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Uniqueness and Comparison of Harmonic Functions under Nonlinear Boundary Conditions

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1. THE PROBLEM

Consider the boundary problem

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{in} \quad S, \qquad \frac{\partial u}{\partial n} = h(s) f_1(u, v) \quad \text{on} \quad \partial S \qquad (1.1)$$

where S is a region in the x, y plane whose boundary ∂S is a simple, closed curve along which s denotes arc length. Here n denotes the outwardly directed normal to ∂S and v is any harmonic conjugate to u in S. For simplicity the functions $h(s)$ and $f(u, v)$, given in advance, are assumed to be analytic in their arguments, although this assumption could be significantly weakened in the sequel (e.g. $h(s) \in C^0, f_1 \in C$ would suffice); moreover, it is assumed that a solution to (1.1) exists.

Martin [3, 7, 8, 9, 10], Levin [6], Dunninger [2, 3, 4], and Cushing [1] have studied various aspects of the uniqueness of solutions to (1.1) and Dunninger [2, 3] has developed theorems which compare solutions of (1.1) to solutions of a second problem

$$\Delta u = 0 \quad \text{in} \quad S, \qquad \frac{\partial u}{\partial n} = h(s) f_2(u, v) \quad \text{on} \quad \partial S. \qquad (1.2)$$

Accompanied by certain hypotheses on the functions f_1 and f_2 these results are usually formulated so as to assert that the existence of a non-constant solution $w_1 = u_1 + iu_3$ to (1.1) satisfying certain conditions implies the non-existence of a non-constant solution $w_2 = u_2 + iu_4$ to (1.2) satisfying certain conditions; quite often these conditions include an assumption concerning the range $M_2 \subseteq R$ of the transformation

$$u_1 = u_1(x, y), \qquad u_2 = u_2(x, y), \qquad u_3 = u_3(x, y),$$
$$u_4 = u_4(x, y), \qquad (x, y) \in S.$$

In this paper we will obtain such comparison theorems for (1.1) and (1.2) (which become uniqueness theorems for (1.1) if $f_1 \equiv f_2$) under the assumption that the manifold M_2 is restricted to a sufficiently small neighborhood of a given point $\bar{r} = (r_1, r_2, r_3, r_4) \in R^4$, i.e., that the solutions w_1, w_2 differ from given constants by a sufficiently small constant $\epsilon > 0$. Under this assumption, our theorems imply many of the uniqueness theorems of Martin and Dunninger without the restriction, which is quite often made in the literature, that f_1 and f_2 are of a highly specialized nature.

Finally, we remark that the application of some of our results to a uniqueness question of Levi-Civita [6] for the problem (1.1) with $f_1 = e^{-3v} \sin u$, $h(s) \equiv$ const. > 0 which arises in the mathematical theory of steady, periodic water waves will be the subject of another paper.

2. The Integral Identity

We begin with an identity developed by Martin and Dunninger

$$\int_{\partial S} \tau \left(\phi_2 \frac{\partial u_1}{\partial n} - \phi_1 \frac{\partial u_2}{\partial n} \right) ds = \int_S Q \, dS \tag{2.1}$$

where

$$Q = ap_1^2 + 2bp_1p_2 + cp_2^2 + ap_3^2 + 2bp_3p_4 + cp_4^2 + 2d(p_2p_3 - p_1p_4) \tag{2.2}$$

is a quadratic form in the variables $p_i = \partial u_i / \partial x$ $(i = 1, 2, 3, 4)$ the coefficients of which are

$$a = \phi_2 \tau_{u_1}, \qquad 2b = (\phi_2 \tau)_{u_2} - (\phi_1 \tau)_{u_1}$$

$$c = -\phi_1 \tau_{u_2}, \qquad 2d = -(\phi_2 \tau)_{u_4} - (\phi_1 \tau)_{u_3}. \tag{2.3}$$

Here $w_1 = u_1 + iu_3$, $w_2 = u_2 + iu_4$ are any two analytic functions on S and $\phi_1 = \phi_1(u_1, u_3)$, $\phi_2 = \phi_2(u_2, u_4)$, $\tau = \tau(u_1, u_2, u_3, u_4)$ may be taken as arbitrary functions of their arguments so long as care is taken to insure the existence of the integrals in (2.1). The identity (2.1) is an easy application of Gauss' Theorem and the Cauchy-Riemann equations for w_1, w_2 (see Dunninger [2]).

