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Strongly Admissible Operators and Banach Space Solutions of Nonlinear Equations

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1. Introduction.

The purpose of this paper is to extend the results of an earlier paper [5] which dealt with the existence of solutions of the operator equation

$$\text{(E)} \qquad Lx=g+p(x)$$

in a Banach space. Under a certain admissibility assumption on the linear operator L and a higher order assumption (near $x=0$) on the perturbation operator p the main result in [5] describes the solution set of (E) (as a function of g) as it is related to that of the linear equation

$$\text{(L)} \qquad Ly=g$$

by means of a homeomorphism near $x=0$. Our goal here is to weaken the assumptions made on p in [5], which we will do at the expense of losing some of this structure on the solution set as well as of having to require some additional hypotheses on L.

The motivation for our approach and results can be found in the study of stable manifolds for perturbed differential [1, 8, 9] and integrodifferential [6, 7] systems. The concept of admissibility, which was originally used in the theory of differential equations to study the solutions of such systems in various function spaces, was soon also used in the theory of integral equations [2, 3, 4, 12] and integrodifferential equations [6, 7, 11]. It was in turn generalized to an abstract setting concerned with linear operators and abstract Banach spaces and many of the results for integral equations are in fact special cases of more general theorems for operator equations on Banach spaces [3, 10, 12]. These latter abstract results and the results for integral equations out of which they developed do not, however, include some of the fundamental results for differential systems which deal with manifolds of solutions in certain function spaces, even though differential systems can be converted by integration to an equivalent integral equation. The reason for this is that the concept of admissibility as defined for integral equations [2] and abstracted to linear operators on Banach spaces [3] does not lend itself to a study of the full structure of the solution space within a given Banach space.

To be a little more specific, in [3] Corduneanu defined L to be (B_1, B_2)-admissible for two Banach spaces B_1, B_2 if $B_2 \subseteq L(B_1)$. If one is interested in solving (E) for x in a given Banach space B_1 for g in a given Banach space B_2, if L is (B_1, B_2)-admissible in this sense and if $p(B_1) \subseteq B_2$ then this definition of admissibility is useful and one can procede as is done in the above references (see particularly [3, 10]). We are interested here, however, (as we were in [5]) in the case of solving (E) in B_1 when L is not (B_1, B_2)-admissible in this sense. While it is true that one could study (E) on B_1 as in the above cited references for g in the Banach space $B_3 = L(B_1) \cap B_2$ (with the norm $|L^{-1}g|_1 + |g|_2$, $g \in B_3$), in which case L is trivially (B_1, B_3)-admissible, this approach would demand that $p(B_1) \subseteq B_3$, a restriction we do not wish to make on p since it again would spoil our hopes of generalizing certain manifold theorems from the theory of differential and integrodifferential equations.

Our approach in [5] was to define a different concept of admissibility which ultimately led not only to the desired generalizations but to generalizations of the abstract results mentioned above. In [5] the main tool was the contraction principle. Here we wish to use instead the Tychonoff-Schauder fiixed point theorem, a change which necessitates a stronger admissibility concept from that defined and used in [5].

2. Results.

Let B_1, B_2 be normed linear subspaces of a Fréchet space F with norms $|\cdot|_1$, $|\cdot|_2$ respectively both of which yield topologies stronger than or equivalent to that induced from the metric on F. Throughout this paper we will use the notation "(Ω_1, Ω_2)-continuous (or closed or compact)" to mean that the operator concerned is continuous (or closed or compact) with respect to the topologies induced on the domain and range by topological spaces Ω_1 and Ω_2 respectively. The following two hypotheses on L will be needed:

H1: L is a closed, one-one linear operator from F onto F;

H2: $\begin{cases} \text{There exist complementary subspaces } B_2^1 \text{ and } B_2^2 \text{ of } B_2 \text{ (i.e., } B_2 = B_2^1 \oplus B_2^2) \\ \text{such that for every } h \in B_2^2 \text{ there exists a } g \in B_2^1 \text{ for which } g + h \in L(B_1). \end{cases}$

Here $L(B_1)$ denotes the range of L restricted to B_1. Note that by the closed graph theorem both L and L^{-1} are (F, F)-continuous.

