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Some Existence Results for Nonlinear Steklov Problems

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We deal here briefly with the following nonlinear problem (N): show there exist (real) values of μ for which $Au = 0$ on D, $u_n = \mu f(x, u, Du, \mu)$ on ∂D has a nontrivial (i.e., $\not\equiv 0$) solution. Here $Au = \sum_{i,j=1}^{m} D_i(a_{ij}(x) D_j u)$, $D_i = \partial / \partial x_i$, $x = (x_1, \ldots, x_m)$; Du is an arbitrary first partial of u; and u_n is the outward conormal derivative of u on the boundary ∂D of a bounded, smooth (say $C^{2+\alpha}$) domain in Euclidean m-space. The coefficients $a_{ij}(x) = a_{ji}(x)$ are in $C^{1+\alpha}(\bar{D})$, $\bar{D} = D + \partial D$; and A is elliptic: $\sum a_{ij}(x) \xi_i \xi_j \geq c \sum \xi_i^2$, c = const.> $x \in D$, $\xi = (\xi_i) \in R^m$. We consider classical solutions $u(x) \in C^{2+\alpha}(\bar{D})$ and assu

H1: $f(x, y, z, \mu) \equiv a(x) y + g(x, y, z, \mu)$, $a(x) \in C^{1+\alpha}(\partial D)$, $\int_{\partial D} a \, dx \neq 0$, $g(x, y, z, \mu) \in C^1(\partial D \times R \times R \times R)$, where $g = o((|y|^2 + |z|^2)^{1/2})$, uniformly in x for bounded μ intervals , near $y = z = 0$.

Thus, $u \equiv 0$ is a solution for all $\mu \in R$.

Nonlinear eigenvalue problems have been studied in many contexts and the general, local bifurcation theory for completely continuous operators can be found in the well known book of Krasnosel'skii [1]. More recently, the importan results in [2, 3] have extended these local results globally as well as studied in detail many aspects of the existence question. Our task here will only be to formulate problem (N) so that these general results apply and to discuss some applications. We expect, of course, bifurcation to occur at $(0, \mu)$ for $\mu \in r(L)$, the spectrum of the linearized problem (L): $Au = 0$, $u_n = \mu a(x)u$. Note that $0 \in r(L)$ and that (N) has the solution branch (u, o), $u \equiv$ const. This is really the source of our difficulties in formulating (N) as an operator equation.

Let B be a sub-Banach space of $C^{1+\alpha}(\bar{D})$ under the usual norm $\|u\|_{1+\alpha}$. Any solution $(u,\mu) \in B \times R$ necessarily satisfies the condition $\int_{\partial D} f(x, u, Du, \mu)\, dx = 0$. To reformulate (N) as an operator equation we do the following: given $(u,\mu) \in B \times R$, denote by $k = k(u,\mu)$ a constant satisfying the equation

$$\text{(H)} \qquad \int_{\partial D} f(x, u+k, Du, \mu)\, dx = 0$$

and by $v = F(u,\mu) \in C^{2+\alpha}(\bar{D})$ the unique solution to the Neumann problem $Lv = 0$, $v_n = \mu f(x, u+k, Du, \mu)$ satisfying $\int_{\partial D} v\, dx = 0$ (see [4, 5]). Thus, $F(u,\mu)$ maps $B \times R \to C^{1+\alpha}(\bar{D})$ provided $k(u,\mu)$ defined by (H) is well-defined. For the moment, we assume

H2: $k(u,\mu): S(\varepsilon) \times R \to R$ with $k(0,\mu) = 0$ is well-defined and continuous on some ball $S(\varepsilon)$ at the origin of radius $0 < \varepsilon \leq +\infty$ in $C^{1+\alpha}(\bar{D})$;

H3: F maps $B \times R \to B$.

If $(u,\mu) \in B \times R$ satisfies (O): $u = \mu F(u,\mu)$ then it is clear that $u^* = u + k$ satisfies (N). Conversely, it can easily be shown that if u^* solves (N) for $\mu \neq 0$ then $u = u^* - k$, $k = s^{-1} \int_{\partial D} u^*\, dx$, $s = \int_{\partial D} dx$ solves the operator equation (O). Thus, (N) and (O) are equivalent for $\mu \neq 0$. Known a priori estimates (see [4, 5]) imply that F is completely continuous and that any solution to (O) in fact lies in $C^{2+\alpha}(\bar{D})$. Moreover, if $w = Lu \in C^{2+\alpha}(\bar{D})$ denotes the unique solution of $Aw = 0$, $w_n = a(x)(u+d)$ satisfying $\int_{\partial D} w\,dx = 0$ where d is the constant such that $\int_{\partial D} a(u+d)\,dx = 0$, then it can be shown that $F(u,\mu) = \mu Lu + G(u,\mu)$, $G(u,\mu) = o(\|u\|_{1+\alpha})$ near 0 on bounded μ intervals and that problem (L) is equivalent to $u = \mu Lu$ for $\mu \neq 0$. The general results of [1-3] are now seen to apply to (O) and we can conclude (see [2]) that <u>if H1 - H3 hold then bifurcating from any eigenvalue $(0,\mu^*)$, $\mu^* \in r(L) - \{0\}$ of odd multiplicity is a continuum of nontrivial solutions $(u,\mu) \in B \times R$ of (N) connecting $(0,\mu^*)$ to either $\partial(B(\varepsilon) \times R)$ or to $(0,\mu^{**})$ for some $\mu^{**} \in r(0) - \{0\}$, $\mu^{**} \neq \mu^*$.</u>