3. Definite Forms Q and Comparison Theorems

To bring out the relationship between this identity (2.1) and comparison theorems for problems (1.1) and (1.2) we set $\phi_1 = f_1(u_1, u_3)$ and

$\phi_2 = f_2(u_2, u_4)$. If $w_1 = u_1 + iu_3$, $w_2 = u_2 + iu_4$ are solutions to (1.1), (1.2) respectively then (2.1) reduces to

$$\int_S Q\, dS = 0 \tag{3.1}$$

where Q is the quadratic form (2.2) whose coefficients (2.3) depend upon the as yet unspecified function $\tau = \tau(u_1, u_2, u_3, u_4)$. Let D denote the set of points in four dimensional Euclidean space R^4 at which Q is positive definite; D clearly depends on the choice of τ.

LEMMA 3.1. *If $w_1 = u_1 + iu_3$ is a nonconstant solution to* (1.1), *then no solution $w_2 = u_2 + iu_4$ to* (1.2) *can exist for which the integrals in* (2.1) *exist and $M_2 \subseteq D + D^*$ where D^* is a manifold in R^4 defined by $u_1 = \psi(u_3)$ for some continuously differentiable function ψ.*

PROOF. Let $R = \{(x, y) \in S : M_2 \subset D\}$ and $R^* = \{(x, y) \in S : M_2 \subset D^*\}$ and assume w_1, w_2 are solutions for which $M_2 \subseteq D + D^*$. This means $S = R + R^*$. If R^* contains an open neighborhood N of a point $(x_0, y_0) \in S$, then $u_1 = \psi(u_3)$ on N and consequently $u_1 = ku_3 + 1$ where k, $1 =$ constants since two harmonic functions related functionally must be linearly related. But in addition u_1, u_3 are conjugates and hence, by an easy application of the Cauchy-Riemann equations, they must be constants on N and thus in S, contrary to assumption. We conclude that any point in R^* is a limit point of R and by continuity (since $Q > 0$ on R) that $Q \geqslant 0$ on S. Identity (2.1) then implies $Q \equiv 0$ on S which in turn implies the contradiction that $p_i = 0$ on R (and, hence, on $S = R + R^*$).

In general the problem is to determine τ so as to make the set D as large as possible; however, we wish to choose τ in such a way that $D + D^*$, for some suitable manifold D^*, contains at least as ϵ-neighborhood

$$N(\bar{r}, \epsilon) = \left\{(u_1, u_2, u_3, u_4) \in R^4 : 0 \leqslant \sum_{i=1}^{4} (u_i - r_i)^2 < \epsilon\right\}$$

of a given $\bar{r} \in R^4$.

REMARK. If one or both of the functions f_1, f_2 do not depend on v, then Lemma 3.1 may be reformulated in an obvious manner. For example, in the important case that both f_1, f_2 are independent of v, then the lemma may be stated in terms of the manifold

$$M_1 : u_1 = u_1(x, y), \qquad u_2 = u_2(x, y), \qquad (x, y) \in S$$

where now $D \subseteq R^2$ and $\bar{r} \in R^2$.

