nonlinear Volterra equations, appearing in "Nonlinear Integral Equations" (P. M. Anselone, ed.), Univ. Wisconsin Press, 1964.

[8] R. K. Miller, On Volterra integral equations with nonnegative integrable resolvents, J. Math. Anal. Appl., **22** (1968), 319–340.

[9] R. K. Miller, J. A. Nohel, and J. S. W. Wong, A stability theorem for nonlinear mixed integral equations, J. Math. Anal. Appl., **25**, No. 2, (1969), 446–449.

[10] R. K. Miller, J. A. Nohel, and J. S. W. Wong, Perturbations of Volterra integral equations, J. Math. Anal. Appl., **25**, No. 3, (1969), 676–691.

[11] Stanley I. Grossman, Existence and stability of a class of nonlinear Volterra integral equations, Trans. AMS, **150** (1970), 541–556.

[12] Aaron Strauss, On a perturbed Volterra integral equation, J. Math. Anal. Appl., **30**, No. 3, (1970), 564–575.

[13] Kenneth Hannsgen and Carol Shilepsky, A boundedness theorem for Volterra equations, J. Diff. Eqns., **10** (1971), 378–387.

[14] T. Sato, Sur l'equation integrale non linearire de Volterra, Compositio Math., **11** (1953), 271–290.

[15] W. A. Coppel, "Stability and Asymptotic Behavior of Differential Equations", D. C. Heath and Co., Boston, 1965.

[16] F. G. Tricomi, "Integral Equations", Interscience Publishers, Inc., N. Y., 1957.

[17] Courant-Hilbert, "Methods of Mathematical Physics", Vol. 1, Interscience, N. Y., 1953.

[18] F. Riesz and B. Sz. -Nagy, "Functional Analysis", F. Ungar Publ. Co., N. Y., 1955.

[19] R. H. Martin, A bound for solutions of Volterra-Stieltjes integral equations, Proc. AMS, **23** (1969), 506–512.

[20] R. K. Miller, Admissibility and nonlinear Volterra integral equations, Proc. AMS, **25** (1970), 45–71.

[21] S. I. Grossman and R. K. Miller, Perturbation theory for Volterra integrodifferential systems, J. Diff. Eq., **8** (1970), 457–465.

[22] J. A. Nohel, Asymptotic relationships between systems of Volterra equations, Annali di Mat. Pura ed Applic., Serie IV, **90** (1971), 149–165.

[23] J. A. Nohel, Perturbations of Volterra equations and admissibility, Japan-U. S. Seminar on ordinary differential and functional equations, Lecture Notes in Mathematics 243, Springer-Verlag, Berlin-Heidelberg-New York, 1971, 40–53.

(Ricevita la 26-an de januaro, 1972)
(Reviziita la 22-an de majo, 1972)