If C is any subspace of B_2^1 complementary to the subspace $B_2^1 \cap L(B_1)$, then it was shown in [5] that under H1 and H2 there corresponds to each $h \in B_2^2$ a unique $g = Ah \in C$ for which $h + g \in L(B_1)$ and that the operator $A: B_2^2 \to C$ so defined is linear. As in [5] we say that L is *(B_1, B_2)-admissible (with respect to the decomposition $B_2=B_2^1\oplus B_2^2$)* if H1 and H2 are satisfied and if the linear operator A is (B_2, B_2)-continuous. For our work here we will need a slightly stronger property: L is called *strongly* (B_1, B_2)-admissible (with respect to the decomposition $B_2=B_2^1\oplus B_2^2$) if it (B_1, B_2)-admissible and if in addition the restriction of A to some B_2^2 neighborhood N of 0 is (F, F)-continuous on N (i.e., if g_n and $g_0 \in N$ are such that $g_n \to g_0$ in F then $Ag_n \to Ag_0$ in F).

The motivation for these definitions are found in applications to differential equations where, roughly speaking, L is the Volterra integral operator obtained from the equivalent integral equation, H1 is the admissibility concept defined and studied by Massera and Schäffer [9] (with $B_2^1=R^n$) and the smoothness assumptions on the operator A follow from the continuity of solutions with respect to initial conditions and forcing terms. These points will be made more explicit in § 3.

Let $S(r)=\{x \in B_1 : |x|_1 \leq r\}$. Concerning the perturbation operator p in (E) we make the assumption:

H3: $p(x)$ is defined on $S(r)$ for some $0<r\leq +\infty$, has range in B_2^2, is (F, F)-continuous and satisfies

$$|p(x)-p(0)|_2 \leq \theta\, |x|_1^q$$

for some reals $0<q\leq 1$ and θ and for every $x \in S(r)$.

Our goal here is to describe the set $\Sigma(p\,; r)=\{x \in S(r) : Lx-p(x) \in B_2^1\}$ as it is related to the corresponding set $\Sigma(r^*)=\{y \in S(r^*) : Ly \in B_2^1\}$ for r^* small.

The following theorem contains our main result.

Theorem 1. *Assume that $p(x)$ satisfies* H3 *and that L is strongly (B_1, B_2)-admissible with respect to the decomposition $B_2=B_2^1\oplus B_2^2$ where B_2^2 is complete. Further, assume that L^{-1} restricted to B_2^2 is (B_2, F)-compact. Then there exist positive constants p_0, r^* and θ_0 such that if $\theta<\theta_0$ and $|p(0)|_2<p_0$ then corresponding to each $y \in \Sigma(r^*)$ there is at least one $x=x(y) \in \Sigma(p\,; r)$ such that $y, y' \in \Sigma(r^*)$, $y\neq y'$ implies $x(y)\neq x(y')$.*

Remark 1. Under assumptions on $p(x)$ stronger than those in H3 it was shown in [5] that the mapping $y\to x$ thus defined is a one-one, bicontinuous map. The compactness of L^{-1} however is not needed in [5]. Under the weaker assumption H3 we cannot guarantee this much structure.

Proof. Consider the linear operator $L^* : B_2^2 \to B_1$ defined by $L^*=L^{-1}(A+I)$ where I is the identity operator on F. First we claim that L^* is (B_2, B_1)-continuous. To see this suppose $g_n \to g_0$ in B_2^2 and $L^* g_n \to g^*$ in B_1 (and hence in F). Then since

B_2^2 is closed we have that $g_0 \in B_2^2$ and hence that Ag_0 is defined; moreover, since A is (B_2, B_2)-continuous we have that $(A+I)g_n \to (A+I)g_0$ in B_2^2 and hence in F. We conclude from the (F, F)-continuity of L^{-1} that $L^* g_n \to L^* g_0$ in F which implies that $L^* g_0 = g^*$; i.e., L^* is (B_2, B_1)-closed. The closed graph theorem implies that L^* is (B_2, B_1)-continuous.

Let $|L^*| > 0$ denote the norm of L^* and set $\theta_0 = |L^*|^{-1} r^{1-q} > 0$. Assume $\theta < \theta_0$ in H3. We take r in H3 smaller (if necessary) so that A is (F, F)-continuous on $p(S(r))$; this is possible by the *strong* (B_1, B_2)-admissibility of L and by H3.

Given any $y \in B_1$ define the operator $T_y: S(r) \to B_1$ by $T_y(\cdot) := y + L^* p(\cdot)$. We first argue that for $|y|_1$ sufficiently small T_y has a fixed point $x \in S(r)$.