It is well known that the quadratic form Q will be positive definite if and only if all the descending principal minors of its associated symmetric matrix are all positive. These turn out to be

$$a, \qquad ac - b^2, \qquad a(ac - b^2 - d^2), \qquad (ac - b^2 - d^2)^2$$

where a, b, c, d are given in (2.3) and consequently Q will be positive definite if and only if the inequalities

$$a > 0, \qquad \Delta = b^2 + d^2 - ac < 0$$

hold. These are partial differential inequalities for the unknown function which can be rewritten as

$$a = f_2\tau_1 > 0 \tag{3.2}$$

$$\Delta = U^2 + V^2 + W < 0 \tag{3.3}$$

where

$$2U = f_1\tau_1 + f_2\tau_2 + (f_2' - f_1')\,\tau, \qquad 2V = f_1\tau_3 + f_2\tau_4 + (f_1^{\cdot} + f_2^{\cdot})\,\tau,$$
$$W = f_1(f_1' - f_2')\,\tau\tau_1 . \tag{3.4}$$

Here we have written

$$\tau_i = \partial\tau/\partial u_i \qquad (i = 1,2,3,4,)$$

and

$$f_1' = \frac{\partial f_1}{\partial u_1}, \qquad f_1^{\cdot} = \frac{\partial f_1}{\partial u_3}, \qquad f_2' = \frac{\partial f_2}{\partial u_2}, \qquad f_2^{\cdot} = \frac{\partial f_2}{\partial u_4}.$$

Suppose now that $\bar{r} \in R^4$ is given and that τ has the form

$$\tau = \sum_{k,l,m,n=0}^{\infty} \frac{\alpha_{klmn}}{k!l!m!n!} (u_1 - r_1)^k (u_2 - r_2)^l (u_3 - r_3)^m (u_4 - r_4)^n \tag{3.5}$$

where α_{klmn} are as yet unspecified constants. We will ultimately choose all but a finite number of the α_{klmn} to be zero so that we will have no problem of convergence for the series (3.5). If into the expressions (3.2)-(3.4) we insert this power series for τ and also the series

$$f_1 = \sum_{k,m=0}^{\infty} \frac{f_1^{(km)}(r_1, r_3)}{k!m!} (u_1 - r_1)^k (u_3 - r_3)^m, \qquad f_1^{(km)}(r_1, r_3) = \frac{\partial^{k+m} f_1(r_1, r_3)}{\partial u_1{}^k\, \partial u_3{}^m}$$

$$f_2 = \sum_{l,n=0}^{\infty} \frac{f_2^{(ln)}(r_2, r_4)}{l!n!} (u_2 - r_2)^l (u_4 - r_4)^n, \qquad f_2^{(ln)}(r_2, r_4) = \frac{\partial^{l+n} f_2(r_2, r_4)}{\partial u_2{}^l\, \partial u_4{}^n}$$

we obtain power series expansions for the quantities a and Δ in terms of the variables $u_i - r_i$, $i = 1, 2, 3, 4$. Letting a zero superscript denote evaluation at the point $\bar{r}$ we find from (3.4) that

$$\begin{aligned} 2\mathring{U} &= \mathring{f}_1\alpha_{1000} + \mathring{f}_2\alpha_{0010} + (\mathring{f}_2' - \mathring{f}_1')\,\alpha_{0000}\,, \\ 2\mathring{V} &= \mathring{f}_1\alpha_{0100} + \mathring{f}_2\alpha_{0001} + (\mathring{f}_1^{\cdot} + \mathring{f}_2^{\cdot})\,\alpha_{0000} \\ \mathring{W} &= \mathring{f}_1(\mathring{f}_1' - \mathring{f}_2')\,\alpha_{0000}\alpha_{1000} \end{aligned}$$

where $\mathring{f}_1 = f_1(r_1, r_3)$, $\mathring{f}_2 = f_2(r_2, r_4)$, etc. If we set

$$\alpha_{0000} = \mathring{f}_1\mathring{f}_2(\mathring{f}_2' - \mathring{f}_1'), \qquad \alpha_{1000} = \mathring{f}_2\,, \qquad \alpha = -\mathring{f}_2(\mathring{f}_1' + \mathring{f}_2')\,(\mathring{f}_2' - \mathring{f}_1'),$$