Concerning H2 and H3, we note: (i) H3 is satisfied for $B = C^{1+\alpha}(\bar{D})$, and an application of the implicit function theorem [6] shows that H2 is always satisfied for some $\varepsilon > 0$ sufficiently small; (ii) it can be shown that $\varepsilon = +\infty$ in H2 with $B = C^{1+\alpha}$ in H3 if the condition $f_y(x, y, z, \mu) > 0$ for all $(x, y, z, \mu) \in \partial D \times R \times R \times R$ holds; (iii) if certain symmetry assumptions are made on D, a_{ij}, and f, then H2 is satisfied with $\varepsilon = +\infty$ (by taking $k = 0$ in (H) for all u) if B is taken to be the subspace of odd functions in $C^{1+\alpha}$ with respect to the symmetry on D (see [5]); (iv) the example $f \equiv u - u^2$ easily shows that (H) may not, in general, have a solution k for all $u \in C^{1+\alpha}$.

An application deals with Levi-Civita's theory of permanent water waves on deep water [7] where the existence of such waves is reduced to the equivalent problem $L = \Delta$ (Laplace's operator), $m = 2$, D = unit disk centered at the origin and $f \equiv \exp(-3Tu)\sin u$ where Tu is the harmonic conjugate of u vanishing at the origin. Nontrivial solutions in $C^{2+\alpha}$ are desired which vanish on the horizontal axis. (Because of the presence of Tu, this problem is not strictly included in the remarks above. Its presence imposses no difficulty, however, when the above techniques are applied. Tu is continuous under $\|u\|_{1+\alpha}$.) Here $\mu = g\lambda/2\pi c^2$ where g is the acceleration due to gravity, λ is the wavelength and c the speed of the water wave, and u is the angle that the flow vector makes with the horizontal. Choosing $B = B_i = \{u \in C^{1+\alpha}$: u vanishes for all polar angles $\theta = \pm j\pi/i$, $j = 0, 1, \ldots, i$ for all $0 \leq r \leq 1\}$, i = positive integer, it can be shown that H3 and H2 are satisfied (with $\varepsilon = +\infty$). Thus, from each $(0, i)$ (the eigenvalues of the linearized problem) there branches an unbounded continuum of solution $(u, \mu) \in B_i \times \mathbf{R}$; and consequently, the local branches shown to exist by Levi-Civita exist globally. Moreover, it can be shown that the only solutions branching from $(0, i)$, $i \geq 2$, lying in B_1 actually lie in $B_i \subset B_1$. This latter statement justifies a conjecture of Levi-Civita to the effect that only solutions branching from $(0, 1)$ are of interest physically. Recent numerical work (by the author at IBM T. J. Watson Research Center) on this problem indicates that although $\|u\|_{1+\alpha}$ is unbounded along the solution branch from $(0, 1)$ the norm $\max_{\partial D} |u|$ is bounded (as are the values of μ); in fact, as $\|u\|_{1+\alpha} \to +\infty$, $\max |u|$ seems to tend to a value near .5 radians (the numerical work is not complete). This, together with the numerical calculation of the boundary values of the solution (which indicates the shape of the wave), suggests that Levi-Civita's theory predicts the empirically observed "peaking" of such waves at an angle of $\pi/6$ [8].

Waves on water of finite depth can also be treated. Here D is the annulus $R^{-1} \leq r \leq R$, $R > 1$.

REFERENCES

1. M. A. Krasnosel'skii, Topological Methods in the Theory of Nonlinear Integral Equations, MacMillan Co., N. Y., 1964.

2. Paul H. Rabinovitz, Some global results for nonlinear eigenvalue problems, J. Func. Anal. 7 (1971), 487-513.

3. Michael G. Crandall and Paul H. Rabinowitz, Bifurcation from simple eigenvalues, J. Func. Anal 8 (1971), 321-340.

4. O. A. Ladyzhenskaya and N. Uraltseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York/London, 1968.

5. J. M. Cushing, Some existence theorems for nonlinear eigenvalue problems associated with elliptic equations, Arch. Rat. Mech. Anal. 42 (1971), 63-76.

6. T. H. Hildebrandt and L. M. Graves, Implicit functions and their differentials in general analysis, Trans. AMS 29 (1927), 127-153.

7. T. Levi-Civita, Determination rigoureuse des on des permanentes d'ampleur fini, Math. Ann. 93 (1925), 264-314.

8. J. J. Stoker, Water Waves, Interscience Publishers Inc., New York, 1957.