The estimates

$$\begin{aligned} |T_y x|_1 &\leq |y|_1 + |L^*|\,|p(x)|_2 \leq r^* + |L^*|\,(|p(x) - p(0)|_2 + |p(0)|_2) \\ &\leq r^* + |L^*|\,(\theta r^q + p_0) \end{aligned}$$

are valid for $x \in S(r)$ and $y \in \Sigma(r^*)$. Using this estimate one easily finds that $|T_y x|_1 \leq r$ for $x \in S(r)$ provided that r^* and p_0 are so small that

$$r^* + |L^*|\, p_0 < r(1 - |L^*|\, \theta r^{q-1}),$$

which is possible since $\theta < \theta_0$. This means that each of the operators T_y, $y \in \Sigma(r^*)$, maps $S(r)$ into itself. The ball $S(r)$ is, of course, a convex subset of the Fréchet space F.

The operator $p(x)$ is (F, F)-continuous as an operator defined on $S(r)$. Thus, using the strong admissibility of L we find that T_y is an (F, F)-continuous operator from $S(r)$ into $S(r)$.

Finally the (B_2, F)-compactness of L^{-1} implies that the F-closure of the image under T_y of $S(r)$ is F-compact. Thus, the Schauder-Tychonoff fixed point theorem [13, p. 32] implies that T_y has a fixed point $x \in S(r)$ for each $y \in \Sigma(r^*)$.

Now $T_y x = x$ implies that $y + L^* p(x) = x$ or $Ly + (A+I)p(x) = Lx$ or finally $Lx = g + p(x)$ where $g = Ly + Ap(x) \in B_2^1$. Thus $x = x(y) \in \Sigma(p; r)$.

Finally, suppose that $y, y' \in \Sigma(r^*)$. Then $x(y) = x(y') = x$ implies $T_y x = T_{y'} x$ which in turn implies $y = y'$ by the definition of T_y and $T_{y'}$. ■

Remark 2. If $p(x)$ is higher order in x near $x = 0$ (i.e., if $|p(x) - p(0)|_2 / |x|_1 \to 0$ as $|x|_1 \to 0$) then the hypotheses on p in Theorem 1 are fulfilled with $q = 1$ provided that $r > 0$ is taken small enough. To see this simply take $r > 0$ so small that $|p(x) - p(0)|_2 / |x|_1 \leq \theta < \theta_0$ all $x \in S(r)$; this is possible since $\theta_0 = |L^*|^{-1}$ in the proof of the theorem is independent of p when $q = 1$. Thus, $q \leq 1$ is no loss in generality in H3.

The following two theorems can be of use in applications (e.g., see § 3) with regard to verifying the continuity requirements on A demanded by the admissibility assumption on L.

Theorem 2. *Assume* H1 *and* H2 *hold. The linear operator A is (B_2, B_2)-continuous if both B_1 and B_2 are Banach spaces and both B_2^2 and C are closed subspaces of B_2.*

Proof. This theorem is simply a restatement of Theorem 1 in [5].

Theorem 3. *Assume that* H1 *and* H2 *holds and that A is (B_2, B_2)-continuous. Assume further that C is finite dimensional and that every B_1-ball $S(r)$ is an F-closed subset of F. Let N be a B_2^2 neighborhood of 0. If g_n and $g_0 \in N$ are such that $g_n \to g_0$ in F, then $Ag_n \to Ag_0$ in F.*

Proof. By the definition of A the sequence $x_n = Ag_n$ lies in the finite dimensional subspace C of the normed space B_2^1. Since A is (B_2, B_2)-continuous and $g_n \in N$ it follows that x_n is a B_2-bounded sequence in C which implies it has at least one B_2-accumulation point in C. Let $x_0 \in C$ be an arbitrary but fixed accumulation point of x_n and select a subsequence$x_{n(i)}$ such that $x_{n(i)} \to x_0$ in B_2. Then $x_{n(i)} = Ag_{n(i)} \to x_0$ in F and hence, by H1, $y_{n(i)} = L^* g_{n(i)} = L^{-1}(g_{n(i)} + Ag_{n(i)}) \to L^{-1}(g_0 + x_0) = y_0$ in F. By the way A was defined we know that $y_{n(i)} \in B_1$; moreover, L^* is (B_2, B_1)-continuous (see the proof of Theorem 1) and consequently $y_{n(i)} \in S(r)$ for some $r > 0$. Since $S(r)$ is assumed to be F-closed we conclude that $y_0 \in B_1$. Thus, $Ly_0 = g_0 + x_0$ and $y_0 \in B_1$ which means that $Ag_0 = x_0$. This implies that the sequence x_n has one and only one B_2-accumulation point (namely Ag_0) and as a result we conclude that $Ag_n \to Ag_0$ in B_2 (and consequently in F). ■

Corollary. *If* H1 *and* H2 *hold then L is strongly (B_1, B_2)-admissible with respect to the decomposition $B_2 = B_2^1 \oplus B_2^2$ if all of the following conditions are met: the spaces B_1, B_2 and B_2^2 are complete, C is finite dimensional and every B_1-ball $S(r)$ is F-closed.*

3. An application to Volterra integral equations.

Our purpose in this last section is to illustrate the use of the results in § 2 in studying the existence of manifolds of bounded solutions of Volterra integral equations. We make no attempt to investigate this question in depth or even to obtain the most general results possible using Theorem 1. We wish only to show how an application can be made in a specific case and for specifically chosen Banach spaces; other choices can obviously be made to result in further applications.