$$\alpha_{0010} = -\mathring{f}_1 - \mathring{f}_1(\mathring{f}_2' - \mathring{f}_1')^2,$$

$$\alpha_{0001} = \alpha_{klmn} = 0 \qquad \text{for} \qquad k + l + m + n \geqslant 2$$

then

$$\mathring{U} = \mathring{V} = 0 \qquad \text{and} \qquad \mathring{\Delta} = -\,[\mathring{f}_1\mathring{f}_2(\mathring{f}_1' - \mathring{f}_2')]^2 < 0, \qquad \mathring{a} = \mathring{f}_2^{\,2} > 0$$

provided $\mathring{f}_1$, $\mathring{f}_2$, and $\mathring{f}_2' - \mathring{f}_1'$ do not vanish. Consequently, there exists an $\epsilon > 0$ such that

$$\Delta = \mathring{\Delta} + \cdots < 0, \qquad a = \mathring{a} + \cdots > 0$$

(where the dots denote, as always in the sequel, terms of higher order in $u_i - r_i$) provided $\sum_{i=1}^{4} (u_i - r_i)^2 < \epsilon$; i.e., $N(\bar{r}, \epsilon) \subseteq D$ for $\epsilon > 0$ sufficiently small. Lemma 3.1 now yields

THEOREM 3.1. *Suppose $f_1(r_1, r_3) \neq 0$, $f_2(r_2, r_4) \neq 0$, and $f_1'(r_1, r_3) \neq f_2'(r_2, r_4)$ for a given $\bar{r} \in R^4$. Then for any $\epsilon > 0$ sufficiently small there cannot exist two solutions, w_1 ($\not\equiv$ const.) to (1.1) and w_2 to (1.2), which satisfy $M_2 \subseteq N(\bar{r}, \epsilon)$.*

In order to study the case $f_1(r_1, r_3) = 0$ we assume $\tau = f_1 T$ where

$$T = T(u_2, u_4) = \sum_{k,l=0}^{\infty} \frac{\beta_{kl}}{k!\,l!}\,(u_2 - r_2)^k\,(u_4 - r_4)^l, \qquad \beta_{kl} = \text{consts.} \tag{3.6}$$

One then readily finds from (3.4) that

$$2V = f_1[f_2T_2 + f_2'T], \qquad 2V = f_1[f_2T_4 + (2f_1^{\cdot} + f_2^{\cdot})\,T]$$

$$W = f_1^2(f_1' - f_2')\,f_1'T^2$$

and, consequently, $\Delta = f_1^{\,2}\Delta^*$ where

$$4\Delta^* = [\mathring{f}_2\beta_{10} + \mathring{f}_2'\beta_{00}]^2 + [\mathring{f}_2\beta_{01} + (2\mathring{f}_1^{\cdot} + \mathring{f}_2^{\cdot})\,\beta_{00}]^2 + 4\mathring{f}_1'(\mathring{f}_1' - \mathring{f}_2')\,\beta_{00}^2\,.$$

If we assume $\mathring{f}_2 \neq 0$, $\mathring{f}_1{}' \neq 0$ and choose

$$\beta_{00} = \mathring{f}_2 \mathring{f}_1{}', \qquad \beta_{10} = -\mathring{f}_1{}' \mathring{f}_2{}',$$
$$\beta_{01} = -\mathring{f}_1{}'(2\mathring{f}_1{}^{\cdot} + \mathring{f}_2{}^{\cdot}), \qquad \beta_{kl} = 0, \qquad k + l \geqslant 2$$

in the expansion (3.6), we find

$$\Delta = f_1{}^2[\mathring{f}_1{}'(\mathring{f}_1{}' - \mathring{f}_2{}')(f_2 f_1{}')^2 + \cdots] \leqslant 0, \qquad a = (\mathring{f}_2 \mathring{f}_1{}')^2 + \cdots > 0$$

provided

$$\mathring{f}_2{}' > \mathring{f}_1{}' \qquad \text{and} \qquad \sum_{i=1}^{4} (u_i - r_i)^2 < \epsilon_1 .$$

Here we have assumed that $\mathring{f}_1{}' > 0$; the case $\mathring{f}_1{}' < 0$ is handled by replacing $-f_1$ and $-h(s)$ for f_1 and $h(s)$ in (1.1) and (1.2). Q is accordingly positive definite at those points in S where $f_1 \neq 0$. Since $\mathring{f}_1{}' \neq 0$ we may solve $f_1(u_1, u_3) = 0$ for $u_1 = \psi(u_3)$ provided $(u_1 - r_1)^2 + (u_3 - r_3)^2 < \epsilon_2$ where ϵ_2 is sufficiently small. Thus, $N(\bar{r}, \epsilon) \subseteq D + D^*$ for $0 < \epsilon < \min(\epsilon_1, \epsilon_2)$ where D^* is the manifold $u_1 = \psi(u_3)$.

THEOREM 3.2. *Suppose*

$$f_1(r_1, r_3) = 0, \qquad f_2(r_2, r_4) \neq 0, \qquad f_2{}'(r_2, r_4) > f_1{}'(r_1, r_3) > 0$$

holds for a given $\bar{r} \in R^4$. *Then the conclusion of Theorem 3.1 holds.*

The possibility of $\mathring{f}_1 = \mathring{f}_2 = 0$ is clearly ruled out in Theorems 3.1 and 3.2 and, since the hypotheses of our theorems depend upon the function τ, one might ask if it is possible in this case to choose τ such that $D + D^*$ contains a neighborhood $N(\bar{r}, \epsilon)$. Unfortunately in the case that $f_1(u_1, r_3)$, $f_2(u_2, r_4)$ change sign across $u_1 = r_1$, $u_2 = r_2$ respectively such a τ does not exist as is shown by the following generalization of a lemma due to Martin [10].

LEMMA 3.2. *If for a given* $\bar{r} \in R^4$ *the functions* $f_1(u_1, r_3)$, $f_2(u_2, r_4)$ *change sign at* $u_1 = r_1$, $u_2 = r_2$ *respectively, then necessarily a vanishes on the manifold* $u_2 = r_2$ *and c on the manifold* $u_1 = r_1$; *moreover, unless* $\tau \equiv 0$ *on some neighborhood of* $\bar{r}$, *then* $a > 0$ *elsewhere in* $N(\bar{r}, \epsilon)$, *for any* $\epsilon > 0$ *sufficiently small, implies* $\Delta > 0$ *somewhere in* $N(\bar{r}, \epsilon)$ *for any function* τ *continuously differentiable in the variables* u_i, $i = 1, 2, 3, 4$.

PROOF. It is understood that all expressions in the following proof are evaluated at $u_3 = r_3$, $u_4 = r_4$ and that all statements are valid in a sufficiently small neighborhood of $u_1 = r_1$, $u_2 = r_2$.

That a, c vanish as stated follows immediately from their definitions (2.3). Assume now that $a > 0$, $\Delta \leqslant 0$ in some $N(\bar{r}, \epsilon_0)$, $\epsilon_0 > 0$, and $u_1 \neq r_1$,

$u_2 \neq r_2$. Since f_1, f_2 change sign at $u_1 = r_1$, $u_2 = r_2$ respectively the conditions $a > 0$, $c \geqslant 0$ (implied by $a > 0$, $\Delta \leqslant 0$) require that τ_1, τ_2 change sign across and, hence, vanish on $u_2 = r_2$, $u_1 = r_1$ respectively. Consequently, $\tau \equiv k = \text{const.}$ on $u_1 = r_1$ and $u_2 = r_2$. We distinguish two cases.