Let F be the Fréchet space of functions continuous for $t \geq 0$ under the topology of uniform convergence on compact intervals. Let $BC \subseteq F$ denote the Banach space, under the supremum norm $|\cdot|_0$, of those functions in F which are bounded for $t \geq 0$.

Let C^1 denote the Banach space of continuously differentiable functions on $t>0$ for which $|f|_1=|f(0)|+|f'|_0<+\infty$ and let C_0^1 denote the Banach subspace of those functions in C^1 vanishing at $t=0$. Let R^n denote n-dimensional Euclidean space. In the notation of § 2 we take $B_1=BC$, $B_2=C^1$ and $B_2^1=R^n$, $B_2^2=C_0^1$.

We consider the problem of finding bounded solution $y(t) \in BC$ of the system

$$(3.1) \qquad y(t)+\int_0^t A(t,s)y(s)ds=f(t), \qquad t\geq 0$$

for $f(t) \in C^1$. We assume that (3.1) is well posed; that is

H4: $\begin{cases} \text{the matrix } A(t,s) \text{ is sufficiently smooth so that (3.1) has, for each } f \in F, \text{ a} \\ \text{unique solution } y \in F \text{ which depends continuously on } f \text{ on finite intervals} \\ \text{(i.e., if } f_n \to 0 \text{ in } F \text{ then } y_n \to 0 \text{ in } F). \end{cases}$

Certainly H4 holds for example if $A(t,s)$ is continuous on $t\geq s\geq 0$. Many other conditions on $A(t,s)$ which are sufficient to guarantee the validity of H4 can be found in standard references (e.g., see [12]; also see H5 below).

We will say that the Volterra integral system (3.1) is *(BC, C^1)-admissible with respect to the decomposition $C^1=R^n\oplus C_0^1$* if for each $g(t) \in C_0^1$ there corresponds at least one initial condition $y_0 \in R^n$ such that the solution of (3.1) with $f(t)=y_0+g(t)$ lies in BC. The connection with the work in § 2 is made via the lemma below. Define the linear operator $L: F\to F$ by

$$(3.2) \qquad Ly\equiv y(t)+\int_0^t A(t,s)y(s)ds.$$

Then H4 implies H1.

Lemma 1. *If* H4 *holds and the system* (3.1) *is* (BC, C^1)*-admissible with respect to the decomposition* $C^1=R^n\oplus C_0^1$ *then the operator* L *defined by* (3.2) *is strongly* (BC, C^1)*-admissible with respect to the same decomposition as was defined in* § 2.

Proof. The spaces $B_1=BC$, $B_2=C^1$ and $B_2^2=C_0^1$ are Banach spaces and $B_2^1=R^n$ (and hence C as defined in § 2 above) is finite dimensional. Suppose y^* is in the F-closure of some ball $S(r)$ in BC. Then there exists a sequence of functions $y_n \in BC$, $|y_n|_0\leq r$ such that $y_n\to y^*$ in F; i.e., $y_n(t)\to y^*(t)$ uniformly on every compact subinterval of $t\geq 0$. Thus, $|y^*(t)|\leq r$ on every compact subinterval of $t\geq 0$ which implies that $|y^*(t)|\leq r$ for all $t\geq 0$ or in other words that $y^* \in S(r)$. This means that every ball $S(r)$ in BC is F-closed. The Corollary in § 2 now implies the lemma. ∎

As a final property of L as defined by (3.2) we need the necessary (C^1, F)-compactness of L^{-1} restricted to C_0^1.

Lemma 2. *Assume that the matrix $A(t,s)$ satisfies the hypothesis*

H5: $\left\{\begin{array}{l} A(t,s) \text{ is measurable in } (t,s) \text{ for } 0 \leq s \leq t < +\infty \text{ and, for every } T>0, \\ \text{satisfies} \\ \qquad \int_t^{t'} |A(t,s)|\,ds \longrightarrow 0, \qquad \int_0^T |A(t',s)-A(t,s)|\,ds \longrightarrow 0 \\ \text{as } t' \to t,\ t' \geq t \text{ uniformly for } t, t' \in [0,T]. \end{array}\right.$

Then the inverse L^{-1} *restricted to* C_0^1 *is* (C^1, F)*-compact.*

Remark 3. That H5 implies H4 follows from the general results in [12, Chapter 2]. In particular H5 holds if $A(t,s)$ is continuous in $t \geq s \geq 0$.

Proof. Let $g_n \in C_0^1$ and $|g_n|_1 \leq r$. We have to show that $y_n = L^{-1} g_n$ has a convergent subsequence in F. This we will do by means of the Ascoli-Arzéla theorem applied on an arbitrary, but fixed interval $[0, T]$.

Firstly, we show that the sequence $y_n(t) \in BC$ is uniformly bounded in n and $t \in [0, T]$. Let $BC(T)$ (or $C^1(T)$) denote the Banach space of functions continuous (or continuously differentiable) on $[0, T]$ under the norm $|y|_T = \sup_{[0,T]} |y(t)|$ (or $|y|_T^1 = |y(0)| + |y'|_T$). If we restrict L defined by (3.2) to $BC(T) \to BC(T)$, then by H4 this restriction L_T is one-one, onto and continuous. As a result Banach's well known theorem implies that L_T^{-1} is bounded. Since a sequence bounded in C_0^1 is necessarily bounded in $BC(T)$ we see that $y_n = L^{-1} g_n$ is bounded in $BC(T)$ uniformly in n; i.e., $|y_n(t)|_T \leq |L_T^{-1}|\, Tr \equiv K(T)$ for all n.

Finally, we prove that the sequence $y_n(t)$ is equicontinuous on $[0, T]$. For $t \in [0, T]$ and every n

$$y_n(t) = g_n(t) - \int_0^t A(t,s) y_n(s)\,ds$$

and hence

$$|y_n(t) - y_n(t')| \leq |g_n(t) - g_n(t')| + K(T) \int_0^T |A(t',s) - A(t,s)|\,ds + K(T) \int_t^{t'} |A(t,s)|\,ds.$$

Since the derivatives of $g_n(t)$ are uniformly bounded, the equicontinuity of $y_n(t)$ on $[0, T]$ follows from this inequality and H5. ■

Suppose now we consider the perturbed Volterra system

$$x(t) + \int_0^t A(t,s) x(s)\,ds = \int_0^t h(t,s,x(s))\,ds + f(t), \qquad t \geq 0. \tag{3.3}$$

In order to apply our main result Theorem 1 above to this system we need the following hypotheses on the perturbation kernel h:

H6: $\left\{\begin{array}{l} h(t, s, z) \text{ is continuous and continuously differentiable in } t \text{ for } 0 \leq s \leq t < +\infty \text{ and } |z| \leq r < +\infty \text{ and satisfies } |h(t,t,z)-h(t,t,0)| \leq \alpha |z|^q \text{ and } |h_t(t,s,z)-h_t(t,s,0)| \leq \beta(t,s)|z|^q \text{ for } |z| \leq r,\ 0<q\leq 1 \text{ and } 0\leq s\leq t \text{ where } \alpha \text{ is a constant and } \beta \text{ is a function satisfying} \\ \sup_{t\geq 0} \int_0^t \beta(t,s)ds = \beta_0 < +\infty . \end{array}\right.$

Let $\theta=\alpha+\beta_0$ and define the constant

$$h_0=|h(t,t,0)|_0+\sup_{t\geq 0}\int_0^t |h_t(t,s,0)|\, ds.$$

From Theorem 1 in § 2 and Lemmas 1 and 2 we obtain the following result.

Theorem 4. *Suppose that $A(t,s)$ satisfies* H5 *and that the linear system* (3.1) *is (BC, C^1)-admissible with respect to the decomposition $C^1=R^n \oplus C_0^1$. Suppose that the perturbation kernel h satisfies* H6 *with θ and h_0 sufficiently small. Let $g(t) \in C_0^1$ be a given function with $|g|_1$ sufficiently small. Then corresponding to each small initial condition $y_0 \in R^n$ for which* (3.1) *has a solution in BC with $f(t)\equiv y_0$ there exists at least one initial condition $x_0=x_0(y_0) \in R^n$ for which the perturbed system* (3.2) *has a solution $x=x(y_0) \in BC$ with $f(t)\equiv x_0+g(t)$; moreover, $y_0^1 \neq y_0^2$ implies that $x(y_0^1)(t) \not\equiv x(y_0^2)(t)$.*

Proof. This result follows immediately from Lemmas 1 and 2 and Theorem 1 of § 2. The operator $p(x)$ in H3 is defined by

$$p(x)\equiv\int_0^t h(t,s,x(s))ds+g(t). \quad \blacksquare$$

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