(i) Suppose f_1, f_2 are both strictly increasing (decreasing) across $u_1 = r_1$, $u_2 = r_2$ respectively. Then $f_1 f_2 > 0$ in the quadrant $Q_1 : u_1 > r_1$, $u_2 > r_2$ and, hence, $\Delta = b^2 + d^2 - ac \leqslant 0$ implies $- ac = f_1 f_2 \tau_1 \tau_2 \leqslant 0$ or $\tau_1 \tau_2 \leqslant 0$ on Q_1. Unless $\tau \equiv k$, this is impossible since $\tau = k$ on $u_1 = r_1$, $u_2 = r_2$. However, if $\tau \equiv k$ then

$$\Delta = k^2[(f_2' - f_1')^2 + (f_1^{\cdot} + f_2^{\cdot})^2] \geqslant 0$$

and as a result $\Delta \equiv 0$; but then $k = 0$, for $f_2' \equiv f_1'$ only if $f_1 = f_2 = \text{const.}$ contrary to assumption.

(ii) Suppose f_1 is strictly increasing while f_2 is decreasing across $u_1 = r_1$, $u_2 = r_2$ respectively; then $f_1' \geqslant 0$, $f_2' \leqslant 0$. In Q_1, $f_1 > 0$ and, moreover, $a = f_2 \tau_1 > 0$ implies $\tau_1 < 0$. Consequently, on Q_1 we have $f_1(f_1' - f_2')\, \tau_1 < 0$ and, hence, $\tau \geqslant 0$ since $\Delta \leqslant 0$ implies $W \leqslant 0$. A similar argument shows that in the quadrant $Q_4 : u_1 > r_1$, $u_2 > r_2$ we have $\tau \leqslant 0$, $\tau_1 > 0$.

But $\tau_1 < 0$, $\tau \geqslant 0$ on Q_1 implies $k > 0$ while $\tau_1 > 0$, $\tau \leqslant 0$ on Q_4 implies $k < 0$. This contradiction concludes the proof.

This lemma explains the difficulty encountered throughout the literature when using these methods in a neighborhood of points $\bar{r}$ where

$$f(r_1, r_3) = f_2(r_2, r_4) = 0.$$

Occasionally, however, this difficulty has been overcome (e.g. in some of the results of Martin, Dunninger and Cushing in [1, 2, 10]) by additional assumptions on the solutions w_1, w_2, viz. that $\lambda = f_1(u_1, u_3)/f_2(u_2, u_4)$ remains smooth in $S + \partial S$. Motivated by such results we choose τ of the form

$$\tau = \lambda T, \qquad \lambda = \frac{f_1}{f_2}$$

where T has the form (3.6). This yields

$$2U = \lambda f_2 T, \qquad 2V = \lambda[f_2 T_4 + 2 f_1^{\cdot} T], \qquad W = \lambda^2 f_1'(f_1' - f_2')\, T^2$$

and $\Delta = \lambda^2 \Delta^*$ where

$$\mathring{\Delta}^* = \beta_{00}^2[\mathring{f}_1^{\cdot 2} + \mathring{f}_1'(\mathring{f}_1' - \mathring{f}_2')].$$

Letting $\beta_{00} = 1/\mathring{f}_1'$, $\beta_{kl} = 0$ for $k + 1 \geqslant 1$ in (3.6) (i.e., $\tau = \lambda/\mathring{f}_1'$) we have

$$\Delta = \left(\frac{\lambda}{\mathring{f}_1'}\right)^2 [\mathring{f}_1^{\cdot 2} + \mathring{f}_1'(\mathring{f}_1' - \mathring{f}_2') + \cdots] \leqslant 0, \qquad a = 1 + \cdots > 0 \tag{3.9}$$

provided

$$\mathring{f}_1'^2 + \mathring{f}_1'(\mathring{f}_1' - \mathring{f}_2') < 0 \tag{3.10}$$

(which implies $\mathring{f}_1' \neq 0$) and $\sum_{i=1}^{4} (u_i - r_i)^2 < \epsilon_1$. Although τ as a function of *independent* variables u_i, $i = 1, 2, 3, 4$, is not defined at $u_2 = r_2$, $u_4 = r_4$ if $f_2(r_2, r_4) = 0$, we may assume τ is C^1 as a function of x, y in $S + \partial S$ in order to insure the validity of identity (2.1). Doing this, we see from (3.10) that Q is positive definite in S except at those points where $f_1(u_1, u_3) = 0$. Since $\mathring{f}_1' \neq 0$, we may solve $f_1(u_1, u_3) = 0$ for $u_1 = \psi(u_3)$ provided

$$(u_1 - r_1)^2 + (u_2 - r_2)^2 < \epsilon_2 \qquad \text{for} \qquad \epsilon_2 > 0$$

sufficiently small and conclude that

$$N(\bar{r}, \epsilon) \subseteq D + D^* \qquad \text{for} \qquad 0 < \epsilon \leqslant \min(\epsilon_1, \epsilon_2)$$

where D^* is the manifold $u_1 = \psi(u_3)$.

THEOREM 3.3. *Suppose* $f_1(r_1, r_3) = f_2(r_2, r_4) = 0$ *and* (3.10) *holds for a given* $\bar{r} \in R^4$. *Then the conclusion of Theorem 3.1 holds for those solutions* w_1, w_2 *such that*

$$\lambda = \frac{f_1}{f_2} \in C'(S + \partial S).$$

The case $f_1 = f_1(u)$, $f_2 = f_2(u)$ in which the harmonic conjugate v does not appear in (1.1), (1.2) is important. The following is a corollary of Theorems 3.1–3.3.

COROLLARY 3.1. *Suppose* $f_1 = f_1(u)$, $f_2 = f_2(u)$ *in* (1.1), (1.2) *respectively.*

(a) *If for a given constant* $\bar{r} \in R^2$ *either*

(i) $f_1(r_1) \neq 0$, $f_2(r_2) \neq 0$, $f_1'(r_1) \neq f_2'(r_2)$ *or*

(ii) $f_1(r_1) = 0$, $f_2(r_2) \neq 0$, $f_2'(r_2) > f_1'(r_1) > 0$

then for any $\epsilon > 0$ *sufficiently small there cannot exist two solutions,* u_1 ($\not\equiv$ const.) *to* (1.1) *and* u_2 *to* (1.2), *for which* $M_1 \subseteq N(\bar{r}, \epsilon)$.

(b) *If* $f(r_1) = f(r_2) = 0, f_2'(r_2) > f_1'(r_1) > 0$ *then the conclusion of part* (a) *holds among those solutions such that*

$$\frac{u_1 - r_1}{u_2 - r_2} \in C'(S + \partial S).$$

We need only note that since $f_2'(r_2) \neq 0$ the expansion

$$\lambda = \frac{f_1}{f_2} = \frac{u_1 - r_1}{u_2 - r_2} \frac{f_1'(r_1) + \cdots}{f_2'(r_2) + \cdots}$$

implies $\lambda \in C'(S + \partial S)$ if and only if $(u_1 - r_1)/(u_2 - r_2) \in C'(S + \partial S)$ at least for ϵ sufficiently small.

The smoothness requirement $\lambda \in C'$ in Theorem 3.1 cannot in general be dropped. Considering the linear problems $h(s) \equiv 1$, $f_1 = u$, $f_2 = nu$ on the unit disk $x^2 + y^2 \leqslant 1$ where $n > 1$ is a positive integer, we find solutions $u_1 = \delta r \sin \theta$, $u_2 = \delta r^n \sin n\theta$ where r, θ are polar coordinates and δ is an arbitrary constant. Clearly $M_1 \subseteq N(0, \epsilon)$ for any $\epsilon > 0$ provided $|\delta|$ is sufficiently small. Note, however, that $\lambda = \sin \theta / r^{n-1} \sin n\theta \notin C'$ on the unit disk.

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